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Fuzzy Proximity and Fuzzy Uniformity

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This is to certify that the subject matter of the thesis is record of work done by Md. Arshaduzzaman himself under my guidance and that the contents of his thesis did not form a basis of the award of any previous degree to him or to the best of my knowledge, to any body else and that the thesis has not been submitted by the candidate for any research degree in any other University.

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I hereby certify that this thesis entitled ***"FUZZY PROXIMITY AND FUZZY UNIFORMITY"*** submitted by Sri MD. ARSHADUZZAMAN, M.Sc. (Math) for the award of DOCTOR OF PHILOSOPHY IN MATHEMATICS, for T.M. Bhagalpur University, Bhagalpur (India), is a record of bonafide research carried out under my supervision. I further certify that major portion of the thesis is his own contribution & no part of it, to the best of my knowledge had been submitted by the candidate for Ph.D. degree or its equivalence in any other University.

P.K. Lal 30/8/05
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INTRODUCTION

Since its inception, the theory of fuzzy sets has evolved in many directions and is finding applications in a wide variety of fields in which the phenomena under study are too complex or too ill defined to be analyzed by conventional techniques. The theory of fuzzy sets have a substantial impact on scientific methodology in years ahead particularly in the realms of psychology, economics, law, medicine, decision analysis, information retrieval & artificial intelligence. Fuzzy set concepts and fuzzy algorithms proposed by L.A. Zadeh have been developed since 1965 and they have been applied to various fields. Zadeh has discussed the advantage of using the fuzzy sets concepts in engineering systems and studied its algorithms.

As a discipline, fuzzy sets have roots in set theory and multivalued logic and generalize along these lines. The abolishment of two valued logic (Yes-No) dogma has led to a series of interesting mathematical insights and investigations that can easily stand on their own. The term fuzzy mathematics goes a bit too far — as the language of mathematics is universal, so is its rigour, from which the development of theoritical foundations benefits greatly.

Fuzziness is a kind of uncertainty. Since the 16th century, probability theory has been studying a kind of uncertainty — randomness, i.e. the uncertainty of the occur of an event: but in this case, the event itself is completely certain, the only uncertain thing is whether the event will occur or not, the casuality is not completely clear now. However, there exist another kind of uncertainty — fuzziness i.e. for some events, it can not be completely determined that which cases these events should be subordinated to (e.g. they have already occurred or have not occured yet), they are in a non-black & non- white state that is to say, the law of excluded middle in logic can not be applied any more. Which case an event should be subordinated to, in mathematical view, is just that which set the "element" standing for the event should belong to. However, in mathematics a set A can be equivalently represented by its characteristic function — a mapping χ_A from the universe X of discourse (region of consideration, i.e. a larger set) containing A to the 2-value set {0,1}, i.e. it is to say, *x* belongs to A iff $\chi_A(x) = 1$. But in "Fuzzy" case "belonging to" relation $\chi_A(x)$ between *x* & A is no longer "0 or otherwise 1", it has a degree of "belonging to" i.e. membership degree such as 0.6. Therefore, the range has to be extended from {0,1} to [0,1]; or more generally, a lattice 1, because all the membership degrees, in mathematical view, form an ordered structure, a lattice.

Thus fuzzy set extended the basic mathematical concept—set. In view of the fact that set theory is the cornerstone of modem mathematics, a new and more general framework of mathematics was established. Fuzzy mathematics in just a kind of mathematics developed in this framework, and fuzzy topology is just a kind of topology developed on fuzzy sets.

We denote the family of all the fuzzy sets on the universe X, which takes [0,1] as the range by I^x, where 1= {0,1}. Substituting inclusive relation by the order relation in I^x, we introduce a topological structure naturally into I^x. So that fuzzy topology is a common carried of ordered structure & topological structure. Fuzzy topology fuses just two large structures — ordered structure & topological structure. Fuzzy topology naturally possessed "point like" structure. This structure is a basic characteristic in fuzzy topology. Fuzzy topology has developed in such an extent that it can react upon its foundation. That is to say, the results obtained thus far in fuzzy topology have important applications in some other branches of fuzzy set theory (for instance, the theory of convex fuzzy sets).

In general fuzzy topological spaces, several problems arise naturally : Is it possible to localize properties, such as for instance continuity & convergence in a consistent way ? Is it possible to characterise the closure of a fuzzy set using a degree of closeness of a point to a fuzzy set by means of a notion of fuzzy neighbourhoods ? so on. The main idea behind the common solution of these problems is the notion of fuzzy neighbourhood systems. R. Lowen, U. Höhle, & R.H. Warren. Earlier attempts at describing fuzzy topologies by means of certain types of fuzzy neighbourhoods can be found in the papers of R. Lowen, U. Höhle, & R.H. Warren. Concepts like prefilterbases, prefilters introduced by Lowen are important tools to study fuzzy neighbourhood systems. Due to this the systems of fuzzy neighbourhoods are some what different but far more significantly the fuzzy closure operators & consequently the associated fuzzy topologies are totally different.

In ordinary topology quasi-uniform spaces are the generalization of metric spaces. A uniformity for X is the quasi-uniformity with additional axiom. It is interesting to note that its fuzzy counterpart has many interesting results. Bruce Hutton extended the notion of quasi-uniformities and uniformities on topological spaces to fuzzy topological spaces. In particular every fuzzy topological spaces is quasi-uniformizable. The fuzzy unit interval plays an essential part in a characterization of uniformizability in terms of a type of complete regularity. To achieve this a natural uniformity on the fuzzy unit interval may be constructed.

Another generalization of metric spaces in ordinary topology is the proximity space. Two sets are near whenever there is a relation of proximity between them.

Fuzzy proximities are natural extension of the classical case with some additional axioms. Every fuzzy uniformity induces a fuzzy proximity & vice-versa, & the induced topologies do not change at any step.

The concept of a fuzzy syntopogenous structure was introduced for the first time in A.K. Katsaras & C.G. Petalas's paper. The fuzzy topologies, the fuzzy proximities & the Hutton fuzzy uniformities are special cases of these structures. In the area of the fuzzy uniformity, the New-Zealand mathematician Hutton has done a piece of profound and penetrating work. The works of Liu Ying-Ming provide useful algebraic tool for investigation of fuzzy uniformities. A definition of a fuzzy uniformity different from that of Hutton was given by Lowen. Also G. Artico & R. Moresco gave a notion of a fuzzy proximity different from that given by A.K. Katsaras. The fuzzy topology induced by a Lowen fuzzy uniformity or by an Artico-Moresco fuzzy proximity has been given by some fuzzy nhd structure. A.K. Katsaras gave a new definition of a fuzzy syntopogenous structure — the fuzzy neighbourhood spaces, the Lowen fuzzy uniformities and the Artico-Moresco fuzzy proximities are special cases of these stuctures. Though every Lowen fuzzy uniformity induces a fuzzy proximity, the correspodence cannot work well, since the two structures do not give the same structure.

Artico-Moresco provided a new definition of fuzzy proximity which differs from the old one slightly, in one axiom fifth ($\delta(\mu,\rho)=o \Longrightarrow \mu < \rho'$) L= {0,1}, the fifth axiom means exactly that if two subsets intersect, then they are proximal. In case L= [0,1]=I, the fifth axiom means that μ & ρ are proximal whenever there exists $x \in X$ s.t. $\mu(x)+\rho(x)>l$. There are various notions of uniformity in fuzzy set theory. Some of them are the Höhle-Katsaras uniformity & the Lowen-Höhle uniformity. The Höhle- Katsaras uniformity (i.e. for T= min & is a straightforward generalization of the uniformity axioms in terms of entourages. It has been found that a saturated Höhle-Katsaras uniformity (i.e. a Höhle-Katsaras T-uniformity which is also a saturated filter on X x X) is clearly a Lowen-Höhle T-uniformity. Probabilistic pseudometrics play an important role to establish the fact that the converse of the above implication is also true.

There are several view points of the notions of a metric & metrizability in fuzzy topology. They can be divided into two main groups. The first group is formed by these papers in which a fuzzy (pseudo) metric on a set X is treated as a map $d : XxX \rightarrow R^+$ where $X \subset I^x$ satisfying some collection of axioms or other that are anologues of the oridnary (pseudo-) metric axioms. Thus is such an approach numerical distances are set up between fuzzy objects. Erceg, Zike Deng, Hu Chang-Ming belong to this group. We include in the second group these papers in which the distance between objects is fuzzy; the objects themselves may be either crisp, or (more seldom) fuzzy. The most interesting papers in this direction are these of Kaleva, Seikkala & Eklund & Gahler.

R.N. Lai & P.K. Lal have introduced the concept of a quasi pseudo n-metroid which is a pseudo n-metroid on a fuzzy lattice & of a pseudo n-metric with some additinal invariance property. Unified theory of spatial structures has been studied by Cs$\bar{a}$sz$\bar{a}$r, Doicinov and P.K. Lal & R.N. Lal with the introduction of a fuzzy m-n syntopogenous structure on a set P in terms of m-n tuple of fuzzy set relations coarser than super fuzzy set relation, a new approach to a syntopogenous structure can be established. A fuzzy symmetrical m-n topogenous structure characterises a fuzzy m-n proximity generalising fuzzy proximity.

Our thesis has been divide into four chapters inclusive of pre-requisite chapter O.

Chapter 1 has three Sections:

Section I is addressed to fuzzy nhd systems prefilter basis & prefilter have been studied in detail. This section also deals with relation between fuzzy nhd systems & fuzzy topologies.

In section II of this chapter, we have continued our investigation of fuzzy nhd syntopogenous structures. Some results with slight modification have been established. New results have been established in section III of this chapter. Relationship between two important types of uniformities has been established with the help of probabilistic pseudo metric.

Chapter 2 has been further divided into two sections. Section I of this chapter deals with fuzzy proximity, fuzzy uniformity & connection between fuzzy proximities & fuzzy uniformities.

Section II of chapter II is mainly concerned with fuzzy syntopogenous structures, correspondence between fuzzy nhd structure and perfect fuzzy topogenous structure, correspondence between fuzzy proximities & symmetrical fuzzy topogenous structure, correspondence between fuzzy quasi uniformities and biperfect fuzzy syntopogenous structures.

Chapter 3 of our thesis is addressed to inverse image of a fuzzy semitopogenous order, inverse image of a fuzzy syntopogenous structure, continuity of function on fuzzy syntopogenous stucture and product fuzzy syntopogenous spaces.

Chaper 4 has two sections :

Section I of this chapter is concerned with fuzzy quasi-proximities, initial fuzzy quasi proximities & product of fuzzy quasi-proximity spaces. Some new proposition have been established. Our contribution to this chapter is section II. The section is concerned with totally new concepts enriching classical fuzzy spatial structures based on fuzzy n-metroid lattice & semi n-uniformity. These concepts are Fuzzy m-n syntopogenous space; n-uniform space, m-n proximity space.

CHAPTER - 0

INVOLUTION :

A function $\theta : P \to Q$ from a poset P to a poset Q is called order preserving or isotone if it satisfies

$x \le y \Rightarrow \theta(x) \le \theta(y)$

An isotone function which has an isotone two sided inverse is called an isomorphism.

An isomorphism from a poset P to istself is called an automorphism. Dual automorphisms are called involutions.

CHANG FUZZY TOPOLOGICAL SPACE :

The basic set carrying the fuzzy topological structure will be denoted by X; the power class of X by 2^X and the fuzzy power class of X by I^X (I=[0,1]). As usual, ϕ and X denote the fuzzy sets given by $\phi(x) = o$, $\forall\, x \in X$ and $X(x) = 1$, $\forall\, x \in X$. Chang fuzzy topological spaces (Shortly Chang fts), is (X, τ) where τ is an ordinary subclass of I^X that contains ϕ, X and is closed under finite (fuzzy) intersections and arbitrary (fuzzy) unions. A fuzzy set s in X is called a fuzzy singleton iff its support (supp s) reduces to a crisp singleton. A fuzzy singleton will often be denoted by x_ε where {x} ($x \in X$) is the support and ε ($\in$] 0,1]) the value of the fuzzy singleton,

FUZZY AND NATURAL FUZZY TOPOLOGIES :

For fuzzy sets, we use the symbols $\le, \vee, \wedge, 1, -$ respectively. Let F be a class of functions $E \to [0,i]$ which satisfies the following conditions :

(i) the constant functions 0 and 1 belong to F,

(ii) if μ_i belong to F for $i \in I$ then $\vee_i\, \mu_i$ belongs to F,

(iii) if μ_1, and μ_2 belong to F, then $\mu_1 \wedge \mu_2$ belongs to F.

Then F is the family of characteristic functions of a class F of fuzzy sets which is a fuzzy topology in the sense of Chang. F contains ϕ and E and respects fuzzy union and finite fuzzy intersection.

The family $F^c = \{l-\mu, \mu \in F\}$ is obviously called the family of closed fuzzy sets of E.

As examples, we can take $F = [0,1]^E$, defining the discrete fuzzy topology, or $F = [0]^E \cup [1]^E$, defining the null fuzzy topology. Other cases are possible, taking, for instance, $F = \{0,1\}^E$.

If (E,T) is a topological space we have a particularly interesting class F : the lower semicontinuous functions (l.s.c) satisfy the conditions (i), (ii) & (iii). The corresponding fuzzy topology is denoted by τ (T) and is called the natural topology; the l.s.c. functions from E to {0,1} are exactly the characteristic functions of the open sets of the topological space E. The closed sets are of course associated with the upper semicontinuous functions (u.s.c).

NORMALITY IN FUZZY TOPOLOGICAL SPACES & FUZZY UNIT INTERVAL :

Normality is one of the few separation axioms which can be defined purely in terms of the properties of the open and closed sets (i.e., with no mention of points) We characterise normality in terms of a "Urysohn" type lemma, and in the process construct a fuzzy topological space which plays the important role in fuzzy topological spaces that the unit interval plays in ordinary topological spaces.

Definition :

A fuzzy topological space is normal if for every closed set K and open set U such that $K \subseteq U$, there exists a set V such that

$K \subseteq V^o \subseteq \overline{V} \subseteq U$

Normality can be characterised in terms of Urysohn Lemma as follows

(Urysohn lemma) :

A fuzzy topological space (X,t) is normal if and only if for every closed set K and open set U such that $K \subseteq U$, there exists a continuous function $f: x \to [0,l](L)$ such that for every $x \in X$

$K(x) \leq f(x)(1-) \leq f(x)(0+) \leq U(x)$

Fuzzy unit Interval :

Under certain lattice conditions the fuzzy topology of the fuzzy unit interval is like the topology of the ordinary unit interval.

Let (L, ≤,') be a completely distributive lattice with orthocomplement. Then there exists a natural 1-1 correspondence between the open sets in the usual topology for [0,1| and the open sets in the fuzzy topology for [0,1] (L) which preserves arbitrary unions and finite intersections.

PROXIMITY & UNIFORMITY STRUCTURES :

A **Proximity structure** in a set X is a relation $<$ in the set of all subsets of X, satisfying the following axioms :

1. $A<B$ implies $A\subset B$.
2. $A\subset B<C\subset D$ implies $A < D$.
3. $A_i < B$ for all $i \in I$, I finite, implies $\bigcup_i A_i < B$; $A < B_i$ for all $i \in I$, I finite implies $A < \bigcap_i B_i$
4, $A < C$ implies that there exists B such that $A<B<C$. Taking I void in axiom 3, we see that $\phi < A < X$ for every set A in X. A proximity structure $<_1$ is called finer than $<$ if $A < B$ implies $A <_1 B$. A set X has a finest proximity structure : the discrete structure in which $A<B$ whenever $A\subset B$ It also has a least fine proximity structure in which $A<B$ only if $A = \phi$ or $B = X$. A set X, together with a proximity structure $<$ in it, is called a proximity space.

The proximity structure $<'$, such that $A <' B$ if and only if $X \setminus B < X \setminus A$, is called the conjugate of $<$. The proximity structure $<$ is called symmetric if $<' = <$. We shall not assume an axiom of symmetry. A proximity structure in X induces a topology in X, a set A being a neighbourhood of a point x if $(x) < A$. A finer proximity structure induces a finer topology.

A Uniform structure in a set X is a family V = {u} of functions from X to the set 2^x of all subsets of X, satisying the following axioms :

1. For each $x \in X$ and each $u \in V$, $x \in u(x)$
2. If $u \in V$ and $u < v$ (i.e., $u(x) \subset v(x)$ for all x), then $v \in V$
3. If $u_i \in V$ for $i \in I$, I finite, then $\cap_i u_i \in V$
4. Given $u \in V$ there exists $v \in V$ such that $v^2 < u$, i.e., if $y \in v(x)$, $v(y) \subset u(x)$.

In axiom 3, $\cap u_i$ is the function which assigns to the point x the set $\cap u_i(x)$. The case of axiom 3 when I is void states that the maximal function **1**, defined by **1** (x) = X for all $x \in X$, belongs to V. Thus V is not empty. A uniform structure W is called finer than V if $V \subset W$. There is a finest uniform structure in X consisting only of the function **1.**

The function v', defined by $v'(x) = \{ y : y \in X, x \in v(y)\}$, is called the conjugate of v. The family V' = {u'} of conjugates of functions u in the uniform structure V is itself a uniform structure, called the conjugate of V. The uniform structure V is called symmetric if V' = V.

A uniform structure V induces a proximity structure $<$ as followes : $A<B$ if there exists $u \in V$ such that $\cup_{X \in A} u(x) \subset B$.

FUZZY UNIFORMITY SPACE :

Let X be a set and I the unit interval. A fuzzy set in X is an element of the set I^x of all functions μ from X into I. If f is a function from X into Y and $\mu \in I^Y$, then $f^{-1}(\mu)$ is the element of I^x which is defined by

$$f^{-1}(\mu)(x) = \mu(f(x))$$

Also for a $\sigma \in I^x$, $f(\sigma)$ is the member of I^Y defined by

$$f(\sigma)(y) = \begin{cases} \sup\limits_{x \in f^{-1}[y]} \sigma(x) & \text{if } f^{-1}[y] \text{ is not empty.} \\ = o & \text{otherwise} \end{cases}$$

A closure operator an I^x is a map $\mu \to \bar{\mu}$ from I^x into I^x such that for μ, ρ in I^x we have

(1) $\mu \leq \overline{\mu}$

(2) $\overline{\overline{\mu}} = \mu$

(3) $\overline{\mu \vee \rho} = \overline{\mu} \vee \overline{\rho}$

Given a closure operator on I^X, the collection

$\{\mu : \overline{1-\mu} = 1-\mu\}$

defines a fuzzy topology on X.

Let $\rho \in I^x$ and $\mu \in I^{XxX}$. We define $\mu < \rho > \in I^X$ by

$$\mu\langle\rho\rangle (x) = \sup_{y \in X} \rho(y) \wedge \mu (y,x)$$

For $\mu, v \in I^{XxX}$, the composition $\mu \Delta v$ defined by

$$\mu \Delta v(x,y) = \sup_{z \in X} v(x,z) \wedge \mu (z,y)$$

A fuzzy uniformity on X is a subset $\mathcal{U}$ of I^{XxX} such that

1) $\mu, v \in \mathcal{U}$ implies $\mu \wedge v \in \mathcal{U}$
2) $\mu \leq v \in \mathcal{U}$ implies $\mu \in \mathcal{U}$
3) For every $\mu \in \mathcal{U}$ and every $x \in X$ we have $\mu(x,x)=l$
4) For every $\mu \in \mathcal{U}$, we have $\tilde{\mu} \in \mathcal{U}$
   where $\tilde{\mu}(x,y)=\mu(y,x)$.

(5) Given $\mu \in \mathcal{U}$ there exists $v \in \mathcal{U}$
   with $v \Delta v \leq \mu$.

A Fuzzy uniformity $\mathcal{U}$ on X defines a Fuzzy topology $\tau(\mathcal{U})$ by

$\tau (\mathcal{U}) = \{\mu \in I^x : \psi(1-\mu)=1-\mu\}$

QUASI-UNIFORMITIES ON FUZZY TOPOLOGICAL SPACES :

In defining a quasi - uniformity for a fuzzy topology, we take our basic elements of the quasi-uniformity to be elements of the set $\mathcal{L}$ of maps $D : L^X \rightarrow L^X$ which satisfy :

(Al) $V \subseteq D(V)$ for $V \in L^x$

(A2) $D(\cup V_\lambda) = \cup D(V_\lambda)$ for $V_\lambda \in L^x$

Definition : A (fuzzy) quasi - uniformity on a set X is a subset $\mathcal{D}$ of $\mathcal{L}$ (the set of all maps) satisfying (Al) and (A2) such that :

(Q1) $\mathcal{D} \neq \phi$

(Q2) $D \in \mathcal{D}$ and $D \subseteq E \in \mathcal{L}$ implies $E \in \mathcal{D}$.

(Q3) $D \in \mathcal{D}$ and $E \in \mathcal{D}$, implies $E \in \mathcal{D}$

(Q4) $D \in \mathcal{D}$ implies there exists $E \in \mathcal{D}$,
Such that

$E \circ E \subseteq D$

We note that (Q3) may replaced by

(Q3') $D_1 \in \mathcal{D}$ and $D_2 \in \mathcal{D}$ imply there exists $D \in \mathcal{D}$ such that $D \subseteq D_1$ and $D \subseteq D_2$

Also we note that any subset $\mathcal{B}$ of $\mathcal{L}$ which satisfies (Q4) generates a fuzzy quasi - uniformity in the sense that the collection of all $D \in \mathcal{L}$ which contain a finite intersection of elements of $\mathcal{B}$ is a quasi-uniformity. Such a set $\mathcal{B}$ is called a sub-basis for the quasi-uniformity generated If $\mathcal{B}$ also satisfies (Q3') then $\mathcal{B}$ is called a basis.

Every fuzzy topology is fuzzy quasi-uniformizable.

A quasi-uniformity $\mathcal{D}$ is a uniformity if it also satifies

(Q5) $D \in \mathcal{D}$ implies $D^{-1} \in \mathcal{D}$

(Q5') $\mathcal{D}$ has a base of symmetric elements.

FUZZY QUASI-UNIFORMITIES:

A fuzzy L-quasi-uniformity (or just a fuzzy quasi-uniformity) is a subset $\mathcal{U}$ of $I^{X \times X}$ which is a prefilter and has the following three properties.

(FUl) $\alpha(x,x)$ for all $\alpha \in \mathcal{U}$ and all $x \in X$

(FU2) for each $\alpha \in \mathcal{U}$ and each $\varepsilon > o$ there exists $\alpha_1 \in \mathcal{U}$ such that $\alpha_1 \text{ o } \alpha_1 - \varepsilon \leq \alpha$

(FU3) $\hat{\mathcal{U}} = \mathcal{U}$, i.e. for every family

$\{\alpha_\varepsilon \in \mathcal{U} : o<\varepsilon<l\}$ we have

$\sup_\varepsilon (\alpha_\varepsilon - \alpha_\varepsilon) \in \mathcal{U}$.

Every fuzzy quasi-uniformity $\mathcal{U}$ on X induces a fuzzy neighbourhood structure $N_{\mathcal{U}}$ where $N_{\mathcal{U}}(x) = \{\alpha_x : \alpha \in \mathcal{U}\}$, $\alpha_x(y) == \alpha(x,y)$.

Also, the mapping

$$\mu \to \bar{\mu},\ \bar{\mu}(x) = \inf_{\alpha \in \mathcal{U}} \ \operatorname{sap}_{y} \ \mu(y) \wedge \alpha(x, y)$$

is a fuzzy closure operator on X & so it induces a fuzzy topology

$$t(\mathcal{U}) = \{\mu : \overline{1-\mu} = 1-\mu\}$$

It is obvious that $t(\mathcal{U}) = t(N_{\mathcal{U}})$

FUZZY PROXIMITY SPACE :

A binary relation δ on the power set of a set X is called a proximity onX if δ satisfies the following axioms :

(PI) A δ B implies B δ A.

(P2) $(A \cup B)\ \delta C$ iff $A\delta C$ or $B\delta C$.

(P3) $A\delta B$ implies $A \neq \phi$, $B \neq \phi$

(P4) $A\delta B$ implies that there exists a subset E of X such that $A\delta E$ and $(X\text{-}E)\delta B$.

(P5) $A \cap B \neq \phi$ implies $A\delta B$.

Generalizing the notion in the case of fuzzy sets, we give the definition of a fuzzy proximity space.

Definition :

A binary relation δ on I^x is called a fuzzy proximity if δ satisfies the following axioms :

(FP1) $\mu\delta\rho$ implies $\rho\delta\mu$

(FP2) $(\mu \vee \rho)\delta\sigma$ iff $\mu\delta\sigma$ or $\rho\delta\sigma$

(FP3) $\mu\delta\rho$ implies $\mu \neq o$ and $\rho \neq o$.

(FP4) $\mu\delta\rho$ implies that there exists a $\sigma \in I^x$ such that $\mu\delta\sigma$ and $(l\text{-}\sigma)\delta\rho$.

(FP5) $\mu \wedge \rho \neq o$ implies $\mu\delta\rho$.

The pair (X, δ) is called a fuzzy proximity space.

PROXIMAL AND DUAL PROXIMAL OPERATION :

Definition :

A proximal operation on a lattice L is a binary operation Δ satisfying for every a,b,............,x of L the properties :

Δ 1. $a \wedge b < a\ \Delta\ b < a$;

Δ 2. $o\ \varepsilon\ \hbar \Rightarrow a\ \Delta\ o = o$;

Δ3. $a < b \Rightarrow x\ \Delta\ a < x\ \Delta\ b$, $a\Delta x < b\Delta x$;

Δ4. $a\Delta(l\ \Delta\ b) < a\Delta(a\Delta b) < a\Delta b$, if 1 exists .

Definition :

A proximal operation Δ on a lattice is called Kuratowskian provided Δ satisfies the properties Δ_1, Δ_2, Δ_4 listed above as well as Δ 3d listed below:

Δ3d. $a\Delta(b \vee c) = (a\Delta b) \vee (a\Delta c)$

$(b \vee c)\ \Delta\ a = (b\Delta a) \vee (c\Delta a)$

A proximal operation leads to the generalisation of a closed elemental structure.

NEARNESS CONCEPT IN FUZZY PROXIMITY SPACES :

Lowen introduced a category of fuzzy uniform spaces and fuzzy unifrom maps, which we denote by FU. He showed that the fts's associated with his fuzzy uniformities are fuzzy neighbourhood spaces. Artico and Moresco introduced a category of fuzzy proximity spaces and fuzzy proximity maps, which is denoted by FP. They showed that FU and FP are compatible to a good extent.

The fts's associated with fuzzy proximities are fuzzy neighbourhood sapces. This adds to the extent to which FU, FP and FNS are related in the desired manner.

Adequately axiomatized fuzzy relations of nearness between crisp subsets is sufficient to define FP.

It can be realized in the following proposition :

A fuzzy proximity on a set X is a function $\delta: I^x \times I^x \to I$ which satisfies, for any $U,V,W \in I^x$, the following conditions :

(PI) $\delta(\mathbf{0},\mathbf{1}) = 0$

(P2) $\delta(U,V) = \delta(V.U)$

(P3) $\delta(U,V) \vee \delta(W,V) = \delta(U \cup W, V)$

(P4) If $\delta(U, V) = \gamma$, for every $\theta \in I_0$ there exist $A,B \in I^x$ such that $A \cup B = 1$, $A \cap B \supseteq \gamma$, $\delta(U,A) \le \gamma + \theta$ and $\delta(B,V) \le \gamma + \theta$

(P5) $\delta(U,V) \ge (U \cap V)(x)$ for every $x \in X$

(P6) If $|V - W| \subseteq \theta$ for $\theta \in I$, then $|\delta(U,V) - \delta(U,W)| \le \theta$ for every $U \in I^x$

The pair (x,δ) is said to be a fuzzy proximity space.

The number $\delta(U,V)$ can be interpreted as the degree of nearness of the fuzzy sets U and V.

PREFILTER :

Definiton **:-** A subset $F \subset I^x$ is a prefilter iff $F \neq \phi$ and

(i) For all $\mu, v \in F$ we have $\mu \wedge v \in F$.

(ii) If $\mu \ge v$ and $v \in F$, then $\mu \in F$.

(iii) $0 \notin F$.

Definition :- *A* subset $B \subset I^x$ is a basis for a prefilter iff $B \neq \phi$ and:

(i) for all $\mu, v \in B$, there is a $\gamma \in B$ such that $\gamma \le \mu \wedge v$

(ii) $0 \notin B$.

The prefilter F generated by B is defined as :

$$F = \{\mu \in I^x : 0 \exists v \in B \text{ s.t. } \mu \ge v\},$$

and is denoted by (B).

A subset B of F is a basi s for F iff for all $\mu \in F$ there is a $v \in B$ such that $v \ge \mu$. For two prefilters F and G such that $F \subset G$, we shall say that F is coarser than G and that G is finer than F.

Definition:

A prefilter F is called a prime prefilter iff for al μ, $\nu \in I^x$ such that $\mu \vee \nu \in F$, we have either $\mu \in F$ or $\nu \in$ F.

For a prefilter F on X the following are equivalent:

(i) F is a prime prefilter.

(ii) For all $A,B \subset X$, if $\chi_{A \cup B} \in F$ then either $\chi_A \in$ For $\chi_B \in$ F.

FUZZY FILTERS :

Let X be an arbitrary nonvoid set and P (X) be the power set of X. A mapping $\mu:P(X) \to [0,1]$ i.e. a fuzzy subset μ of P (X) is called a fuzzy filter on X iff μ satisfies the following conditions

(F1) $\mu(\phi)=o$, $\mu(X)=l$

(F2) $\mu(A)+\mu(B) \leq \mu(A \cup B)+\mu(A \cap B)$

A fuzzy filter μ is a fuzzy ultrafilter iff μ fulfills the additional property

(F3) $l-\mu(A) \leq \mu(A)$ $\forall A \in P(X)$

From (F1) - (F3) we infer that the set of all fuzzy ultrafilters oh X coincides with the set of all finitely additive probability measures on P (X).

FUZZY METRIC:

We define a fuzzy metric on a set X as a map d : $XxX \to \Im$ 3 (R), where $\Im$ (R) is the interval real line, satisfying the axioms :

(i) d (x,y) = o iff x = y

(ii) d (x,y) = d (y,x)

(iii) $d(x, z) \leq d(x, y) + d(y, z)$

x, y, z $\in$ X.

A number d (x,y) (t) is treated in this connection as the “Possibility” that the distance between x and y is equal to t. The pair (X,d) is called a fuzzy metric space. A more general definition may be given according to which a fuzzy metric space is a quadruple (X,d,L,R), where L,R : $I^2 \to$ I are symmetric decreasing functions with L (0,0) = 0,R (1,1) = 1 In the case where L=Min, R = Max, this definition is equivalent to the one presented above.

FUZZY N-METRIC LATTICE :

Let L be an atomic lattice, with its carrier as the atomic set L_o of atoms, denoted by the letters p,q,r,s,t ....

Then a mapping $a:L_0 \to I=[0,l]$ is a fuzzy element of L, with its carrier as the atomistic L_0, which is a fuzzy atom, if it has a singleton support, say p, of degree $\alpha \in (0,1) = I_0$ denoted by α p or p^{α}, satisfying a = $\vee$ $v\ \alpha$ p/αp $\leq$ a

An (m,n)-tuple of fuzzy (f) atoms will be denoted by $\langle \alpha_i\ p_i,\ \beta_j\ p_j \rangle$ $i \in m,\ j \in n$, and in particular an (1,n) - tuple by (αp;α_j p_j). An (n+1)-tuple $\langle \alpha_i\ p_i \rangle$ $i \in n+1$, in which the mth term $\alpha_m\ p_m$ is replaced by a fixed α p, will be denoted by $\langle \alpha_m\ p_m \to \alpha \rangle$ and in particular $\langle \alpha_0\ p_0 \to \alpha p \rangle$ and $\langle \alpha_n\ p_n \to \alpha\ p \rangle$ will represent $\langle \alpha\ p,\ \alpha_1 p_1, \ldots\ldots\ldots, \alpha_n\ p_n \rangle$ and $\langle \alpha_0\ p_0,\ \alpha_1 p_1, \ldots\ldots\ldots, \alpha_{n-1}\ p_{n-1},\ \alpha_n\ p_n \rangle$.

The set I^{Lo} of fuzzy elements of L, called fuzzy element set, is a pseudocomplemented lattice, containing fuzzy atomic set $\mathcal{A}$.

A mapping d : $\mathcal{A}^{n+1} \to \Re^+$ is a fuzzy (f) n-metric, provided

d1. (a) d $(\overline{\alpha_i\ \ p_i})$=0; (b) d $\langle \alpha_i\ p_i \rangle = 0 \to \langle \alpha_i\ p_i \rangle = (\overline{\alpha_i\ \ p_i})$

d2. d $\langle \alpha_i\ p_i \rangle < e \to \ni \beta_0 > \alpha_0$, with d $\langle \alpha_0\ p_0 \to \beta_0\ p_0 \rangle$ < e ;

d3. d $\langle \alpha_i\ p_i \rangle = d\ \langle 1 - \alpha_{n-i}\ p_{n-i} \rangle$;

d4. d$\langle \alpha_i\ p_i \rangle \leq +_{i=0}^{n}$ d $\langle \alpha_i\ p_i \to \alpha\ p \rangle$, for every α p, which is called equilateral, provided d $\langle \alpha_i\ p_i \rangle = \alpha$, a positive real number.

A lattice L, with a fuzzy n-metric d is called a fuzzy n-metric lattice <L,$\mathcal{A}$,d>.

L-FUZZY PRETOPOLOGICAL SPACES :

Given a lattice L (Complete, with infimum 0 & supremum 1, equipped with an order reversing involution) and a non-empty set X, the L-fuzzy sets of X are just the elements of L^X, i.e., the functions from X to L. ϕ is the L-fuzzy set defined by ϕ: X$\to$L,ϕ(x) = o for each x$\in$X. For A,B $\in L^X$, the intersection A$\cap$B, union A$\cup$B, and the order reversing involution A', respectively, are defined by :

$(A \cap B)\ (x) = A(x) \wedge B(x),\ \forall x \in X.$

$(A \cup B)\ (x) = A(x) \vee B(x),\ \forall x \in X.$

$A'(x) = \xi(A(x)), \quad \forall x \in X.$

Let A,B $\in L^X$. A is included in B (A $\leq$ B) provided that A (x) $\leq$ B (x) holds for every x$\in$X. A fuzzy singleton p in X is an L-fuzzy set defined by : p (x) = t for -x=x_o and p (x) = 0 otherwise.

The point x_o is the support of p and o<t$\leq$l.

An L-fuzzy pretopology on a set X is a funaction a : $L^X \to L^X$
stich that

(P1) a(ϕ)=ϕ,

(P2) a(A) $\geq$ A

are satisfied for every A $\in L^X$
The pair (X,a) is said to be an L-fuzzy pretopological space (for short L-f pts).

Chapter-1

Section-I

Fuzzy neighbourhood (nhd) systems

&

Relation between fuzzy nhd systems and fuzzy topologies

❖

Section-II

Fuzzy nhd syntopogenous structures

❖

Section-III

Relationship between Lowen-Höhle uniformity & Höhle-Katsaras uniformity

Chapter -1

Section - I

In general fuzzy topological spaces several problems are encountered with :

Problem 1.

We consider a fuzzy set μ and a constant fuzzy set α. We can cut off μ at level α & then its closure $\overline{\mu \wedge \alpha}$ or we can first take the closure of μ & then cut it off at level α, $\bar{\mu} \wedge \alpha$. Are these two fuzzy sets same ?

Problem 2.

If $(\bar{\mu}_n)_{n\in\wedge}$ is a sequence of fuzzy sets which converges uniformly to a fuzzy set μ; do we then have $(\bar{\mu}_n)_{n\in\wedge}$ converges uniformly to $\bar{\mu}$?

Problem 3.

Is it possible to characterise the closure of a fuzzy set using a degree of closeness of a point to a fuzzy set by means of a notion of fuzzy neighbourhoods ?

Problem 4.

Is it possible to localize properties, such as for instance continuity & convergence in a consistent way ?

***Problem* 5.**

For convergent prefilter, does one have $F \subset F'$, F convergent $\Rightarrow F'$ convergent ?

A positive answer to problem 1 & 2 would be of interest for approximation problem since both questions are concerned with the interchangeability of performing certain operations on fuzzy sets & fuzzy closure i.e. adding limit points.

A positive answer to problem 3 would bring the notion of fuzzy closure out of its abstract form— the closure of a fuzzy set is the infimum of all closed fuzzy sets which are large γ & make it more tangible. In ordinary topology too this abstract form only came after the theory was axiomatized, the original form being by means of neighbourhoods. It is this form which in the most natural fashion conveys the idea of closure as the adding of limit points.

In exactly the same way a solution to problem 4 would give us a characterization of continuity which again is closer to the classical notion & to our intuition.

A positive answer to problem 5 would ensure the elimination of some pathological results in properties characterized by means of convergent prefilters. Unfortunately, in general fuzzy topological spaces the answer to all the questions posed is negative.

There exist a class of fuzzy topological spaces which solves all these questions in the positive.

The main idea behind the common solution of these problems is the notion of fuzzy neighbourhood systems.

Remarks :

Earlier attempts were madeby R. Lowen, H. Ludescher, E. Roventa & R.H. Warren at describing fuzzy topologies by means of certain types of fuzzy nhds. Ludescher & Roventa gave a definition which is such that different fuzzy topologies can have the same systems of fuzzy nhds which is rather unpleasant.

Warren's definition is linked in a straightforward manner to the notion of open fuzzy set which makes it possible to describe every fuzzy topological space in a unique way by a system of fuzzy nhds & vice versa. Höhle defined so called probabilistic topological spaces by means of a system of L-fuzzy nhds.

Notations & Preliminaries:

The unit interval shall be denoted by I. I_0 mean the interval [0,l] & I_1 stands for [0, 1].

Filters will be denoted by capital script letters.

If X is a set & $Y \subset X$, we shall denote by I_Y the characteristic function of Y.

A fuzzy closure on X is a map $^-: I^X \to I^X$ satisfying the following properties

(clos 1) - For all α contant, $\overline{\alpha} = \alpha$

(clos 2) - For all $\mu \in I^X$; $\overline{\mu} \geq \mu$

(clos3) - For all $\mu, \xi \in I^X$, $\overline{(\mu \vee \xi)} = \overline{\mu} \vee \overline{\xi}$

(clos 4) - For all $\mu \in I^X$; $\overline{\overline{\mu}} = \mu$

Prefilter and Prefilter basis:

A prefilter F is a non-empty subset of I^X not containing 0, stable for finite intersections & such that if $\mu \in F$ & $\xi \geq \mu$ then $\xi \in F$.

A prefilter basis B is a non empty subset of I^X not containing 0 & such that for all $\mu, \xi \in B$, there exist $\theta \in B$ s.t. $\theta \leq \mu \wedge \xi$

If B is a prefilter basis then we denote the prefilter generated by it

$[B] = \{\mu \in L\ I^X : \exists \beta \in B, \beta \leq \mu\}$ & we denote by $\hat{B}$ the following family

$$\hat{B} = \left\{ \sup_{\in \in I_0} (\beta_\in - \in) : (\beta_\in)_\in \in_{I_0} \in B^{I_0} \right\}$$

We shall only mention here few propositions on prefilter & prefilter basis as per our main aim is to study the nhd systems.

Proposition 1 :

If B & B' are prefilter bases then :

(i) $B \subset \hat{B}$

(ii) $\hat{B} \subset [\hat{B}]$

(iii) $B \subset B' \Rightarrow \hat{B} \subset \hat{B}'$

Proposition 2 :

If B is a prefilter basis then $[\hat{B}]=[\hat{B}]$

We shall denote $\overline{B}$ the prefilter $[\hat{B}]= [\hat{B}]$

Proposition 3 :

If B & B' are prefilter bases, then

(i) $B \subset \overline{B}$ (») $B = B$

(ii) $\tilde{\tilde{B}} = \tilde{B}$

(iii) $B \subset B' \Rightarrow \tilde{B} = \tilde{B}'$

1. FUZZY NEIGHBOURHOOD SYSTEMS

Definition 1: A collection of prefilter $(B(x))_{x\in X}$ is called a fuzzy neighbourhood system iff the following conditions are satisfied.

(NH1) - For all $x \in X$ & for all $v \in B(x)$, $v(x) = 1$

(NH2) - For all $x \in X$;

$\hat{B}_{(x)} = B(x)$ i.e. for all family $(v_\in)_{\in\in I_0}$ of elements of B(x) we have $\underset{\in\in I_0}{\text{Sup}}(v_\in - \in) \in B(x)$

(NH3) - for all $x \in X$, for all $v \in B(x)$ & for all $\in \in I_0$ there exists a family

$(v_Z^\in)_{z\in X}$ s.t. for all $y \in X$; $\underset{z\in X}{\text{Sup}}\ v_X^\in(z) \wedge v_Z^\in(y) - \in \leq v(y)$

B(x) is called a fuzzy nhd prefilter in x & the elements of B(x) are called fuzzy nhds of x.

Remark :

An equivalent way of expressing (NH3) obviously is to say that for all $x\in X$ & for all $v\in B(x)$, there exists a double indexed family $(v_Z^\in)_{z\in X\ \in\in I_0}$ such that for all $z\in X$, & for all $\in \in I_0$, $v_Z^\in \in B(z)$ & such that for all $y \in X$,

$$\sup_{\in\, \in\, I_0} \left(\sup_{z \in X} v_x^{\in}(z) \wedge v_z^{\in}(y) - \in\right) \leq v(y)$$

The family $(v_z^{\in})_{z\in X}$ will be called an $\in$-kernel for v & the family $(v_z^{\in})_{z\in X,\, \in\in I_0}$ a kernel for v.

Without any confusion & for simplicity a fuzzy nhd system $(B(x))_{x\in X}$ will be denoted by B.

Definition 2 :

A collection of prefilter bases $(B(x))_{x\in X}$ is called a fuzzy neighbourhood base iff the following conditions hold

(PBI) - for all $x \in X$ & $\beta \in B(x)$;

$$\beta(x) = 1$$

(PB2) - for all $x \in X$,

$$\beta \in B(x) \ \& \in \in I_0$$

there exists a family $(\beta_z^{\in})_{z\in X}$ such that for all $z \in X$, $\beta_z^{\in} \in B(z)$ & such that for all $y \in X$.

$$\sup_{z\in X} \beta_x^{\in}(z) \wedge \beta_z^{\in}(y) - \in \leq \beta(y)$$

B(x) is called a fuzzy nhd base in x & the elements of B(x) are called basic fuzzy nhds.

Analogously we say the family $(\beta_z^{\in})_{z\in X}$ is an $\in$ - kernel & the family $(\beta_z^{\in})_{z\in X,\, \in\in I_0}$ is a kernel for β. We denote a fuzzy nhd base $(B(x))_{x\in X}$ simply by B.

Definition 3 :

If B is a fuzzy nhd system, then we shall say that B is a basis for B iff for all $x \in X$; B(x) is a prefilter basis & $\tilde{B}(x) = B(x)$. We then also say that B(x) is a basis for B(x).

Remark :

The difference between a basis B for some prefilter F & a basis B(x) for some fuzzy nhd prefilter B(x). In the first case F is generated by B while in the second case B(x) is generated by $\hat{B}(x)$.

Theorem 1 :

If $(B(x))_{x\in X}$ is a fuzzy nhd base, then $(\tilde{B}(x))_{x\in X}$ is a fuzzy nhd system with $(B(x))_{x\in X}$ as a basis.

Proof **:**

The last part of the theorem is clear. We shall only show that $(\tilde{B}(x))_{x\in X}$ is indeed a fuzzy nhd system. (NH_1) is obvious & (NH_2) follows at once from the fact that ~ is idempotent. To prove (NH_3) let $x \in X$, $v \in \tilde{B}(x)$ & $\in \in I_0$ then there exists a family

$(B_\delta)_{\delta \in I_0} \in B(x)^{I_0}$ such that $v \geq \sup_{\delta \in I_0} (B_\delta - \in)$.

In particular, $v \geq \beta_{\in/2} - \frac{1}{2} \in$.

Since B(x) fulfills (PB_2) there exist an $\frac{1}{2} \in$-kernel $(\beta_z^{\in/2})_{z \in X}$ for $\beta_{\in/2}$ (in B). From the fact that

$$\sup_{z \in X} \beta_x^{\in/2}(z) \wedge \beta_z^{\in/2}(y)$$

$$\leq \beta_{\in/2}(y) + \frac{1}{2} \in$$

$$\leq v(y) + \in$$

For all $y \in X$ & from the fact that $B(x) \subset \tilde{B}(x)$. It follows that $(\beta_z^{\in/2})_{z \in X}$ is an $\in$-kernel for v.

Theorem 2 :

If $(B(x))_{x \in X}$ is a basis for the fuzzy nhd system $(B(x))_{x \in X}$ then $(B(x))_{x \in X}$ is a fuzzy nhd base.

Proof:

(PB_1) is obvious. To show (PB_2), let $\beta \in B(x) \subset B(x)$ & let $\in \in I_0$. Then there exists an $\frac{1}{2} \in$ kernel $(v_z^{\in/2})_{z \in X}$ for β (in B). For each $z \in X$ there exists a fuzzy set

$\beta_z^{\in} \in B(z)$ s.t.

$v_z^{\in/2} \geq \beta_z^{\in} - \frac{1}{2} \in$.

We then have for all $y \in X$.

$$\sup_{z \in X} \beta_x^{\in}(z) \wedge \beta_z^{\in}(y)$$

$$\leq \sup_{z \in X} v_x^{\in/2}(z) \wedge v_z^{\in/2}(y) + \frac{1}{2} \in$$

$$\leq \beta(y) + \in$$

Which implies that $(\beta_z^{\in})_{z\in X}$ is an $\in$ - kernel for β (in B).

We shall now show that in which way a fuzzy nhd system on X determines a fuzzy topology on X by means of a fuzzy closure operator.

Theorem 3 :

If B is a fuzzy nhd system on x, then the operation $^{-}: I^x \to > I^x$ where for all $\mu\in I^x$ & $x \in X$

$$\bar{\mu}(x) = \inf_{v\in B_{(x)}} \sup_{y\in X} \mu \wedge v(y) \text{ is a fuzzy closure operator.}$$

Proof:

(Clos 1) is obvious from the definition of — . (Clos 2) follows at once from (NH_1). For (Clos 3), let μ, $\xi \in I^x$ then for any $x\in X$, we have

$$\overline{\mu \vee \xi}(x) = \inf_{v\in B_{(x)}} (\sup_{y\in X} \mu \wedge v(y) \vee \sup_{y\in X} \xi \wedge v(y))$$

$$\geq \inf_{v\in B_{(x)}} \sup_{y\in X} \mu \wedge v(y) \vee \inf_{v\in B_{(x)}} \sup_{y\in X} \xi \wedge v(y)$$

$$= \bar{\mu} \vee \bar{\xi}\ (x)$$

on the other hand

$$\bar{\mu} \vee \bar{\xi}\ (x) = \inf_{v,v' \in B_{(x)}} (\sup \mu \wedge v\ (y) \vee \sup_{y\in X} \xi \wedge v'\ (y))$$

$$\geq \inf_{v,v' \in B_{(x)}} \sup_{y\in X} (\mu \wedge v) \vee (\xi \wedge v')\ (y)$$

(& since for all v, v' $\in$ B(x) also $v \wedge v' \in$ B(x) we have,)

$$\geq \inf_{v\in B_{(x)}} \sup_{y\in X} (\mu \vee \xi) \wedge v(y)$$

$$= \overline{\mu \vee \xi}(x)$$

For (Clos 4) let $\mu \in I^x$ & $x \in X$, we have $\bar{\bar{\mu}}(x) = \inf_{v\in B_{(x)}} \sup_{y\in X} \bar{\mu} \wedge v(y)$

$$= \inf_{v\in B_{(x)}} \sup_{y\in X} \inf_{v'\in B_{(y)}} \sup_{z\in X} \mu(z) \wedge v'(z) \wedge v(y)$$

on the other hand it follows from (NH_3) that for any $v \in B(x)$ & $\in \in I_0$ there exists an $\in$ - kernel $(v_z^{\in})_{z\in X}$ for v. Then

$$\sup_{z\in X} \mu \wedge v(Z) + \in \geq \sup_{z\in X} \mu(z) \wedge (v(z) + \in)$$

$$\geq \sup_{z\in X} \mu(z) \wedge (\sup_{y\in X} v_x^{\in}(y) \wedge v_y^{\in}(z))$$

$$= \sup_{z,y\in X} \mu(z) \wedge v_x^{\in}(y) \wedge v_y^{\in}(z)$$

Now if we put

$$H = \{h:X \to \bigcup_{y\in X} B(y), \hbar(y)\in B(y)\}$$

We obtain for $\overline{\overline{\mu}}(x)$

$$\overline{\overline{\mu}}(x) \leq \sup_{y\in X} \inf_{z'\in B(y)} \sup_{z\in X} \mu(z) \wedge v'(z) \wedge v_x^{\in}(y)$$

$$= \inf_{\hbar\in H} \sup_{z,y\in X} \mu(z) \wedge \hbar(y)(z) \wedge v_x^{\in}(y)$$

$$\leq \sup_{x,y\in X} \mu(z) \wedge v_y^{\in}(z) \wedge v_x^{\in}(y)$$

$$\leq \sup_{z,y\in X} \mu \wedge v(z) + \in$$

The last inequality is true for all $v \in B(x)$ & $\in \in 4\, I_0$. It follows that

$\overline{\overline{\mu}}(x) \leq \overline{\mu}(x)$ which together with (Clos 2) proves (Clos 4).

If B is a fuzzy nhd system, then the fuzzy topology generated by the fuzzy closure it determines is denoted by $\tau(B)$.

In accordance with theorem 1 of (2) if B is a fuzzy nhd base, we shall denote the fuzzy nhd systems $\widetilde{(B(x))}_{x\in X}$ generated by it by $\tilde{B}$ & the fuzzy topology generated by it by $\tau(\tilde{B})$.

If a fuzzy topological space has a fuzzy topology which is generated by a fuzzy nhd system, we shall call it a fuzzy nhd space.

Remark :

Not every fuzzy topological space is a fuzzy nhd space.

If B & B' are fuzzy nhd systems on X, then we shall say that B is finer than B' or B' is coarser than B iff for all $x\in X$, $B'(x)\subset B(x)$ & we denote this by writing $B'\subset B$.

Remark :

In general fuzzy topological spaces it is a drawback that it is virtually impossible to define local properties such as for instance continuity in a point.

This however poses no problem in fuzzy nhd spaces which is clear below.

3. *Definition 1* :

Let (X, τ(B))&(X', τ(B')) be fuzzy nhd spaces & f : X$\rightarrow$ X'. Then we say that *f* is continous in $x_0 \in X$ iff for all $v' \in B'(f(x_0))$ we have $f^{-1}(v') \in B(x_0)$ or equivalently iff for all $v' \in B'(f(x_0))$ there exists a, $v \in B(x_0)$ s.t.

$f(v) \leq v'$

Theorem 1:- We have now the following theorem :

If (X,τ(B)) & (X', τ(B')) are fuzzy nhd spaces, B & B' are bases for B & B' respectively & f : X$\rightarrow$X'; then *f* is continous in $x_0 \in X$ iff for all $\beta' \in B'(f(x_0))$ & for all $\in \in I_0$ there exists $\beta \in B(x_0)$, s.t. $\beta - \in \leq r^{-1}(\beta')$ or equivalently iff for all $\beta' \in B'(f(x_0))$ we have $f^{-1}(\beta') \in B(x_0)$

Following is the immediate consequence of the above theorem.

If (X,τ(B)&(X', τ(B')) are fuzzy nhd spaces & f : X' $\rightarrow$ X', then f is continuous in $x_0 \in X$ iff for all $v' \in B'(f(x_0)$ & $\in \in I_0$ there exists $v \in B(x_0)$ s.t. $v - \in \leq f^{-1}(v')$.

It follows at once that each fuzzy nhd system is a basis for itself.

***Remarks* :**

In ordinary topology usually a map is defined to be continuous if it is continuous in each point. Since in general fuzzy topological spaces local continuity could not be defined in a satisfactory way as continuity was defined saying inverse images of open sets were open.

R.H, Warren provides not so much a localization of fuzzy continuity in x but rather in X x [0,1].

The following theorem is an important theorem in this connection.

Theorem 2 :

Let (X, τ (B)) & (X', τ (B')) be fuzzy nhd spaces & f : X $\rightarrow$ X'.

Then f is continuous iff it is continuous in each point of X.

2. RELATION BETWEEN FUZZY NHD SYSTEMS & FUZZY TOPOLOGIES :

We have already mentioned, not every fuzzy topological space is a fuzzy nhd space. It has been shown that a fuzzy topology determines a unique fuzzy closure & vice versa.

The most natural way to study the fuzzy topological aspects of a fuzzy nhd space then is by studying the fuzzy closure it determines.

Section - II

FUZZY NHD SYNTOPOGENOUS STRUCTURES

Fuzzy syntopogenous structures are studied as a unified theory of fuzzy topologies, fuzzy uniformities & fuzzy proximities. Lowen introduced the category of fuzzy nhd spaces, which can be considered as a universal frame work within which his earlier fuzzy uniform spaces can be accommodated.

Artico & Moresco introduced a concept of fuzzy proximities which are compatible with Lowen fuzzy uniformities. Morsi showed that those fuzzy proximities are particular to the fuzzy nhd spaces only and he introduced for them their associated fuzzy proximal nhd systems.

By identifying the common features of the fuzzy nhd systems underlying the above three notions, we present here the theory of fuzzy nhd syntopogenous structures. We then specify the above three notions as special types of the new structures. The formulation parallels both classical theory & Katsaras theory.

For every $r \in I$ we denote by r the constant fuzzy set which takes the value r for every $x \in X$. For a fuzzy set U and for every $r \in I$, the r - cut (r* - cut) of U is the crisp subset.

$U^r = \{x \in X : U(x) > r\}$

$U_{r*} = \{x \in X : U(x) \geq r\}$

of X.

Prefilters & prefilter bases were introduced by Lowen. A prefilter base in X is a non empty collection $B \in I^x$ which satisfies $O \notin B$ and every finite intersection of members of B contains an element of B. A prefilter is a prefilter base which contains all the fuzzy subsets of its individual members. We denote by [B] the prefilter generated by the prefilter base B. Lower introduced the operator $\wedge$ on prefilter bases which is called the presaturation operator. It is defined on a prefilter base B in X by

$\hat{B} = \{V_{\theta \varepsilon I_0} (\cup_\theta - \theta) : \cup_\theta \in B\} \subseteq I^x$. Lowen showed that $\hat{B}$ is also a prefilter base unless it contains O. We have also B c $\hat{B}$ c $\hat{B}$ c $[\hat{B}] = [\hat{B}]$.

The saturation operator ~ is defined on prefilter bases by $\tilde{B} = [\hat{B}]$.

A prefilter base B is called presaturated (saturated) when $\hat{B} = B (\tilde{B} = B)$.

Definition 1: A fuzzy nhd system on a set X is a family $v = (v(x))_{x \in X}$ of prefilters in X which satisfies:

(N_1) $V \in v(x) \Rightarrow V(x) = 1$ for $x \in X$

(N_2) v is presaturated,

i.e. $v = \hat{v} = (\hat{v}(x))_{x \in X}$

(N_3) given $x \in X$ and $\in\ \in I_0$ every $V \in v(x)$ has a θ - kernel in V.

This consists of a family $({}^{\theta}V_z)_{z \in x}$.

Such that for all $y, z \in X$, ${}^{\theta}V_z \in v(z)$ &

${}^{\theta}V_x(z) \wedge {}^{\theta}V_z(y) \le V(y) + \theta$.

For every fuzzy nhd system on X, an associated fuzzy closure operator on X is defined by : for $U \in I^X$ & $x \in X$, $U^-(x) = \inf_{V \in v(x)} \sup (U \wedge V)$.

The associated fuzzy topology on X is denoted by t(v) and the topological space (X, t(v)) is called a fuzzy nhd space (fns).

Proposition 1: An operator $^-: I^X \to I^X$ is the fuzzy closure operators of a fuzzy nhd space (X, $^-$) iff, '—' satisfies the following five axioms, the first four of which are properties of the restriction $^-: 2^X \to I^X$

For all $M, N \in 2^X$.

(a) $\overline{O} = O$

(b) $\overline{M} \ge M$

(c) $(M \cup N)^- = M^- \vee N^-$

(d) $([M^-])_{r^\bullet} = M^- \vee [M^-]_{r^\bullet}$
for all $r \in I_o$

(e) $^-: I^X \to I^X$ is retrieved from its restriction to crisp subsets by the formula $U^- = \vee_{r \in I} [r \wedge (U_{r^\bullet})^-]$, $U \in I^X$.

Proposition 2 : Let $v = (v(x))_{x \in X}$ be a fuzzy nhd system on X and let $^-: 2^X \to I^X$ be the restriction of the closure operator of (X, t (v)). Then for all $x \in X$,

$v(x) = \{U \in I^X : (X - U^{\alpha})^- (x) \le \alpha \text{ for all } \alpha \in I_1\}$

Because of the above two propositions, we can construct a fuzzy nhd space as a pair $(X, {}^-: 2^X \to I^X)$ satisfying the above four axioms (a), (b), (c) & (d).

Proposition 3: Let B be a saturated prefilter in a non empty set X, $U \in I^X$. Then $U \in B$ iff $U^1 \vee r \in B$ for all $r > t$ in I_1.

Definition 2:- A fuzzy proximity on a set X is a function $\delta : I^X \times I^X \to I$ which satisfies for any U,V,W $\in I^X$ the following conditions

(P_1) $\delta(0,1) = 0$

(P_2) $\delta(U, V) = \delta(V, U)$

(P_3) $\delta(U, V) \vee \delta(W, V) = \delta(U \vee W, V)$

(P_4) if $\delta(U,V) = t$ for every

$\theta \in I_O$ there exists $A, B \in I^X$ such that $A \vee B = 1$,

$A \wedge B \ge t$, $\delta(U,A) \le t + \theta$ and

$\delta\{B,V) \le t + \theta$

(P_5) $\delta(U,V) \ge (U \wedge V)(x)$ for every $x \in X$.

(P_6) if $|V-W| \leq \theta$ for $\theta \in I$, then
$|\delta(U, V) - \delta(U,W)| \leq \theta$ for every $U \in I^X$

A fuzzy proximity is completely determined by its behaviour on crisp subsets. We shall abbveviate fuzzy nhd to N.

Definition 3 : A N. topogenous order on a set X is a relation << between the crisp subsets & fuzzy subsets of X which satisfies for all $M, N \in 2^X$ & $u, v \in I^X$.

(T_1) O << O & 1 << 1

(T_2) if M<<u then $M \leq u$

(T_3) if $M \subseteq N << u \leq v$, then M << v.

(T_4) if M << u & N << v then
$M \cap N << u \wedge v$
& $M \cup N << u \vee v$

A N. topogenous order << will be called symmetric if it satisfies: "if $M, N \in 2^X$ & $r \in I$ are such that $M << (X\text{-}N) \vee r$ then $N << (X\text{-}M) \vee r$ ".

Definitions 4: *A* N. syntopogenous structure on a set X is a family S of N. topogenous orders on X satisfying.

(S_1) S is directed in the sense that for every $<<_1, <<_2 \in S$ there exist $<< \in S$ which is finer than $<<_1$ & $<<_2$ (i.e. is a finer relation).

(5_2) if $<< \in S$, then for every $\theta \in I_O$ there is $<<_\theta \in S$ such that whenever $M << N \vee r$ for M, $N \in 2^X$ & $r \in I$, there exists $C \in 2^X$ such that $M <<_\theta C \vee r + \theta$ & $C <<_\theta N \vee r + \theta$

(5_3) Saturation axiom :- For each family $\{<<_\theta \in S : \theta \in I_O\}$ there is $<< \in S$ such that whenever a set $M \in 2^X$ & a family $\{u_\theta \in I^X : \theta \in I_O]$ satisfy
$M <<_\theta u_\theta$ for all $\theta \in I_O$ then $M << V_{\theta \in I_o} [u_\theta - \theta]$

A N. syntopogennus space (X, S) is a set X with N. syntopogenous structure S on X. In case S consists of a single N topogenous order, it is called N. topogenous structure & (X, S) is called N. topogenous space.

Definition 5: (a) A N. syntopogenous structure S on X is said to be symmetric if all its members are symmetric orders.

(b) A N. topogenous order << is called perfect if it satisfies : $M_i << u_i$ for every $i \in J$, implies $\bigcup_{i \in j} M_i << \bigwedge_{i \in j} . u_i$ and is called biperfect if it is perfect and satisfies $M_i << u.$ for every $i \in J$ implies $\bigcap_{i \in j} M_i << \bigwedge_{i \in j} u_i$.

(c) A N. syntopogenous space (X, S) is called perfect (biperfect) iff every member of S is perfect (biperfect).
(d) A biperfect N. syntopogenous structure S on X is said to be full (it is called saturated) if every biperfect N. topogenous order on X which is coarser than some member of S belongs to S.

Definition 6: Let S_1 and S_2 be two families of N. topogenous orders on X, we say that S_1 is finer (or S_2 is Coarser than S_1) iff for each member $<<_2$ of S_2 there exist a member $<<_1$ of S_1 which is finer than $<<_2$. If S_2 is also finer than S_1 then they are called equivalent, written as $S_1 \cong S_2$. This relation is clearly an equivalence relation.
Using the above definition we then have :

Lemma 1: Let S and S_o be two equivalent families of N. topogenous orders on X. If S_o is a N. syntopogenous structure on X then S is also a N. syntopogenous structure on X. *Correspondence between the fuzzy nhd systems and the perfect N. topogenous structures:*

We shall now show that there is a one to one correspondence between the fuzzy nhd systems on a set and the perfect N. topogenous structures on the same set.

For a given N. syntopogenous space (X, S) and for every crisp point $x \in X$, we define the family $v_s(x) = \{\beta \in I^x : x << \beta \text{ for some } << \in S\}$

Clearly $v_s(x)$ is a prefilter which we call the prefilter of fuzzy nhds of x in S. Also given a fuzzy nhd space (X, t (v)) & $M \in 2^X$, we put $v(M) = \bigcap_{x\in M} v(x)$.

Evidently, this v(M) is a saturated prefilter & if $u \in v(M)$, then $u \geq M$.
We also have:
$v(0)=\bigcap_{x\in 0} v(x) = I^X$

Proposition 4: Let N & M be crisp subsets of a fuzzy nhd space (X, t (v)) & let $t \in I$, then $N \vee t \in v(M)$ iff $(X\text{-}N)^- \wedge M \leq t$.

Proof: For all r, $t \in I$ & $N \in 2^X$ we have :

$$\left(X-\overline{(N\vee t)^r}\right)^- = \begin{cases} 0 & \text{if } r < t \\ (X-N) & \text{if } r \geq t \end{cases} \quad (1)$$

Hence for all N, $M \in 2^X$ & $t \in I$,
$N \vee t \in v(M)= \cap \{v(x):x\in M\}$
This means for all $r \in I$ & $x \in M$,
$(X-(N \vee t)^r)^-(x) \leq r$ (Theorem... 2, Section-II)
or in other words for all $r \geq t$ &
$x\in M, (X - N)^-(x) \leq r$ using (1)
This is equivalent to :

For all
$x \in M, (X - N)^-(x) \leq t$
i.e. $(X - N)^- \wedge M \leq t$

***Proposition 5*:** In theorem 1 (Section II) in the presence of (b) & (c), the axiom (d) can be replaced by any one of the following two axioms
($d^/$) for all $M \in 2^X$, $t \in I_0$ & $x \in X$, if $M^-(x) < t$ then $((M^-)_{t^*})^-(x) < t$
($d^{//}$) for all $M \in 2^X$ & $t \in I_0$ $([(M^-)_t]^-)_{t^*} = (M^-)_{t^*}$

Proof: Clearly ($d^/$) is equivalent to ($d^{//}$) & they are weaker than (d). ($d^{//}$) follows by taking the t_* - cuts of both sides in (d).

Now let us suppose that $(X, ^-)$ satisfies (b), (c) & ($d^/$). We need only to show that it satisfies (d).

Let $M \in 2^X$ and $t \in I_O$. Then by (b), (c),

$\left((M^-)_{t^*}\right)^- \geq (M_{t^*})^- = M^-$

and $\left((M^-)_{t^*}\right)^- \geq (M^-)_{t^*}$

Hence $\left((M^-)_{t^*}\right)^- \geq (M^-)_{t^*} \vee M^-$

To prove the inverse inclusions, let us choose any $x \in X - (M^-)_{t^*}$;
i.e. x satisfies $M^-(x) < t$.
Now we put $r = M^-(x)$, then for every $s \in\ ]r, 1[$ we get by ($d^/$)

$\left((M^-)_{s^*}\right)^-(x) < s$

Since $(M^-)_{t^*} = \bigcap_{S \in]r,t[} (M^-)_{s^*}$

& since the operator $^-$ is isotonoe (by(c)), Hence

$\left((M^-)_{t^*}\right)^-(x) \leq \bigwedge_{S \in]r,t[} \left((M^-)_{s^*}\right)^-(x)$

$\leq \bigwedge_{S \in]r,t[} S = r = M^-(x)$ (2)

Therefore since $(M^-)_{t^*} \cup (X - (M^-)_{t^*}) = X$, then

$$\left((M^-)_{t^*}\right)^- = \left[((M^-)_{t^*})^- \wedge (M^-)_{t^*}\right] \vee \left[((M^-)_{t^*})^- \wedge (X - (M^-)_{t^*}\right] \leq (M^-)_{t^*} \vee M^- \quad \text{(by 2)}$$

Hence equality holds which proves (d).

Proposition 6 : Let (X, {<<}) be perfect N. topogenous space & we define the operator $^- : 2^X \to I^X$ by setting

$M^{-}(x) = \inf\{r \in I:(X - M) \vee >> x\}$
for all $M \in 2^X$ & $x \in X$.
Then $(X, ^{-})$ is a fuzzy nhd space,

Proof: For all $x \in X$,
$\mathbf{O}^{-}(x) = \inf\{r \in I : (X - O) \vee r >> x\}$
$= 0$
Hence $\mathbf{O}^{-} = 0$.

(b) For $M \in 2^X$ & $x \in M$.
$M^{-}(x) = \inf\{r \in I:(X - M) \vee r >> x\}$
$= 1$ (from T_2 in def ^{n} 3 of section II)
This proves $M^{-} >> M$.

(c) For $M, N \in 2^X$ & $x \in X$.
$(M \cup N)^{-}(x) = \inf\{r \in I:(X - (M \cup N)) \vee r >> x\}$
$= \inf\{r \in I:((X - M) \vee r) \wedge ((X - N) \vee r) >> x\}$
{From T_4}
$= \inf\{r \in I:(X - M) \vee r >> x \;\&(X - N) \vee r >> x\}$
$= \max\{\inf\{r \in I:(X - M) \vee r >> x\} , \inf\{r \in I:(X - N) \vee r >> x\}\}$
$= M^{-}(x) \vee N^{-}(x)$
Thus $(M \cup N)^{-} = M^{-} \vee N$

(d) Nex let us suppose that $M \in 2^X$, $t \in I_O$ & $x \in X$ are such that $M^{-}(x) < t$. Now we put $r=M^{-}(x)$ & choose $v, w \in I$ such that $r < v < w < t$. Then from the definition of the Operator $^{-}$ $(X-M) \vee v >> x$. Hence, from axiom (S_2) in definition 4 (section II), there is $C \in 2^X$ Such that $(X-M) \vee w >> C$ & $C \vee w >> x$. Hence from the definition of -, we have again $M^{-} \wedge C \leq w$ & $(X - C)^{-}(x) \leq w$. This implies $(M^{-})_{t^*} \cap C=0$, & hence
$(M^{-})_{t^*} \leq X - C$ & consequently
$\left((M^{-})_{t^*}\right)^{-}(x) \leq (X - C)^{-}(x) \leq w < t.$

This proves that $(X, ^{-})$ satisfies the axiom (d') of prosposition 5 (section II). Hence from that proposition, it satisfies (d). By theorem 1 (section II), this completes the proof that $(X, ^{-})$ is a fuzzy nhd space.

Theorem 7 : Let $(X, \{<<\})$ be a perfect N. topogenous space & let $(X, ^{-})$ be the fuzzy nhd space of previous theorem. Then $(X, ^{-})$ has the fuzzy nhd system.
$v = (v(x))_{x \in X}$ where $v(x) = (u \in I^x ; u >> x\}$

Proof : Since >> is saturated, then $v(x)$ is a saturated prefilter for all $x \in X$. Let us suppose that $v' = (v'(x))_{x \in X}$ be the fuzzy nhd system of $(X, ^{-})$. Then for all $u \in I^X$ & $x \in X$, $u \in v'(x)$, i.e. $(X-u')^{-}(x) \leq t$ for all $t \in I_1$ (from theorem 2 (section II)). It means $\inf\{r \in I: u' \vee r >> x\} \leq t$ for all

$t \in I_1$ (by definition of —); or $u' \vee r >> x$ for all $t, r \in I_1$ such that $r > t$ (from axion T_3 in definition 3 (section II)) which is equivalent to $u \in v(x)C$ from lemma 1 (section II), because v(x) is a saturated prefilter.

This proves that $v' = v$.

Prposition 8 : For a given fns (X, t (v)), we define a binary relation $<<_v$ by setting $M <<_v u$ iff $u \in v(M)$, for every $M \in 2^X$ & $u \in I^X$. Then $(X, \{<<_v\})$ is a perfect N. topogenous space.

Proof : We write $<<_v$ to $<<$

(T_1) is obvious.

(T_2) follows from (N_t) & definition 1 (section II)

(T_3) let $M \subseteq N << u \leq v$. Then $u \in v(N) \subseteq v(M)$. But v(M) is a prefilter. Hence $v \in v(M)$ i.e. $M << v$.

Perfectness : For every j, let us suppose that $M_j \in 2^X$ & $u_j \in I^X$ are such that $M_j << u_j$. Then

$\vee_j u_j \geq u_{j_0} \in v(M_{J_o})$ for all $j_O \in J$.

Since every $v(M_{J_o})$ is a prefilter, then

$\vee_j u_j \in \bigcap_j \vee (M_j) = \bigcap_j \bigcap_{x \in M_j} v(x)$

$= \bigcap_{x \in \bigcup_j M_j} v(x) = \vee(\bigcup_j M_j)$

This means that $\bigcup_j M_j << \vee_j u_j$

(T_4) If $M << u$ & $N << v$, then by (T_3), $M \bigcap N << u, v$

Since $v(M \bigcap N)$ is a prefilter, then $M \bigcap N << u \wedge v$.

i.e. $M \bigcup N << u \vee v$ follows from perfectness.

(S2) Next let us suppose that $M, N \in 2^X$ & $t \in I_1$ are such that $M << N \vee t$,

i.e. $N \vee t \in v(M)$. for each $r \in]t, 1[$, we take $C = x - ((X-N^{-})_{r*}$

Since from theorem 4 (section II)

$(X\text{-}N)^{-} \wedge M \leq t$ .... (3)

then

$(X–C) \wedge M = \left(((X-N)^{-}_{t*}) \wedge M\right)$

$= \left[((X–N)^{-}_{t*} \vee (X–N) \wedge M\right.$

(from theorem 1 (section II))

$= \left[((X–N)^{-}_{t*}, \bigcap M\right] \vee \left[(X–N)^{-} \wedge M\right] < O \vee t = t$ (from (3))

Hence from proposition 4 (section II)

$C \vee t \in v(M)$ & hence $C \vee r \in v(M)$ which means $M << C \vee r$.

On the other hand

$(X-N) \wedge C = (X-N) \wedge [X-((X-N))_{r^*}]$
$\leq [((X-N)^{-}_{r^*} \vee r] \wedge [X-((X-N)^{-})_{r^*}]$
$< O \vee r = r$

Hence by proposition 4 (section II),
$N \vee r \in v(C)$, i.e. $C << N \vee r$.
(S3) Saturation follows directly from the saturation of prefilter v (M) for all $M \in 2^X$.
This completes the proof that {<< v} is a perfect N. topogenous structure on X.

Proposition 9 : With the notation in proposition (8(section II)) i.e. in the previous theorem the association (X, t (v)) → (X, $\{<<_v\}$) is a one to one correspondence between fuzzy nhd spaces & perfect N. topogenous spaces.

Proof: By previous theorem the mapping defined there between fns's & perfect N. topogenous spaces, is well defined. But by proposition (7 (section II)) & perfectness, this mapping is evidently the inverse mapping to that given in proposition (6 (section II)).

Hence each of them is a one to one correspondence.

Remark: This theorem provides fuzzy nhd systems with a new alternative definition, equivalent to Lowen's original definition by replacing axiom (N_3) by :

(N3): If $x \in X$, $M \in 2^X$ & $r \in I_1$ are such that $M \vee r \in v(x)$, then for every $t \in]r, 1[$, there exists $C \in 2^X$ such that

$M \vee t \vee v(C)$ & $C \vee t \in v(x)$.

Section - III

In this section we shall discuss two important types of uniformities : The Lowen -Höhle uniformity and the Höhle Katsaras uniformity which is a straightforward generalization of the uniformity axioms in terms of entourages. By means of probabilistic pseudometrics it would be interesting to establish some relationship between them. Result in this section hold for a completely distributive lattice L=[0,1] with certain conditions.

In this section, a[0,l] - fuzzy topological space will be called, for simplicity, a fuzzy topological space, not a [0,1] - topological space. A fuzzy topology on a set X we mean a subset $\Delta \subseteq I^X$ which is closed under finite infs & arbitrary sups and contains all the constant fuzzy sets, i.e., a stratified Chang fuzzy topology. Clearly the construct FTS of fuzzy topological spaces in a well-fibred topological construct. Lowen and Wuyts showed that the topological construct FTS contains a lot of concretely both reflective and coreflective full subconstructs. Since FTS is a topological construct, each of such subconstructs of FTS is closed with respect to the formation of initial and final structures in FTS, hence gives rise to a perfectly viable and natural autonomous theory of topology. This means a theory of topology for each of such subconstructs can be developed. The best example is the theory for the construct FNS of fuzzy neighborhood spaces initiated by Lowen. This phenomenon sharply distinguishes fuzzy topology from classical topology on the categorical level. Since the construct Top of topological spaces contains no nontrivial such subconstructs.

For simplicity, a both concretely reflective and coreflective full subconstruct of a topological construct. A will be called a subuniverse of A. In this section we will show that the various notions of uniformity in fuzzy set theory, which were introduced from different motivations, can be put into a coherent and clear picture if we interpret them as notions of uniformity for suitable subuniverses of FTS or some superconstruct of FTS.

Here we recall some basic ideas about triangular norms and categorical terminologies needed in this section. We recall the basic results about prefilters and fuzzy filters.

Preliminaries :

As usual, I denotes the unit interval; I_0 and I_1 the intervals (0,1], [0,1). For a set X and $\alpha \in I$, we also write α to denote the constant function $X \to I$ with value α . For a fuzzy set λ and $\lambda \in I$, the strong α-cut of λ, denoted by λ_α, is the crisp set $\{x \in X : \lambda(x) > \alpha\}$.

Let us suppose that U is a subset of X and $\alpha \in [0,1]$. We write $\alpha \wedge U$ for the element in I^X defined by $\alpha \wedge U(x) = \alpha$ if $x \in U$ and $\alpha \wedge U(x) = 0$ if $x \notin U$. Such a fuzzy set will be called a one-step function, or a levelled characteristic function. When $\alpha=1$, we simply write U for $\alpha \wedge U$.

A function $f:M \to N$ between two lattices is said to be increasing (decreasing) if $f(m_1) \leq f(m_2)$ whenever $m_1 \leq m_2$ ($m_1 \geq m_2$).

A triangular norm, a t- norm for short, on [0,1] is a function $T:[0,1]x[0,1] \to [0,1]$ such that:

(1) T is increasing on each variable;

(2) T is associative, i.e., $T(T(x,y),z)=T(x,T(y, z))$;

(3) T is commutative, i.e. $T(x,y)=T(y, x)$;

(4) For all $x \in [0,1]$, $T(x, 1)=T(1, x) = x$.

If T is moreover a continous function with respect to the usual topology, we will call it a continous t-norm.

Every t-norm T can be extended to a binary operation, still denoted by without any confusion T on I^X for every set X pointwisely.

i.e. $T(\lambda,\mu,)(x) =T(\lambda(x), \mu(x))$ for all $\lambda,\mu \in I^X$. Suppose T is a continous f-norm. By continuity of T we can define a binary function $\to : I \times I \to I$ by $\alpha \to \mu = \vee \{\beta \in [0,l] | T(\alpha,\beta) \leq \mu\}$ $\to$ is called the residuation corresponding to T.

Definition 1 :

(Schweizer and Sklar, Höhle). A probabilistic pseudometric on a set X is a mapping $F: X \times X \to D(R^+)$ such that for all x, y, z in X we have

(1) $F(x, x) = \in_0$

(2) $F(x, y) = F(y,x)$;

(3) $F(x,y) \oplus_T F(y, x) \leq F(x, z)$.

If in addition $F(x,y) \neq \epsilon_0$ for all $x \neq y$, then (X, F) is called probabilistic metric space. Let (X, F) be a probabilistic pseudometric (metric) space. For any $t > 0$, the strong t-vicinity is the subset D(t) of XxX given by $D(t)=\{(x,y)|F(x,y)(t)>1-t\}$.

Then the strong vicinity system $D = \bigcup_{t>0} D(t)$ is a base for some (Hausdorff) uniformity on X, called the strong uniformity induced by F. The topology corresponding to this uniformity is called the strong topology. The neighborhood system at a point x for the strong topology is given by $N_x = \{N_x(t)|t>0\}$, where $N_x(t)=\{y \in X|F(x,y)(t)>1-t\}$.

A functor $F:A \rightarrow B$ is called topological provided that every F-source $(X \xrightarrow{f_i} F(A_i))_{i \in J}$ has a unique F-initial lift $(A \xrightarrow{g_i} Ai)_{i \in J}$.

A concrete category over a base category B, i.e., a pair (A, U), where $U:A \rightarrow B$ is a forgetful functor, is called initially complete, provided that U is topological. A construct (A, U), i.e., a concrete category over Set, is called a topological construct provided that it is initially complete and fibre-small. A topological construct, A is called well-fibred provided that on any set of cardinality at most 1 there is exactly one A-structure on it. Clearly the construct of fuzzy topological spaces is a well-fibred topological construct.

Let A be a fibre-small topological construct, X be a set, a cotower of A-structures on X is a function Γ from [0,1] to the complete lattice of A structures on X such that $\left\{(X,\Gamma(\alpha)) \xrightarrow{id_x} (X,\Gamma(\beta))\right\}_{\beta>\alpha}$ is an initial source for each $\alpha \in [0,1]$ or equivalently Γ is a sup-preserving mapping from [0,1] to the complete lattice of A-structures on X. (X,Γ) is called a cotower sace and $\Gamma(\alpha)$ is called the α - level structure of (X, Γ).

By definition Γ (1) is the indiscrete structure on X and Γ (O) is completely determined by $\Gamma(\alpha)$ for each $\alpha \in (0,1)$. Thus it suffices to specify $\Gamma(\alpha)$ for each $\alpha \in (0,1)$ when we describe a cotower.

A morphism between cotower spaces $(X, \Gamma) \rightarrow (Y,k)$ is a function $f:X \rightarrow Y$ such that $f:(X,\Gamma(\alpha)) \rightarrow (Y,k(\alpha))$ is a morphism in A for each $\alpha \in I$. The construct of cotower spaces and morphisra is denoted $A^C(I)$, called the cotower extension of A.

The idea of tower (cotower) extension of a topological construct traces back to the work of Frank on probabilistic topological spaces. Frank defined a probabilistic topological space to be an object in the tower extension of pretopological spaces with certain conditions. Brock and Kent used the same idea to introduce limit tower spaces and other constructs. A similar idea was employed to define approach uniform spaces. Moreover, the construct of approach spaces can also be characterized as a subconstruct of the tower extension (or cotower extension) of pretopological spaces.

Let T be a continous t-norm and μ, $v \in I^{X\times X}$. The element $\mu o_T v$ in $I^{X\times X}$ is given by $\mu o_T v(x,y) = \bigvee_{z\in X} T(\mu(x, z), v(z, y))$.

Definition 2 : (Höhle, Katsaras). A Höhle-Katsaras T-uniformity on a set X is a prefilter on X x X satisfying the following conditions :

(HK1) For all $\mu \in \mathcal{U}$ and $x \in X$, $\mu(x, x) = 1$

(HK2) For all $\mu \in \mathcal{U}$ and ${}_s\mu \in \mathcal{U}$, where , $\mu(x,y) = \mu_s(y, x)$.

(HK3) For all $\mu \in \mathcal{U}$ and there exist $v \in \mathcal{U}$ such that $v\, o_T v \le \mu$.

We call (X, $\mathcal{U}$) a Höhle-Katsaras *T*-uniform space. A subset of $\mathcal{U}$ satisfying (HK3) is called base of $\mathcal{U}$.

A morphism between two Höhle-katsaras *T*-uniform spaces f :(X, $\mathcal{U}_X$) → (Y, $\mathcal{U}_Y$) is a mapping f : X → Y such that $(fxf)^{-1}(\mu) \in \mathcal{U}_X$ for each $\mu \in \mathcal{U}_Y$. It is obvious that the construct of Höhle-Katsaras *T*-uniform spaces is a well-fibred topological construct.

Note :- The Höhle-Katsaras uniformity was used for *T*=min and it is a straightforward generalization of the uniformity axioms in terms of entourages.

Definition 3 : (Lowen, Höhle). A Lowen-Höhie T-uniformity on X is a prefilter $\mathcal{U}$ on XxX with the following conditions :

(LH1) $\mathcal{U}$ is saturated.

(LH2) For all $\mu \in \mathcal{U}$ and $x \in X$, $\mu(x,x)=1$.

(LH3) For all $\mu \in \mathcal{U}_s v \in \mathcal{U}$, where ${}_s v(x,y)=v(y,x)$.

(LH4) For all $\mu \in \mathcal{U}$ and all $\varepsilon \in I_0$ there exists $v_\varepsilon \in \mathcal{U}$ such that $v_\varepsilon o_T v_\varepsilon \le v$.

We call (X, $\mathcal{U}$) a Lowen-Höhle *T*-uniform space, or simply a fuzzy *T*-uniform space. A subset of $\mathcal{U}$ satisfying (LH4) is called a base of $\mathcal{U}$. In the case *T*=rnin, a Lowen-Höhie *T*-uniform space will be called simply a fuzzy uniform space.

Morphisms between Lowen-Höhle *T*-uniform spaces are defined in an obvious way. Clearly the construct of Lowen-Höhie *T*-uniform spaces is also a well-fibred topological construct, denoted *T*-FUS. In the ease *T*=min, *T*-FUS is simply denoted FUS.

Note. The Lowen-Höhle *T*-uniformity was introduced by Lowen for *T*=min and by Höhle in the form presented here.

We shall establish a relation between the Lowen-Höhle *T*-uniformities and the Höhle-Katsaras *T*-uniformities.

A saturated Höhle-Katsaras *T*-uniformity (i.e., a Höhle-Katsaras *T*-uniformity which is also a satuarated prefilter on X×X) is clearly a Lowen-Höhle *T*-uniformity.

By means of probabilistic pseudometrics, we will see that the converse is also true. That is to say, the Lowen-Höhle T-uniformities are just those saturated Höhle-Katsaras T-uniformities.

If F is a probabilistic preudometric, then the family $\{F_t \mid 1>0\}$, F_t (x,y) = F (x,y) (t), is a base for a Höhle-Katsaras T-uniformity on X and it is also a base for a Lowen-Höhle T-uniformity $\mathcal{U}$ (F) since $F_{t_1} \oplus_T F_{t_2} \leq F_{t_1+t_2}$. A Lowen-Höhle T-uniformity (or a Höhle - Katsaras T-uniformity) $\mathcal{U}$ on a set X is called probabilistic pseudometrizable if there exists a probabilistic pseudometric F on X such that $\{F_t \mid t > 0\}$, F_t (x,y) = F(x,y)(t) is a base for $\mathcal{U}$, i.e., $\mathcal{U} = \mathcal{U}$ (F).

We discuss below few theorems without proof.

Theorem (1): (Höhle, Katsaras for T=min). A Lowen-Höhle T-uniformity on a set X is probabilistic pseudometrizable iff it has a countable base.

Theorem (2): (T = $\wedge$, Katsaras). Let $\mathcal{U}$ be a Lowen- Höhle T-uniformity on X and $\mu \in \mathcal{U}$. Then there exists a probabilistic pseudometric F_μ on X such that

(1) $\mu \in \mathcal{U}(F_\mu) \subseteq \mathcal{U}$;

(2) F (x, y) (t) = 1 for all x,y$\in$X and t > 1.

Therefore, a Lowen-Höhle T-uniformity can be described as a collection of probabilistic pseudometrics.

Immediate consequence of the above thearem is :

Corollary (2:1):- (T =Min, Katsaras). Let $\mathcal{U}$ be a fuzzy T-uniforrnity on a set X and μ be an element in $\mathcal{U}$. Then there is some $\mu^* \in \mathcal{U}$ such that $\mu^* o_T \mu^* \leq \mu$. Hence a Lowen-Höhle T-uniformity is a just a saturated Höhle - Katsaras T-uniformity (2:2):- The Saturation of a Höhle - Katsaras T-uniformity is a Lowen - Höhle T-uniformity , hence also a Höhle - Katsaras T-uniformity.

(2:3) :- The construct of Höhle - Katsaras T-uniform spaces contains that of Lowen - Höhle T-uniform spaces as a concretely coreflective full subconstruct. Precisely, for each Höhle - Katsaras T-unifrom space (X,$\mathcal{U}$), its T-FUS-coreflection is (X,$\mathcal{U}$), Where $\mathcal{U}$ is the saturation of $\mathcal{U}$.

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Chapter-2

Section-I

Fuzzy proximity

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Section-II

Fuzzy syntopogenous structures

Chapter-2

Section - I
FUZZY PROXIMITY:

Many concepts of general topology were extended to fuzzy set theory after the papers of Zadeh & Chang. Fuzzy uniformities were introduced by Lowen & Hutton. The two approaches are quite different. The one praposed by Hutton suits in a better manner to Fuzzy set theory.

The concept of Fuzzy proximity till then was unsatisfactory :
Its "Fuzzyness" was rather poor since these proximities were in a canonical one-one correspondence with the usual proximities.

Moreover the open sets of the induced topologies are crisp and though every Lowen fuzzy uniformities induces a fuzzy proximity, this correspondence cannot work well since the two structures do not give the same fuzzy topology. For the reasons, another definition of fuzzy proximity was given by Artico & Moresco which enables to associate a topology in a completely different way. Moreover every fuzzy uniformity induces a fuzzy proximity & vice-versa.

Notations & Preliminaries:

(L, $\vee$, $\wedge$') will be a (complete) completely distributive lattice with order reserving involution $'$ (= complementation).

Given a set X any element of L^X is called fuzzy "set" & will be denoted by $\gamma,\mu,\nu,\rho,\sigma,\tau$ etc. 0 & 1 denote the infimum & supremum of L respectively. If Y is a subset of X, we shall use the same letter Y to indicate the element of L^X so defined

f (x) = 1 if $x \in Y$

f (x) = 0 otherwise,

i.e. $a \in L$, $x \in X$; ax denote the elements of L^X which takes the value 'a' at the point x & 0 elsewhere. ax is said to be a fuzzy point & x its support. Also 1x = x. If $\mu \in L^X$. We say that ax belongs to μ or that ax is a fuzzy point of μ if $a \leq \mu$ (x).

L^X inherits a structure of lattice with order reversing involution in a natural way, by defining $\vee$, $\wedge$' pointwise (same notation of L are used).

If f : X $\rightarrow$ Y is a function & μ, v belong to L^x, L^Y respectively, are usual we put

$f^{\leftarrow}(v)(x) = v(f(x)) = (vof)(x)$ for $x \in X$

$f(\mu)(y) = \sup\{\mu(x) : x\in X, f(x) = y\}$ for $y \in Y$

Clearly $f^{\leftarrow}(v) \in L^X$, $f(\mu) \in L^Y$ and we then have obviously

$f\left(f^{\leftarrow}(v)\right) = v \wedge f(x)$ & $f^{\leftarrow}\left(f(\mu)\right) \geq \mu$

Moreover $f^{\leftarrow}$ preserves complementation, arbitrary unions & arbitrary intersections & that:

$f(\vee_{i\in I}\, \mu_i) = \bigvee_{i\in I} f(\mu_i)$

A fuzzy topologieal spaces is a pair (X, τ) where $\tau \in L^X$ contains the constants O & μ & is closed under finite intersection & arbitrary unions. The elements of τ are called open & their complements closed.

Given a fuzzy topological space (X, τ) a fuzzy set $\mu \in L^X$ is said to be τ -nhd (or simply nhd) of ax if there exists $v \in \tau$ such that $ax \leq v \leq \mu$. Clearly a fuzzy set is open iff it is a nhd of any of its points, interior & closure of fuzzy sets are defined in the usual way.

If (X, τ) & (Y, τ') are fuzzy topological spaces a function $f : X \rightarrow Y$ is said to be continuous if $f^{\leftarrow}(v) \in \tau$ for every $v \in \tau'$

For the sake of brevity we shall write "f topology" or simply "topology" instead of "fuzzy topology". Similarly for fuzzy uniformities, fuzzy proximities & so on.

When we shall refer to classical cases, we shall write it explicitely, using words such as 'usual' or classical.

Now we use the definition of fuzzy uniform space given by Hutton. We denote by Z the set of maps.

$U : L^X \rightarrow L^X$ which satisfy

$U(\underline{O}) = \underline{O}$ -------- (i)

$U(\mu) \geq \mu$ ------- (ii)

$U\left(\bigvee_{i \in I} \mu_i\right) = \bigvee_{i \in I} U(\mu_i)$ for $\mu, \mu_i \in L^X$ -------- (iii)

If U, V belongs to Z, we define $U \wedge V$ to be the infimum of U & V in Z which turns out to satisfy

$(U \wedge V)(\mu) = \bigwedge_{\mu_i \vee \mu_2 = \mu} (U(\mu_1) \vee V(\mu_1))$

Moreover we define

$U^{-1}(\mu) = \inf \{\rho : U(\rho') \leq \mu'\}$

an element U such that

$U = U^{-1}$ is called symmetric.

Definition 1: A fuzzy uniformity on X is a subset $\mathcal{U}$ of Z

such that $\mathcal{U} \neq \phi$ ------ (U_1)

$U \in \mathcal{U}$ & $U \leq V \in Z$ implies $V \in \mathcal{U}$ ------------------ (U_2)

$U, V \in \mathcal{U}$ implies $U \wedge V \in \mathcal{U}$ ---------------- (U_3)

$U \in \mathcal{U}$ implies there exists $V \in \mathcal{U}$ such that V O V $\leq$ U --------- (U_4)

$U \in \mathcal{U}$ implies $U^{-1} \in \mathcal{U}$ -------------(U_5)

Subbasis & basis of a uniformity get the obvious significance.

Clearly (U_5) may be replaced by : $\mathcal{U}$ has a basis of symmetric elements ----- (U5')

Given a function $f: X \to Y$, for any $V : L^Y \in L^Y$, we define
$f^{\leftarrow}(V) : L^X \to L^X$ by
$f^{\leftarrow}(V)(\mu) = f^{\leftarrow}(V(f(\mu)))$
for any $\mu \in L^X$.

It is clear that V satisfies (i) — (iii), then $f^{\leftarrow}(V)$ satisfies (i) — (iii) too.

If $(X, \mathcal{U})$ & $(Y, \mathcal{V})$ are uniform spaces, a function $f : X \to Y$ is said to be a uniform map if for every $V \in \mathcal{V}$, the element $f^{\leftarrow}(V)$ belongs to $\mathcal{U}$.

Hutton showed that any Fuzzy uniformity $\mathcal{U}$ induces a fuzzy topology by putting $\mu \in \tau_{\mathcal{U}}$ iff
$\mu = \sup (\rho \in L^X : U(\rho) \le \mu$ for some $U \in \mathcal{U}\}$

Moreover every uniform map from $(X, \mathcal{U})$ to $(Y, \mathcal{V})$ is continuous equipping X & Y with the induced fuzzy topologies.

Proposition 1 : let $(X, \mathcal{U})$ & $(Y, \mathcal{V})$ be uniform spaces, $f : X \to Y$ a function and τ' a subbasis of $\mathcal{V}$. Then f is a uniform map iff $f^{\leftarrow}(S) \in \mathcal{U}$ for every $S \in \tau'$.

Proof: The ‘only if part is trivial. For the converse let us suppose that if S_1 & S_2 belong to τ', then $f^{\leftarrow}(S_1 \wedge S_2)$ belongs to $\mathcal{U}$; namely we show that $f^{\leftarrow}(S_1 \wedge S_2) = f^{\leftarrow}(S_1) \wedge f^{\leftarrow}(S_2)$.

First we observe that first member of the equality is less than or equal to the second one. For the other inequality we have for $\mu \in L^X$ & $x \in X$,

$(f^{\leftarrow}(S_1) \wedge f^{\leftarrow}(S_2))(\mu))(x)$

$= \bigwedge_{\mu_1 \vee \mu_2 = \mu} (S_1(f(\mu_1)) \vee S_2(f(\mu_2))(f(x))$

& $f^{\leftarrow}(S_1 \wedge S_2)(\mu)(x) = (S_1 \wedge S_2)(f(\mu))(f(x))$

$= \bigwedge_{v_1 \vee v_2} (S_1(v_1) \vee S_2(v_2))(f(x))$

we see that $\inf v_1 \vee v_2 = f(\mu)$, we then have

$(f^{\leftarrow}(v_1) \wedge \mu) \vee (f^{\leftarrow}(v_2) \wedge \mu)$
$= (f^{\leftarrow}(v_1) \wedge f^{\leftarrow}(v_2)) \wedge \mu$
$= f^{\leftarrow}(v_1 \wedge v_2) \wedge \mu = f^{\leftarrow}(f(\mu) \wedge \mu = \mu$

Moreover for i = 1, 2

$f(f^{\leftarrow}(v_i) \wedge \mu)(y)\ v_2) = \sup \{(f^{\leftarrow}(v_i) \wedge \mu)(x) : (x) = y\}$
$= \sup\{v_i(f(x) \wedge \mu(x) : f(x) = y\}$
$= v_i(y) \wedge \sup \{\mu(x) : f(x) = y\}$
$= v_i(y) \wedge f(\mu)(y) = v_i(y)$

Hence if we take $\mu_i = f^{\leftarrow}(v_i) \wedge \mu$, we have $\mu_1 \vee \mu_2 = \mu$ & $f(\mu_i) = v_i$ and the conclusion follows.

Definition 2 : A Fuzzy proximity on a set X is a function $\delta : L^X \times L^X \to \{0,1\}$ which satisfies for any $\mu, v, \rho \in L^X$ the following conditions:

(P_1) $\delta(\underline{0}, \underline{1}) = 0$
(P_2) $\delta(\mu, \rho) = \delta(\rho, \mu)$
(P_3) $\delta(\mu, \rho) \vee \delta(v, \rho) = \delta(\delta \vee v, \rho)$
(P_4) if $\delta(\mu, \rho) = 0$ there exists
$\gamma \in L^X$ Such that $\delta(\mu,\gamma) = 0$, $\delta(\rho,\gamma') = 0$
(P_5) $\delta(\mu, \rho) = 0$, implies $\mu \leq \rho'$

The pair (X, δ) is said to be a fuzzy proximity space.

If $\delta(\mu,\rho) = 0$ we say that μ & ρ are far, otherwise we say that they are proximal. $(P_1 — P_4)$ are the natural extensions of classical case. (P_5) needs some comment since A. Katsaras formulated the analogous axiom in a different manner. In the case L = {0, 1}, (P_5) means exactly that if two subsets intersect then they are proximal. In the case L = {0,1} = 1, (P_5) means that μ & ρ are proximal whenever there exist $x \in X$ such that $\mu(x) + \rho(x) > l$.

Definition 3*:* Let (X, δ) & (Y, η) be fuzzy proximity spaces. A function f is a proximity map if one of the following equivalent condition holds :

a) For every, $v, \sigma \in L^Y$, $\eta(v,\sigma) = 0$ implies $\delta(f^{\leftarrow}(v), f^{\leftarrow}(\sigma)) = 0$

b) For every $\mu, \rho \in L^x$, $\delta(\mu,\rho) = 1$ implies $\eta(f(\mu), f(\rho))=l$

To see that conditions (a) & (b) are equivalent, we may use part (i) of the following Lemma.

Leema 1: Let (X, δ) be a fuzzy proximity space. `

(i) For every $\mu,\rho \in L^X$, $\delta(\mu, \rho) = 1$
implies $\eta(f(\mu), f(\rho)) = l$

(ii) If $\delta(\mu_i, \rho_i) = 0$ for i = 1, ... , n.

then $$\left(\wedge_{i=1,....n} \mu_i \,,\, \vee_{i=1....n} \rho_i\right) = 0$$

Proof: We use (P_3) to prove (i) & (i) and (P_3) to prove (ii).

Remark : Clearly the set of all proximities on a given set X can be equipped with a partial order by defining δ_1 finer than δ_2 (or δ_2 coarser than δ_1) if the identity of X is a proximity map from (X, δ_1) to (X, δ_2).

We shall define the fuzzy topology induced by a fuzzy proximity.

We take a proximity space (X, δ) & for any $\mu \in L^X$, we put

$\text{int}(\mu) = \sup\{\rho : \delta(\rho, \mu') = 0\}$

& denote it by $\overset{0}{\mu}$ or int $(\overset{0}{\mu})$.

Theorem 2 : The function int : $L^X \rightarrow L^X$ satisfies the interior axioms namely, we have for $\mu,\rho \in L^X$.

(I_1) int ($\underline{1}$) = $\underline{1}$
(I_2) int (μ) $\leq \mu$
(I_3) int (int (μ)) = int μ
(I_4) int ($\mu \wedge \rho$) = int (μ) $\wedge$ int (ρ)

Proof: (I_1) & (I_2) follow trivially from (P_1) & (P_5) respectively.
(I_3) clearly int (int (μ)) $\leq$ int (μ);

We now take ρ such δ (ρ, μ') = 0.

By (P_4) there exist γ such that $\delta(\rho,\gamma') = 0$

& $\delta(\gamma,\mu') = 0$; hence $\rho \leq$ int (γ), $\rho \leq$ int (μ) & int (γ) $\leq$ int (int(μ)) because int is monotone, therefore $\gamma \leq$ int (int (μ)) for every ρ, such that $\delta(\rho,\mu') = 0$.
So that int (int μ) $\geq$ int (μ)
(I_4) Trivially int ($\mu \wedge \rho$) $\leq$ int (μ) $\wedge$ int (ρ).

For the converse, we see that in a completely distributive lattice, the infinite distributive law holds, hence we have

int (μ) $\wedge$ int (ρ) = sup{v:δ(v, μ') = 0} $\wedge$ sup {σ:$\delta(\sigma,\rho') = 0$}
= sup {v $\wedge$ σ:δ(v,μ') = 0 = $\delta(\sigma;\rho')$}
$\leq$ sup{t:δ(t,$\mu' \vee \rho'$) = 0}
= sup {t:δ(t, $(\mu \wedge \rho)'$) = 0}
= int ($\mu \wedge \rho$)

Definition 4 : The f topology induced by f. proximity δ is denoted by τ_δ & consists of all fuzzy sets $\mu \in L^X$ such that μ = int (μ) .

Clearly the closure of μ in the topology τ_δ denoted by $C1_{\tau\delta}$ (μ) or Cl(μ) is given by (int (μ'))'.

Remark I : If L = I then μ is a τ_δ - nhd of ax iff for every b<a we have

$$\delta\,(bx,\ 1-\mu) = 0$$

(II) : If (X, δ) is a classical proximity space, for any $\mu \in L^X$, we put

coz (μ) = (x $\in$ X : μ (x) > 0} &

define $\hat{\delta}$ (μ, ρ) = 0 iff coz (μ)δ coz (ρ).

Then $\hat{\delta}$ is a fuzzy proximity & τ_δ open fuzzy sets are exactly the characteristic functions of the sets which are open in the topology induced by δ.
(III) The fuzzy proximities introduced by Katsaras satisfy conditions (P_1 - P_5) & the δ of the example above is a Katsaras proximity. Furthermore, given a Katsaras proximity η, it is clear to prove that there exists a classical proximity δ such that $\hat{\delta} = \eta$; indeed for A,B subset of X. We put A δ B iff A η B.

To prove that δ is a usual proximity & $\hat{\delta} = \eta$, we consider the fact that for every $\mu, \rho \in I^X$ we have that the closure of μ introduced by Katsaras (denoted by $\bar{\mu}$ in this example) is a characteristic function and

$\mu\eta\rho$ iff $\bar{\mu}\,\eta\,\bar{\rho}$ iff coz $(\mu)\eta$ coz(ρ) iff coz$(\mu)\delta$ coz(ρ)

iff $\hat{\delta}(\delta, \rho) = 1$

Thus Katsaras proximities are in a cononical 1-1 correspondence with the usual proximities.

Proposition 3 : Let (X, δ), (Y, η) be f. proximity spaces.

If $f : X \to Y$ is proximity map, then it is continuous equipping X & Y with the induced f. topologies.

Proof : Let v be τ_η - open set

i.e. $v = \sup\{\sigma : \eta(\sigma, v') = 0\}$

Hence $f^{\leftarrow}(v) = \sup\{f^{\leftarrow}(\sigma) : \eta(\sigma, v') = 0\}$

$\leq \sup\{\rho : \delta(\rho, f^{\leftarrow}(v))' = 0\}$

i.e. $f^{\leftarrow}(v) = \text{int}\,(f^{\leftarrow}(v))$ is a τ_δ-open set.

Proposition 4: Let δ be a fuzzy proximity on X. Then,

(i) $\delta(\mu,\rho) = 0$ iff $\delta(\bar{\mu},\rho) = 0$

(ii) $\mu = \sup\,(v.\delta\,(\mu,\rho) = \delta(v,\rho)$ for every $\rho \in L^X\}$.

Proof: (i) The ‘if part’ is trivial, for the converse let us take γ such that $\delta(\gamma',\mu) = 0 = \delta(\rho, \gamma)$. Hence $\gamma' \leq \text{int}\,(\mu')$ so that $\gamma \geq (\text{int}\,(\mu'))' = \bar{\mu}$ & $\delta(\rho, \mu) = 0$

(ii) By (i) we get that $\bar{\mu} \leq \sup\{v : \delta(\mu,\rho) = \delta(v,\rho)$ for every $\rho \in L^X\}$.

We then take $v \leq \bar{\mu}$ such that $\delta(\mu,\rho) = \delta(v,\rho)$ for every $\rho \in L^X$ & we put

$t = \mu \vee v$. We see that $t >_{\neq} \bar{\mu}$ & $\delta(t,\rho) = \delta\,(\mu,\rho)$ for every $\rho \in L^X$.

Since $t' <_{\neq} (\bar{\mu})' = \text{int}(\mu')$; by the definition of int there exists $\sigma \leq$ cutting t' .

such that $\delta(\mu,\sigma) = 0$ while (P5) implies $\delta(t,\sigma) = 1$ which is a contradiction.

Thus the theorem follows.

CONNECTION BETWEEN FUZZY PROXIMITIES & FUZZY UNIFORMITIES :

Now we shall study some connection between fuzzy uniformities & fuzzy proximities : namely we shall show that any f. uniformity induces a f. proximity in a cononical way & vice-versa.

Let $\mathcal{U}$ be a f. uniformity and for $\mu, \rho \in L^X$ we define
$\delta_{\mathcal{U}}(\mu, \rho) = 0$ iff there exists
$U \in \mathcal{U}$ s.t. $U(\mu) \leq \rho'$.

Theorem 5: $\delta_{\mathcal{U}}$ as defined above is a f. proximity.

Proof: We shall verify properties (P1 — P5).

(P_1) – is trivial.

(P_2) $\delta_{\mathcal{U}}(\mu, \rho) = \delta_{\mathcal{U}}(\rho, \mu)$
Since for $U \in \mathcal{U}$ $U(\mu) \leq \rho'$ iff $U^{-1}(\rho) \leq \mu$

(P_3) It is sufficient to prove that
$\delta_{\mathcal{U}}(\mu, \rho) = 0 = \delta_{\mathcal{U}}(v, \rho)$ implies
$\delta_{\mathcal{U}}(\mu \vee v, \rho) = 0$ since the converse is trivial.
If $U(\mu) \leq \rho'$, $V(v) \leq \rho'$,
we have $(U \wedge V)(\mu \vee v) \leq \rho'$ then $\delta_{\mathcal{U}}(\mu \vee v, \rho) = 0$.

(P_4) Let $\delta_{\mathcal{U}}(\mu, \rho) = 0$, $\exists\, U \in \mathcal{U}$
such that $U(\mu) \leq \rho'$.

We take $V \in \mathcal{U}$, then $V = V^{-1}$, $VoV \leq U$, then $V(V(\mu)) \leq \rho' \Rightarrow V(\rho) \leq (V(\mu))'$.
Hence for $\gamma = V(\rho)$ we have $\delta_{\mathcal{U}}(\mu, \gamma) = 0 = \delta_{\mathcal{U}}(\rho, \gamma')$

(P_5) Trivial.

Remark: We say that a f. uniformity $\mathcal{U}$ is separated if for given points ax, by such that $ax \leq (by)'$ there exists $U \in \mathcal{U}$ such that:

$$U(ax) \leq (by)'$$

Theorem : Let $\mathcal{U}$ be a f. uniformity $\mathcal{U}$ & $\delta_{\mathcal{U}}$ induce the same topology.

Proof : Given a fuzzy set μ, we see that
$\{v : \exists\, U \in \mathcal{U}$ such that $U(v) \leq \mu\}$
$= \{v : \delta_{\mathcal{U}}(v, \mu') = 0\}$ & the supermum of the first member of the equality is the interior of μ in the topology induced by $\mathcal{U}$, while the supermum of the second one is the interior of μ in the topology induced by $\delta_{\mathcal{U}}$.

Section II

FUZZY SYNTOPOGENOUS STRUCTURES

Csāszār gave a new method for foundation of general topology based on the theory of syntopogenous structures. Special cases of these structures are the topologies, the proximities and the uniformities. In the case of fuzzy structures, there are at least two notions of fuzzy uniformities, one of them is due to Hutton & the other due to Lowen. There are also two definitions of fuzzy proximities. The definition of a fuzzy proximity given by Katsaras is closely connected with Hutton fuzzy uniformities. The other definition of a fuzzy proximity was given by Artico-Moresco & it is closely connected with Lowen Fuzzy uniformities.

We now study the fuzzy syntopogenous structure which agrees very well with the fuzzy nhd structures, the Lowen fuzzy uniformities & the Artico-Moresco fuzzy proximities. There is a one to one correspondence between the fuzzy nhd structures & the so called perfect fuzzy topogenous structures. Also there is a one to one correspondence between the Artico-Moresco fuzzy proximities & the symmetrical fuzzy topogenous structures.

A fuzzy set in a set X is an element of the set I^X of all functions from X to the unit interval I. μ,σ,ρ will denote fuzzy sets. If $A \subset X$, we will use also A to denote fuzzy set which is equal to the characteristic function of A. For $x \in X$ we will let x denote also the signleton $\{x\}$. For $\theta \in [0,1]$ we will dentoe by θ the fuzzy set which assumes the value θ at each $x \in X$.

A fuzzy topological space is a pair (X, τ) where τ is a subset of I^X containing 0,1 & closed under finite intersection & arbitrary unions. All fuzzy topological spaces which we consider here will contain the constant fuzzy sets.

If $(\gamma_j)_{j \in J}$ is a set of real numbers we will denote by $\wedge_{j\in J} \gamma_j$ & $\vee_{j\in J} \gamma_j$

the $\inf_{j \in J} \gamma_j$ & $\sup_{j \in J} \gamma_j$ respectively.

For a family $\{\mu_j\}_{j\in j}$ of fuzzy sets in X, the fuzzy sets $\mu = \vee_j \mu_j$ & $\rho = \wedge_j \mu_j$ are defined by

$\mu(x) = \sup_j \mu_j(x)$, $\rho(x) = \inf_j \mu_j(x)$.

A fuzzy topological space (X, τ) is a fuzzy nhd space if the fuzzy topology τ is induced by some fuzzy nhd structure. A fuzzy L-quasi uniformity $\mathscr{U}$ (or simply a fuzzy quasi-uniformity) has all the properties of a Lowen fuzzy uniformity except that we do not require that $\alpha^{-1} \in \mathscr{U}$ when $\alpha \in \mathscr{U}$, where $\alpha^{-1}(x, y) = \alpha(y,x)$. ;

In this section, by a fuzzy uniformity (resp. a fuzzy L-quasi-uniformity) we will mean a Lowen fuzzy uniformity (resp. a fuzzy L-quasi-uniformity). The fuzzy topology $\tau(\mathscr{U})$ induced by a fuzzy quasi uniformity $\mathscr{U}$ given by the closure operator.

$\bar{\mu}(x) = \inf_{\alpha \in \mathscr{U}} \alpha^{-1} < \mu > (x)$ where

$\beta < \mu > (x) = \sup_y \mu(y) \wedge \beta(y,x)$

Definition I: A function $\delta: I^X \times I^X \to I$ is a fuzzy proximity in the sense of Artico-Moresco (or) just a fuzzy proximity in this section, if it satisfies the following axioms.

(P_1) $\delta(0,1) = 0$

(P_2) $\delta(\mu,\rho) = \delta(\mu,\rho)$

(P_3) $\delta(\mu_1 \vee \mu_2, \rho) = \delta(\mu_1, \rho) \vee \delta(\mu_2, \rho)$

(P_4) if $\alpha = \delta(\mu,\rho)$, then for each $\in > o$ there are $\gamma, \gamma' \in I^X$ such that
$\gamma \vee \gamma' = 1, \gamma \wedge \gamma' \geq \alpha;$
$\delta(\mu, \gamma) \vee \delta(\gamma', \rho) \leq \alpha + \in$

(P_5) $\delta(\mu,\rho) \geq (\mu \wedge \rho)(x)$ for $x \in X$.

(P_6) if $\|\rho - \rho'\| = \sup_{x \in X} |\rho(x) - \rho'(x)|$

then $|\delta(\mu,\rho) - \delta(\mu,\rho')| \leq \|\rho - \rho'\|$. ($P_4$) can be replaced by : (P4') If $\delta(\mu,\rho) < \theta$, then there is $A \subset X$ such that

$\delta(\mu,A) \vee \delta(\rho, A^C) < \theta$

where A^C is the complement of A. The fuzzy topology $\tau(\delta)$ induced by δ is given by the closure operator $\bar{\mu}(x) = \delta(x,\mu)$ $(\mu \in I^X, x \in X)$.

Definition 2 : An order relation << on the subsets of a set X is called a semitopogenous order on X if it satisfies the following conditions.

a) $X << X$ & $\phi << \phi$

b) $A << B$ implies that $A \subset B$

c) $A_1 \subset A << B \subset B_1$ implies $A_1 << B_1$

The semi-topogenous order << is called topogenous if it satisfies also

(d_1) $A_i << B$, $i = 1,2$, imply that
$A_1 \cup A_2 << B$

(d_2) $A << B_i$, $i = 1, 2$ imply that
$A << B_1 \cap B_2$

We now give the following definitions :

Definition 3 : A fuzzy semi-topogenous order on X is a function,
$\zeta : I^X \times I^X \to I$ whcih satisfies the following conditions.

(i) $\zeta(0,0) = \zeta(1, 1) = 1$

(ii) $\zeta(\mu,\rho) \leq [1 - \mu(x)] \vee \rho(x)$ for every $x \in X$.

(iii) $\mu_1 \leq \mu$ & $\rho < \rho_1$ may imply that

(iv) $|\zeta(\mu,\rho) - \zeta(\mu', \rho')|$
$\leq \|\mu - \mu'\| + \|\rho - \rho'\|$
where for $\sigma_1, \sigma_2 \in I^X$,
$\|\sigma_1 - \sigma_2\| = \sup_{x \in X} |\sigma_1(x) - \sigma_2(x)|$

The fuzzy semi-topogenous order ζ is called topogenous if it satisfies also.

(v) $\zeta(\mu_1 \vee \mu_2, \rho)$
$= \zeta(\mu_1, \rho) \wedge \zeta(\mu_2, \rho)$
$\zeta(\mu, \rho_1 \wedge \rho_2) = \zeta(\mu, \rho_1) \wedge \zeta(\mu, \rho_2)$

Remarks I : We see that (i), (ii), (iii) & (v) are natural extensions of (a), (b), (c) & (d) respectively. The condition (iv) is trivially satisfied in the classical case & it says that the value of ζ at (μ,ρ) varies little when the fuzzy sets μ,ρ vary little.

(iv) is equivalent to (iv)' $|\zeta(\mu,\rho)-\zeta(\mu',\rho)| \le \|\mu - \mu'\|$

$\& \; |\zeta(\mu,\rho)-\zeta(\mu,\rho')| \le \|\rho - \rho'\|$

Lemma 1 : Let ζ be a fuzzy semi-topogenous order on X & let ζ^C be defined by

$\zeta^C(\mu,\rho) = \zeta(1-\rho,\ 1-\mu)$.

Then ζ^C is a fuzzy semi-topogenous order on X which is topogenous if ζ is topogenous.

Lemma 2 : Let ζ be a fuzzy semi-topogenous order on X &

let $\theta \in [0, 1]$ & $\mu, \rho \in I^X$; then

(1) $\zeta(1,\theta) = \theta$ & $\zeta(\theta,0) = 1 - \theta$

(2) $\theta \le \zeta(\mu,\theta) \le \theta \vee (1-\mu(x))$ for all x.

(3) $1-\theta \le \zeta(\theta,\rho) \le (1-\theta) \vee \rho(x)$ for all x.

(4) $\theta \le \zeta(\theta,\theta) \le (1-\theta)$ if $\theta < \frac{1}{2}$ and $\zeta(\theta,\theta) = \theta$ if $\theta \ge \frac{1}{2}$

(5) If $\mu(x) \ge 1 - \theta$ for some x, then $\zeta(\mu, \theta) = \theta$

(6) If $\rho(x) \le 1 - \theta$ for some x then $\zeta(\theta, \rho) = 1 - \theta$

(7) $\zeta(x,\theta) = \theta$ & $\zeta(\theta,x^C) = 1 - \theta$

Proof :

(2) $\zeta(\mu, \theta) \ge \zeta(1, \theta)$

$= [\zeta(1,\theta)-\zeta(1,1)] + 1 \ge 1 - (1 - \theta) = \theta$

Moreover, $\zeta(\mu, \theta) \le \theta \vee (1 - \mu(x))$ by (ii) of def. (3) (Section II)

(3) $\zeta(\theta, \rho) \ge \zeta(\theta, 0) = [\zeta(\theta, 0) - \zeta(0, 0)] + 1 \ge 1 - \theta$

Moreover, $\zeta(\theta,\rho) \le (1 - \theta) \vee \rho(x)$

Now (4) & (5) follows from (2), (6) from (3) & (1) & (7) from (2) & (3).

Definition 4: A fuzzy semi-topogenous order ζ on X is called :

(i) Perfect if $\zeta\left(\bigvee_{j\in J} \mu_j,\rho\right) = \bigwedge_{j\in J} \zeta(\mu_j,\rho)$

(ii) Biperfect if it is perfect & $\zeta\left(\mu, \bigvee_{j\in J} \rho_j\right) = \bigwedge_{j\in J} \zeta(\mu,\rho_j)$

(iii) Symmetrical if $\zeta = \zeta^C$

ζ is biperfect iff both ζ & ζ^C are perfect.

Deftnition 5 : If ζ_1, ζ_2, are fuzzy semi-topogenous order on X, then $\zeta = \zeta_1$ o ζ_2 is defined by $\zeta(\mu,\rho) = \vee_{A\subset X}\ \zeta_2(\mu,A) \wedge \zeta_1(A, \rho)$.

Lemma 3 : Let ζ_1, ζ_2 be fuzzy topogenous orders on X & let $\zeta = \zeta_1$ o ζ_2 then

(1) ζ is a fuzzy semi-topogenous order.

(2) If both ζ_1, & ζ_2 are topogenous so is ζ.

(3) If ζ_0 is a fuzzy topogenous order with $\zeta_0 \geq \zeta_1, \zeta_2$ then $\zeta_0 \geq \zeta$.

(4) $\zeta^C = \zeta_2^C o \zeta_1^C$

(5) If ζ_1, ζ_2 are perfect (resp. biperfect) than ζ is perfect (resp. biperfect).

Proof:

(1) We see that ζ satisfies (i) & (ii) of definition (3) of Chapter II.

Let $A\subset X$. If $x\notin A$ than $\zeta_2(\mu,A) \leq 1 - \mu(x)$. For $x \in A$, we have $\zeta_1(A,\rho)\leq\rho(x)$. Thus for all $A\subset X$ & all $x\in X$ we have $\zeta_2 (\mu, A) \wedge \zeta_1 (A,\rho) \leq [1- \mu(x)] \vee \rho(x)$ which implies that $\zeta(\mu,\rho)\leq[1 - \mu (x)] \vee \rho(x)$.

Finally, let $\gamma = \| \mu - \mu' \|$ & let us suppose that $\zeta (\mu, \rho) - \zeta (\mu', \rho) > \gamma$.

We choose θ such that $\zeta(\mu,\rho) > \theta$, $\zeta(\mu',\rho) < (\mu' - \rho)$.

Let $A \subset X$ be such that $\zeta_2(\mu, A) \wedge \zeta_1(A,\rho) > \theta$. Since

$\zeta_2(\mu',A) \geq \zeta_2(\mu,A)-\gamma>\theta-\gamma$, we have $\zeta (\mu',\rho) \geq \zeta_2(\mu',A) \wedge \zeta_1(A,\rho) > \theta - \gamma (\mu',\rho)$ which is a contradiction. This proves that $|\zeta(\mu,\rho)-\zeta(\mu',\rho)|\leq\| \mu-\mu'\|$. Similarly we can show that $|\zeta(\mu,\rho)-\zeta(\mu,\rho')|\leq\|\rho-\rho'\|$.

(2) If $\mu \geq \mu'$, then for each $A \subset X$ we have

$\zeta_2(\mu, A) \wedge \zeta_1(A, \rho) \leq \zeta_2(\mu',A) \wedge \zeta_1(A, \rho) \leq \zeta(\mu', \rho)$ and so $\zeta(\mu,\rho) \leq \zeta(\mu', \rho)$.

Hence $\zeta(\mu_1 \vee \mu_2,\rho) \leq \zeta(\mu_1,\rho) \wedge \zeta(\mu_2,\rho)$. Let $\theta < \zeta(\mu_1,\rho)\wedge\zeta(\mu_2,\rho)$. There are $A_1,A_2\subset X$ such that $\zeta_2(\mu_i,A_i) \wedge \zeta_1(A_i,\rho) > \theta$, i = 1,2.

If $A = A_1 \cup A_2$, then $\zeta(\mu_1 \vee \mu_2,\rho) \geq \zeta_2(\mu_1, \vee \mu_2,A) \wedge \zeta_1(A, \rho)$

$= \zeta_2 (\mu_1,A) \wedge \zeta_2 (\mu_2,A) \wedge \zeta_1(A_1, \rho) \wedge \zeta_1(A_2,\rho)\geq\zeta_2(\mu_1,A_1)\wedge\zeta_2(\mu_2,A_2)\wedge\zeta_1(A_1, \rho) \wedge \zeta_1(A_2,\rho) > \theta$

This proves that $\zeta(\mu_1 \vee \mu_2,\rho) \geq \zeta(\mu_1,\rho) \wedge \zeta(\mu_2, \rho)$

Similarly we can show that $\zeta(\mu, \rho_1 \wedge \rho_2) = \zeta(\mu,\rho_1) \wedge \zeta(\mu, \rho_2)$

(3) Let $\mu, \rho \in I^X$ we need to show that $\zeta_0 (\mu,\rho) \geq \zeta(\mu,\rho)$. If $\zeta(\mu,\rho) = 0$, we have nothing to prove. Let us suppose that $\zeta(\mu,\rho) > 0$ & let $0 < \theta < \zeta((\mu,\rho)$. There exists $A \subset X$ such that $\zeta_2 (\mu, A) \wedge \zeta_1 (A,\rho) > \theta$. If $x \notin A$ then $\theta < \zeta_2(\mu, A) \leq 1 - \mu(x)$. Thus $\mu \wedge A^C \leq 1- \theta$ & so $\zeta_0(\mu \wedge A^C, \rho) \geq \zeta(1 -\theta, \rho) \geq \theta$. Also we have

$\zeta_0(\mu\wedge A,\rho)\geq\zeta_0(A,\rho) > \theta$ & hence $\zeta_0(\mu,\rho) = \zeta_0(\mu\wedge A, \rho)\wedge\zeta_0(\mu\wedge A^C,\rho) > \theta$.

This clearly proves that $\zeta_0(\mu,\rho) \geq \zeta(\mu,\rho)$.

(4) $\zeta^C(\mu,\rho) = \zeta(1 - \rho, 1 - \mu) = \operatorname{Sup}_{A\subset X} \zeta_2 (1 - \rho, A) \wedge \zeta_1 (A, 1- \mu) = \operatorname{Sup}_{B\subset X} \zeta_2^C (B,\rho) \wedge \zeta_1^C (\mu,B)$

$=(\zeta_2^C \circ \zeta_1^C)(\mu,\rho)$

(5) Let us suppose that ζ_1, ζ_2 are perfect & let $\mu = \bigvee_{j\in J} \mu_j$. If $\bigwedge_{j\in J} \zeta(\mu_j,\rho) > \theta > 0$; then there are $A_j \subset X$ such that $\zeta_2(\mu_j, A_j) \wedge \zeta_1(A_j,\rho) > \theta$ for all $j \in J$.

If $A = \bigcup_j A_j$ then $\zeta_2(\mu,A) = \bigwedge_j \zeta_2(\mu_j, A) \geq \bigwedge_{j\in J} \zeta_2(\mu_j, A_j) \geq \theta$ &

$\zeta_2(A, \rho) = \bigwedge_j \zeta_1(A_j, \rho) \geq \theta$

& so $\zeta(\mu, \rho) \geq \theta$. This proves that $\zeta(\mu, \rho) = \bigwedge_j \zeta(\mu_j, \rho)$ & so

$\zeta(\mu,\rho) = \bigwedge_j \zeta(\mu_j, \rho)$ which shows that ζ is perfect.

If ζ_1, ζ_2 are biperfect then ζ_1^C, ζ_2^C are perfect so $\zeta^C = \zeta_2^C \circ \zeta_1^C$ is perfect. Thus both ζ & ζ^C are perfect & hence ζ is biperfect.

Definition 6 : *A* fuzzy syntopogenous structure on a non empty set X is a non empty family S of fuzzy topogenous orders on X satisfying the following axioms.

(FS1) S is directed in the sence that $\zeta_1,\zeta_2 \in S$ there exist $\zeta \in S$ with $\zeta \geq \zeta_1, \zeta_2$.

(FSI) Given $\zeta \in S$ & $\in > 0$ there exist $\zeta' \in S$ such that $\zeta' \circ \zeta' + \in \geq \zeta$.

Let now S be a fuzzy syntopogenous structure on X.

Wew define $^{-}: I^X \to I^X$, $\mu \to \bar{\mu}$ where $\bar{\mu}(x) = 1 - \operatorname{Sup}_{\zeta\in S} \zeta(x, 1-\mu)$.

Theorem 1 : The mapping $\mu \to \bar{\mu}$ defined above is a fuzzy closure operator on X with $\bar{\theta} = \theta$ for every $\theta \in I$.

Proof : Clearly $\mu \leq \bar{\mu}$, since $\zeta(x, 1-\mu) \leq 1-\mu(x)$. Since $\zeta(x, 1-\theta) = 1-\theta$. (by Lemma (2) of Section (II)). We get $\bar{\theta} = \theta$. It is also clear that $\overline{\mu_1} \leq \overline{\mu_2}$ when $\mu_1 \leq \mu_2$. It follows from this that $\overline{\rho_1 \vee \rho_2} > \overline{\rho_1} \vee \overline{\rho_2}$.

On the other hand, given $x \in X$, & $\in > 0$ there are $\zeta_1,\zeta_2 \in S$ such that $\overline{\rho_i}(x)+\in>1-\zeta_i(x, 1-\rho_i)$; $i = 1,2$.

Taking $\zeta \in S$, $\zeta \geq \zeta_1, \zeta_2$ we have

$$\begin{aligned}\overline{\rho_1 \vee \rho_2}(x) &\leq 1-\zeta(x, 1-\rho_1 \vee \rho_2)\\ &= 1-\zeta(x, 1-\rho_1)\wedge\zeta(x, 1-\rho_2)\\ &< \in + \overline{\rho_1}(x) \wedge \overline{\rho_2}(x)\end{aligned}$$

This proves that $\overline{\rho_1 \vee \rho_2}\,(x) \le \overline{\rho_1}\,(x) \wedge \rho_2(x)$ & so $\overline{\rho_1 \vee \rho_2} = \overline{\rho_1} \vee \overline{\rho_2}$.

Finally it remains to show that $\overline{\overline{\mu}} = \overline{\mu}$. Since $\overline{\mu} \le \overline{\overline{\mu}}$, it is sufficient to show that for every x $\in$ X & each $\in$ > O we have $\overline{\overline{\mu}}(x) \le \gamma + 2\in$ where $\gamma = \overline{\overline{\mu}}(x)$.

This clearly holds if $\gamma + 2 \in \ge 1$.

Now we suppose that $\gamma + 2 \in < 1$ & let $\zeta \in S$ be such that $\gamma > 1 - \zeta(x, 1-\mu) - \in$. Let $\zeta_1 \in S$ and $A \subset X$ be such that $\zeta_1(x,A) \wedge \zeta_1(A, 1-\mu) > 1-\gamma-\in$, for $y \in A$; we have $1 - \zeta_1(y, 1-\mu) \le 1-\zeta_1(A, 1-\mu) < \gamma + \in$.

Hence, if $\sigma = (\gamma+\in) \vee A^C$ then $\overline{\mu}(y) \le 1-\zeta_1(y, 1-\mu) \le \sigma(y)$ for all $y \in X$. Also we have $|\zeta_1(x, A\wedge(1-\gamma)) - \zeta_1(x, 1-\sigma)|$

$$\le \|A\wedge(1-\gamma)-(1-\sigma)\| \le \in.$$

Hence $\zeta_1(x, 1-\sigma) \ge \zeta_1(x, A\wedge(1-\gamma)) - \in$

$$= -\in + \zeta_1(x, A) \wedge \zeta_1(x, 1-\gamma) \ge 1-\gamma-2\in$$

Then $\overline{\mu}(x) \le 1- \zeta_1(x, 1-\overline{\mu}) \le 1 - \zeta_1(x, 1-\sigma) \le \gamma + 2\in$. This completes the proof.

Remark : Given a fuzzy syntopogenous structure on X, we will denote by $\tau(S)$ the fuzzy topology induced by the fuzzy closure operator of the previous proposition. We [will see that $(X,\tau(s))$ is a fuzzy nhd space i.e. $\tau(S)$ is induced by some fuzzy nhd system.

Theorem 2: If S is a fuzzy syntopogenous structure on X, then for each subset M of X & each $\theta \in I$ we have $\overline{\theta \wedge M} = \theta \wedge \overline{M}$ where the closures are taken with respect to the fuzzy topology $\tau(S)$.

Proof: Since the constant fuzzy set θ is closed we have $\overline{\theta \wedge M} = \theta + \overline{M}$. Let $x \in X$ & we suppose that $\overline{\theta \wedge M(x)} < \theta + \overline{M}(x)$.

Case I: $\theta > \overline{M}(x)$.

Since $\overline{\theta \wedge M(x)} < \theta$, there exist $\zeta \in S$ such that

$1 - \zeta(x, 1-\theta \wedge M) = 1-\zeta\left((x, 1-\theta) \vee M^C\right) < \theta$

Let $\zeta' \in S$ & $A \subset X$ be such that

$\zeta'(x, A) \wedge \zeta'\left(A, (1-\theta) \vee M^C\right) > 1 - \theta$

Since $1 - \theta < \zeta'\left(A, (1-\theta) \vee M^C\right) \le A^C(y) \vee (l-\theta) \vee M^C(y)$

for all y, it follows that $A \subset M^c$ & so

$\overline{M}(x) \le 1-\zeta'(x, M^C) \le 1 - \zeta'(x, A) < \theta$ which is a contradiction.

Case II: $\theta > \overline{M}(x)$

Since $\overline{\theta \wedge M(x)} < \overline{M}(x)$, there exists $\zeta \in S$ such that $\zeta(x, (l-\theta) \vee M^C) > 1 - \overline{M}(x)$.

Let $\zeta' \in S$ & $B \subset X$ be such that $\zeta'(x,B) \wedge \zeta'(B, (1-\theta) \vee M^C) > 1 - \overline{M}(x)$.

If $y \in B$ then $1- \overline{M}(x) < \zeta'\left(B,(1-\theta)\vee M^C\right) \leq (1-\theta) \vee M^C(y)$ & so $y \in M^C$, since $\theta > \overline{M}(x)$. Thus $B \subset M^C$ & therefore we have

$$\overline{M}(x) \leq 1-\zeta'(x,M^C) \leq 1-\zeta'(x,B) < \overline{M}(x).$$

which is a contradiction. This completes the proof.

Now we state the following theorem without proof.

Theorem 3 : If S is a fuzzy syntopogenous structure on X, then (X, $\tau(S)$) is a fuzzy nhd space.

Definition 7 : If a fuzzy syntopogenous structure S on X consists of a single fuzzy topogenous order, then S is called a fuzzy topogenous structure & (X, S) a fuzzy topogenous space.

Remark : If $S = \{ \zeta \}$ is a fuzzy topogenous structure on X, then $\zeta o \zeta + \in \geq \zeta$ for every $\in > 0$ & so $\zeta o\zeta \geq \zeta$. Since $\zeta o \zeta \leq \zeta$ (by (3) of Lemma (3) (Section II), we have $\zeta o \zeta = \zeta$.

Definition 8 : A fuzzy syntopogenous structure S is called perfect (resp. biperfect, resp. symmetrical) if every member of S is perfect (resp. biperfect, resp. symmetrical).

Theorem 4 : Let S be a fuzzy syntopogenous structure on X & ζ_S be defined by

$$\zeta_S(\mu,\rho) = \text{Sup}\,\{\zeta(\mu,\rho):\zeta \in S\}.$$

Then $S' = \{\zeta_S\}$ is a fuzzy topogenous structure on X with $\tau(S') = \tau(S)$.

Proof: It is obvious that ζ_S satisfies (i) - (iv) of definition (3)(Section II). To prove (v), let $\theta < \zeta_S(\mu_1,\rho) \wedge \zeta_S(\mu_2, \rho)$. There are $\zeta_1,\zeta_2 \in S$ such that $\zeta_i(\mu_i, \rho) > \theta$; i = 1,2 .
If $\zeta \in S$ is such that $\zeta \geq \zeta_1, \zeta_2$ then

$\zeta_S(\mu_1 \vee \mu_2, \rho) \geq \zeta(\mu_1 \vee \mu_2, \rho)$

$= \zeta(\mu_1, \rho) \wedge \zeta(\mu_2, \rho) > \theta$

This proves that $\zeta_S(\mu_1 \vee \mu_2, \rho) \geq \zeta_S(\mu_1, \rho) \wedge \zeta_S(\mu_2, \rho)$

and so $\zeta_S(\mu_1 \vee \mu_2, \rho) = \zeta_S(\mu_1, \rho) \wedge \zeta_S(\mu_2, \rho)$

Similarly we can show that $\zeta_S(\mu, \rho_1 \wedge \rho_2) = \zeta_S(\mu, \rho_1) \wedge \zeta_S(\mu, \rho_2)$

Finally, let $\mu,\rho \in I^X$ & $\in > 0$.

Then there exists $\zeta \in S$ such that $\zeta(\mu,\rho) > \xi_S(\mu,\rho) - \in$.

Let $\zeta' \in S$ & $A \subset X$ be such that $\zeta'(\mu,A) \wedge \zeta'(A, \rho) > \gamma = \zeta_S(\mu,\rho) - \in$.

Now $\zeta_S(\mu,A) \wedge \zeta_S(A, \rho) > \zeta_S(\mu,\rho) - \in$.

This proves that S' is a fuzzy topogenous structure.

Also for $\mu \in I^X$

$$\overset{-t(S')}{\mu}(x) = 1 - \zeta_S(x, 1-\mu)$$

$$= 1 - \sup_{\zeta \in S} \zeta(x, 1-\mu)$$

$= \overset{-\tau(S')}{\mu}(x)$

This implies that $\tau(S)=\tau(S')$. Hence the theorem follows.

For a fuzzy syntopogenous structure S on X we will denote by S^t the fuzzy syntopogenous structure $\{\zeta_S\}$.

Lemma 4 : Let ζ be a fuzzy semi-topogenous order on X & let $A \subset X$ & $\theta \in I$. If $\mu \in I^X$, then

(C1) $\zeta o \zeta(\theta \wedge A, \mu) \le (1-\theta) \vee \zeta(A, \mu)$

(C2) $\zeta o \zeta(\mu, \theta \vee A) \le \theta \vee \zeta(\mu, A)$

(C3) If $\zeta o \zeta = \zeta$, then we have equality in both (C1) & (C2).

Proof: (Cl) Let us suppose that $\zeta o \zeta(\theta \wedge A, \mu) > \gamma = (1-\theta) \vee \zeta(A, \mu)$

Case I : $1-\theta \le \zeta(A, \mu)$

Let $B \subset X$ be such that $\zeta(\theta \wedge A, B) \wedge \zeta(B, \mu) > \gamma = \zeta(A, \mu)$. If $x \in A$, then $\gamma < \zeta(\theta \wedge A, B) \le (1-\theta) \vee B(x)$ & so $x \in B$ since $1-\theta \le \gamma$.

Thus $A \subset B$ & hence $\zeta(A,\mu) \ge \zeta(B,\mu) > \zeta(A,\mu)$ which is a contradiction.

Case II : $\zeta(A,\mu) < 1-\theta$.

Let $M \subset X$ be such that $\zeta(\theta \wedge A, M) \wedge \zeta(M,\mu) > \gamma = 1-\theta$.

If $x \in A$, then $1-\theta < \zeta(\theta \wedge A, M) \le (1-\theta) \vee M(x)$ & so $x \in M$.

Thus $A \subset M$ & therefore $\zeta(A,\mu) \ge \zeta(M,\mu) > 1-\theta$ which is a contradiction. This proves (Cl).

(C2) let $\zeta_1 = \zeta o \zeta$. Then $\zeta_1^C = \zeta^C o \zeta^C$ (by Lemma (3)(Section II). Hence using (Cl.) we get $\zeta o \zeta(\mu, \theta \vee A) = \zeta^C o \zeta^C \left((1-\theta) \wedge A^C, 1-\mu\right) \le \theta \vee \zeta^C(A^C, 1-\mu) = \theta \vee \zeta(\mu, A)$.

(C3) We have $\zeta(\theta \wedge A,\mu) \ge$
$\zeta(\theta,\mu) \vee \zeta(A, \mu) \ge (1-\theta) \vee \zeta(A, \mu)$ & $\zeta(\mu, \theta \vee A) \ge \zeta(\mu,\theta) \vee \zeta(\mu, A) \ge \theta \vee \zeta(\mu, A)$ Thus (C3) follows from (Cl) & (C2).

Definition 9 : A fuzzy, syntopogenous structure S on X is said to be finer than another one S' if for each $\zeta' \in S'$ & each $\in > 0$, there exists $\zeta \in S$ with $\zeta \ge \zeta' - \in$. In this case we also say that S' is coarser than S. The fuzzy syntopogenous structures S & S' are said to be equivalent & we write $S \cong S'$ if S is both finer & coarser than S'.

Theorem 5 : (1) If S is finer and S' then $\tau(S') \subset \tau(S)$.

(2) If $S \cong S'$ then $\tau(S) = \tau(S')$.

Proof: (1) It suffices to show that $\overset{-}{\mu}^{\tau(S)} \cong \overset{-}{\mu}^{\tau(S')}$ for each $\mu \in I^X$. So, let $\mu \in I^X$, $x \in X$ & $\in > 0$. Let us choose $\zeta' \in S'$ such that $1-\zeta'(x, 1-\mu) < \overset{-}{\mu}^{\tau(S)} + \in$.

Let $\zeta \in S$ with $\zeta \ge \zeta' - \in$.

Now $\overset{-}{\mu}^{\tau(S)}(x) \le 1-\zeta(x, 1-\mu) \le \in + 1 - \zeta'(x, 1-\mu) \le 2\in + \overset{-}{\mu}^{\tau(S')}(x)$

Thus the result follows. Since $\in > 0$ is arbitrary

(2) follows immediately from (1).

CORRESPONDENCE BETWEEN FUZZY NHD STRUCTURES & PERFECT FUZZY TOPOGENOUS STRUCTURE

Let (X, τ) be a fuzzy nhd space i.e. the fuzzy topology τ is given by some fuzzy nhd system.

Let us define $\zeta = \zeta_\tau$ by $\zeta(\mu = \rho) = \inf_{x\in X} [1-\mu(x)] \vee \rho^\circ(x)$ $(\mu,\rho\in I^X)$

Where ρ° denotes the τ interior of ρ. Then we have the following.

(T1) ζ is perfect fuzzy topogenous order. It is obvious that ζ satisfies (i), (ii) & (iii) of definition (3) (Section II). Next let us suppose that $\zeta(\mu,\rho) - \zeta(\mu', \rho) > \gamma=\|\mu-\mu'\|$.

Let θ be such that $\zeta(\mu, \rho) > \theta$ & $\zeta(\mu',\rho) < \theta - \gamma$. There exists $x \in X$ such that $\{1-\mu'(x)] \vee \rho^0(x) < \theta - \gamma$

Since $1 - \mu(x) = [1-\mu'(x)]+[\mu'(x)- \mu(x)] < \theta$, we have

$\zeta(\mu,\rho) \le [1-\mu(x)] \vee \rho^0(x) < \theta$,

which is a contradiction.

This proves that $|\zeta(\mu,\rho) - \zeta(\mu'\ \rho)|\le\|\mu-\mu'\|$.

Since $\|\rho^0- \rho_1^0\| \le \| \overline{1-\rho_1} - \overline{1-\rho}\|\le\|\rho_1 - \rho\|$;

We get in the same way that $|\zeta(\mu,\rho)- \zeta(\mu,\rho_1)|\le\| \rho - \rho_1\|$

Finally, if $\mu = \bigvee_{j\in J} \mu_j$, then

$$\zeta(\mu,\rho) = \inf_x \left[1-\operatorname{Sup}_j \mu_j(x)\right] \vee \rho^0(x)$$

$$= \inf_j \left[\inf_x [1-\mu_j(x)] \vee \rho^0(x)\right]$$

$$= \inf_j \zeta(\mu_j,\rho)$$

Hence ζ is perfect.

(T2) $S_\tau =\{\zeta\}$ is a fuzzy topogenous structure with $\tau = \tau(S_\tau)$. In fact, let $\mu, \rho \in I^X$ & $\in>0$. We put $\gamma = \zeta(\mu,\rho)- \in$. We need to show that there exists $A \subset X$ such that $\zeta(\mu,A\} \wedge \zeta(A,\rho) > \gamma$. If $\gamma < 0$, there is nothing to prove. Let us suppose that $\gamma > 0$. Since $\zeta(\mu,\rho) > \gamma + \frac{1}{2} \in$, we have $[1 - \mu(x)] \vee \rho^0(x) > \gamma + \frac{1}{2} \in$. for every x.

We set $A = \{x:\rho^0(x) > \gamma + \frac{1}{2} \in$

Since τ is given by some fuzzy nhd structure, we have $A \wedge \rho^0 \in \tau$.

Now $\zeta(\mu,A) \wedge \zeta(A,\rho) \geq \gamma + \frac{1}{2} \in$.

In fact, let $x \in X$.

If $1-\mu(x) < \gamma + \frac{1}{2} \in$ then $\rho^0(x) > \gamma + \frac{1}{2} \in$ & so $x \in A$ which implies that

$A^0(x) \geq (A \wedge \rho^0)(x) \geq \gamma + \frac{1}{2} \in$

Hence $\zeta(\mu,A) \geq \gamma + \frac{1}{2} \in$.

Similarlly $A^C(x) \vee \rho^0(x) > \gamma + \frac{1}{2} \in$ for x & so

$\zeta(A,\rho) \geq \gamma + \frac{1}{2} \in$.

This proves that S_τ is a fuzzy topogenous structure.

Finally for $\mu \in I^X$, we have

$\mu^{\tau(S_\tau)}(x) = 1 - \zeta(x, 1-\mu)$

$= 1 - \inf_{y} x^C(y) \vee (1-\mu)^0(y)$

$= 1 - (1-\mu)^0(x)$

$= \overline{\mu}^{\tau}(x)$

This implies that $\tau(S_\tau) = \tau$.

We have now the following result.

Theorem 6 : The mapping $\tau \to S_\tau$ from the set of all fuzzy topologies on X which are given by fuzzy nhd structures to the set of all perfect fuzzy topogenous structures on X, is one to one & onto. Moreover $\tau = \tau(S_\tau)$.

Proof : Since $\tau =(S_\tau) = \tau$, the mapping in one to one. To show that it is onto, let $S = \{ \zeta \}$ be a perfect fuzzy topogenous structure on X. The fuzzy topology $\tau = \tau(S)$ is given by some fuzzy nhd system (by theorem (3)(Section II). The proof will be complete if we show that $S = S_\tau$. Since ζ is perfect, we have

$\zeta(\mu,\rho) = \zeta(\bigvee_{x\in X} \mu(x) \wedge x,\rho)$

$= \bigwedge_{x} \zeta(\mu(x) \wedge x, \rho)$

Since $\zeta \circ \zeta = \zeta$, we have $\zeta(\mu(x) \wedge x, \rho) = [1 - \mu(x)] \vee \mu(x,\rho)$

[by (C3) of Lemma (4)(Section II)]

Also $\rho^{O}(x) = 1-\overline{1-\rho^{\tau}(x)} = 1-\overline{1-\rho^{\tau(S)}(x)} = \zeta(x,\rho)$

Hence

$\zeta(\mu,\rho) = \bigwedge_{x} [1-\mu(x)] \vee \rho^{O}(x)$

$= \xi_{\tau}(\mu,\rho)$

and so $\zeta = \zeta_{\tau}$. This proves that $S = S_{\tau}$ & the result follows,

CORRESPONDENCE BETWEEN FUZZY PROXIMITIES & SYMMETRICAL FUZZY TOPOGENOUS STRUCTURES

Let δ be a fuzzy proximity on X. We define ζ by $\zeta(\mu,\rho) = 1 - \delta(\mu, 1 - \rho)$. It is obvious that ζ is a symmetrical fuzzy topogenous order on X & $S_{\delta} = \{\zeta\}$ is a fuzzy topogenous structure with $\tau(S_{\delta}) = \tau(\delta)$. The mapping $\delta \to S_{\delta}$ is clearly one to one. If $S = \{\zeta\}$ is a symmetrical fuzzy topogenous structure on X, then the function

$\delta : I^{X} \times I^{X} \to I;$

$\delta(\mu,\rho) = 1 - \zeta(\mu, 1 - \rho)$ is a fuzzy proximity on X with $S = S_{\delta}$.

Thus we have the following theorem without proof.

Theorem 7: The mapping $\delta \to S_{\delta}$ from the set of all fuzzy proximities on X to the set of all symmetrical fuzzy topogenous structures on X, is one to one & onto. Moreover

$\tau(\delta) = \tau(S_{\delta})$.

CORRESPONDENCE BETWEEN FUZZY QUASI UNIFORMITIES & BIPERFECT FUZZY SYNTOPOGENOUS STRUCTURES

For an $\alpha \in I^{X \times X}$ with $\alpha(x, x) = 1$ for all $x \in X$, we define

$\zeta = \zeta_{\alpha}$ by setting

$\zeta(\mu,\rho) = 1 - \sup_{x} [\alpha < u > \wedge \alpha^{-1} < 1 - \rho >](x) \quad (\mu,\rho \in I^{X})$

we have already

$\alpha^{-1}(x,y) = \alpha(y,x)$ & that

$\alpha < \mu > (x) = \operatorname{Sup}_{y} \mu(y) \wedge \alpha(y,x)$

Lemma 5 : The function ζ_{α} is a biperfect fuzzy topogenous order on X.

Moreover $\zeta_{\alpha}(x, y^{C}) = 1 - \alpha \circ \alpha(x,y)$

$\& \zeta_{\alpha}(\mu,\rho) = \bigwedge_{x,y} [1-\mu(x)] \vee \rho(y) \vee [1 - \alpha o \alpha(x,y)]$

(ζ_{α} satisfies (i) & (ii) of definition (3)(Section II).

Lemma 6: Let $\alpha,\beta \in I^{X\times X}$ with $\alpha(x,x) = \beta(x, x) = 1$ for every x.

Then we have

(1) for $\in \geq o$ we have $\zeta_\alpha \geq \zeta_\beta - \in$
iff $\alpha o \alpha \leq \in + \beta o \beta$

(2) If $\alpha \leq \beta + \in (\in \geq o)$, then $\zeta_\alpha \geq \zeta_\beta - \in$.

(3) If $\gamma = \alpha o \alpha$, then $\zeta_\alpha \ o \ \zeta_\alpha \geq \zeta_\gamma$

(4) $\zeta_{\alpha-1} = (\zeta_\alpha)^C$

Theorem 8: If $\mathcal{U}$ is a fuzzy quasi uniformity on X, then the family $S = S_{\mathcal{U}} = (\zeta_\alpha : \alpha \in \mathcal{U}\}$ is a biperfect fuzzy syntopogenous structure on X with $\tau(S) = \tau(\mathcal{U})$.

Proof: If $\alpha_1, \alpha_2 \in \mathcal{U}$, then by taking $\alpha \in \mathcal{U}$, $\alpha \leq \alpha_1$, we have $\zeta_\alpha \geq \zeta_{\alpha_1}, \zeta_{\alpha_2}$ by the preceeding Lemma. Also given $\alpha \in \mathcal{U}$ & $\in > 0$, we choose $\beta \in \mathcal{U}$ with $\beta o \beta - \in \leq \alpha$.

Now $\zeta_\alpha \leq \zeta_{\beta o \beta} + \in \leq \zeta_\beta o \zeta_\beta + \in$.

This proves that S is a fuzzy syntopogenous structure which is biperfect (by Lemma (5) (Section II). Now it remains to show that $\tau(S) = \tau(\mathcal{U})$.

So let $\mu \in I^X$ & $x \in X$.

We have

$$\bar{\mu}^{\tau(S)}(x) = 1 - \sup_{\alpha \in \mathcal{U}} \zeta_\alpha(x, 1-\mu)$$

$$= \inf_{\alpha \in \mathcal{U}} \sup_y [\alpha < x > \wedge \alpha^{-1} < \mu >] (y)$$

$$\& \ \bar{\mu}^{\tau(\mathcal{U})}(x) = \inf_{\alpha \in \mathcal{U}} \alpha^{-1} < \mu > (x)$$

Let now $\alpha \in \mathcal{U}$ & $\in > 0$. We choose $\beta \in \mathcal{U}$ such that $\beta o \beta - \in \leq \alpha$.

For $y \in X$, we have

$$[\beta < x > \wedge \beta^{-1} < \mu >] (y)$$

$$= [\beta(x,y) \wedge [\sup_z \mu(z) \wedge \beta(y,z)]$$

$$\leq \sup_z \mu(z) \wedge \beta o \beta(x,z)$$

$$\leq \in + \sup_z \mu(z) \wedge \alpha(x,z)$$

$$= \in + \alpha^{-1} < \mu > (x)$$

Thus $\bar{\mu}^{\tau(S)}(x) \leq \in + \alpha^{-1} < \alpha > (x)$

Since $\in > 0$ & $\alpha \in$ e $\mathcal{U}$ were arbitrary, we get

$$\bar{\mu}^{\tau(S)}(x) \leq \bigwedge_{\alpha \in \mathcal{U}} \alpha^{-1} < \mu > (x) = \bar{\mu}^{\tau(\mathcal{U})}(x)$$

on the other hand,

$$\sup_y [\alpha < x > \wedge \alpha^{-1} < \mu >] (y)$$

$$\geq [\alpha (x) \wedge \alpha^{-1} < \mu > (x)]$$

$$= \alpha^{-1} < \mu > (x) \text{ \& therefore}$$

$$\overline{\mu}^{\tau(S)} (x) \geq \inf_{\alpha \in \mathscr{U}} \alpha^{-1} < \mu > = \overline{\mu}^{\tau(\mathscr{U})} (x)$$

Thus $\overline{\mu}^{\tau(S)} = \overline{\mu}^{\tau(\mathscr{U})}$

which implies that $\tau(S) = \tau(\mathscr{U})$.

We have seen that the fuzzy syntopogenous structure S is called (a) topogenous if S consists of a single element, (b) perfet (resp. biperfect) if each element of S is perfect (resp. biperfect).

If $\zeta_S = \sup_{\zeta \in S} \zeta$, then $S' = \{\zeta_S\}$ is a fuzzy topogenous structure. The fuzzy topology $\tau(S)$ induced by a fuzzy syntopogenous structure S, is given by the closure operator

$$\overline{\mu} (x) = 1 - \sup_{\zeta \in S} \zeta (x, 1 - \mu)$$

$$= 1 - \zeta_S (x, 1 - \mu)$$

or equivalently by the interior operator $\mu^{\circ}(x) = \zeta_s(x, \mu)$.

Clearly $\tau(S) = \tau(S')$. We have also seen that a fuzzy syntopogenous structure S is finer than another one S' (or that S' is coarser than S) if for each $\zeta \in S'$ & $\in > 0$, there exist $\zeta \in S$ with $\zeta \geq \zeta' - \in$.

Definition 10 : A fuzzy semi-topogenous order ζ on a set X is said to be finer than another one ζ' if $\zeta' \leq \zeta$.

In this case we also say that ζ' is coarser than ζ .

Theorem 9 : Given a fuzzy semi-topogenous order ζ on X, there exists a fuzzy topogenous order ζ^q finer than ζ & coarser than any other fuzzy topogenous order on X which is finer than ζ. For μ,ρ fuzzy sets in X, we have $\zeta^q (\mu,\rho) = \sup \inf_{i,j} \zeta(\mu_i,\rho_j)$ .... (*).

The supremum is taken over all finite families $(\mu_i),(\rho_j)$ of fuzzy sets with $\mu = \vee_i \mu_i$, $\rho = \wedge_j \rho_j$.

Proof: Let ζ^q be defined as in(*). Clearly $\zeta \leq \zeta^q$ & so $\zeta^q (0,0) = \zeta^q(l,l) = 1$.

Let $x \in X$ & $\theta > [1-\mu(x)] \vee \rho(x)$.

If $\mu = \vee_i \mu_i$ & $\rho = \wedge_j \rho_j$ then there are i, j such that $1-\mu_i (x) < \theta$, $\rho_j(x) < \theta$ & so $\zeta(\mu_i,\rho_j) \leq [1 - \mu_i (x)] \vee \rho_j (x) < \theta$.

It follows that $\zeta^q (\mu,\rho) \leq \theta$ which clearly proves that

$\zeta^q(\mu, \rho) \leq [1 - \mu(x)] \vee \rho(x)$ for all $x \in X$.

Next, let $\mu' \leq \mu$ & $\rho \leq \rho'$.

If $\mu = \vee \mu_i$, & $\rho = \wedge \rho_j$, then $\mu' = \vee (\mu_i \wedge \mu')$ & $\rho' = \wedge (\rho_j \vee \rho')$ & hence

$$\zeta^q(\mu', \rho') \geq \inf_{i,j} \zeta(\mu_i \wedge \mu', \rho_j \vee \rho')$$

$$\geq \inf_{i,j} \zeta(\mu_i, \rho_j)$$

which proves that

$\zeta^q(\mu', \rho') \geq \zeta^q(\mu, \rho)$.

Finally, let $\delta = \|\mu - \mu'\|$.

We need to show that

$|\zeta^q(\mu, \rho) - \zeta^q(\mu', \rho)| \leq \delta$.

In fact, let us suppose that $\zeta^q(\mu,\rho) > \zeta^q(\mu' \rho)$ and let $\theta < \zeta^q(\mu,\rho) - \zeta^q(\mu',\rho)$. We choose γ such that $\zeta^q(\mu,\rho) > \gamma$, $\zeta^q(\mu',\rho) < \gamma - \theta$. There are finite fimilies (μ_i), (ρ_j) of fuzzy sets such that

$\mu = \vee \mu_i$, $\rho = \wedge \rho_j$

& $\wedge_{i,j} \zeta(\mu_i,\rho_j) > \gamma$

If $\mu'' = (\mu+\delta) \wedge 1$, then $\mu' \leq \mu''$.

Putting $\mu_i' = (\mu_i + \delta) \wedge 1$ we have $\mu'' = \vee \mu_i'$.

Also we have

$\zeta(\mu_i', \rho_j) \geq - \delta + \zeta(\mu_i,\rho_j) > \gamma - \delta$

Since $\|\mu_i' - \mu_i\| \leq \delta$ & so

$$\zeta^q(\mu'', \rho) \geq \inf_{i,j} \zeta(\mu'_i - \rho_j) \geq \gamma - \delta.$$

Since $\zeta^q(\mu'', \rho) \leq \zeta^q(\mu', \rho) < \gamma - \theta$ we must have $\delta > \theta$. This proves that

$|\zeta^q(\mu,\rho) - \zeta^q(\mu', \rho)| \leq \|\mu-\mu'\|$.

In an analogous way, we can show that $|\zeta^q(\mu,\rho) - \zeta^q(\mu,\rho')| \leq \| \rho - \rho''\|$ & so ζ^q is a fuzzy semi-topogenous order. To show that ζ^q is topogenous, let $\mu = \mu_1 \vee \mu_2$ & let $\eta < \zeta(\mu_1, \rho) \wedge \zeta(\mu_2,\rho)$.

There are finite finilies (σ_i), (ρ_j), (τ_m), (ϕ_K) of fuzzy sets such that

$\mu_1 = \vee \sigma_i$, $\rho = \wedge \rho_j = \wedge \phi_K$; $\mu_2 = \vee \tau_m$, $\wedge_{i,j} \zeta(\sigma_i, \rho_j) > \eta$,

& $\wedge_{m,K} \zeta(\tau_m,\phi_K) > \eta$

Since $\mu = [\vee \sigma_i] \vee [\vee \tau_m]$;

$\rho = \wedge_{j,K} [\rho_j \vee \phi_K]$ & since

$\zeta[\sigma_i, \rho_j \vee \phi_K] \geq \zeta(\sigma_i, \rho_j) > \eta$

& $\zeta(\tau_m, \rho_j \vee \phi_K) > \eta$, we have $\zeta^q(\mu,\rho) > \eta$. This clearly proves that

$\zeta^q(\mu_i, \vee \mu_2 \rho) \geq \zeta^q(\mu,\rho) \wedge \zeta^q(\mu_2, \rho)$

whcih implies that

$\zeta^q(\mu_1 \vee \mu_2, \rho) = \zeta^q(\mu_1,\rho) \wedge \zeta^q(\mu_2, \rho)$.s

Analogously we can show that

$\zeta^q(\mu, \rho_1 \wedge \rho_2) = \zeta^q(\mu,\rho_1) \wedge \zeta^q(\mu, \rho_2)$

Thus ζ^q is topogenous.

Finally if ζ' is topogenous and finer than ζ & if $\mu = \vee \mu_i$ & $\rho = \wedge \rho_j$, then

$\zeta'(\mu,\rho) = \bigwedge_{i,j} \zeta'(\mu_i,\rho_j) \geq \inf_{i,j} \zeta(\mu_i,\rho_j)$

& so $\zeta'(\mu,\rho) \geq \zeta^q(\mu,\rho)$.

This completes the proof.

Corollary **:**

(i) A fuzzy semi-topogenous order ζ on X is topogenous iff $\zeta = \zeta^q$.

(ii) $\zeta^{qq} = \zeta^q$

Definition 11 : The fuzzy topogenous order ζ^q is called topogenous cover of ζ. We have therefore the following result:

Theorem 10 : Let ζ be a fuzzy semi-topogenous order on X.

Then,

(i) $\zeta^{q^C} = \zeta^{C^q}$

(ii) If ζ is symmetrical, so is ζ^q.

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Chapter - 3

Image of fuzzy syntopogenous structures

Chapter - 3

In the previous chapter we have discussed fuzzy syntopogenous structures. We have already discussed the conditions under which such structures are topogenous, perfect & biprfect.

Some operations on fuzzy syntopogenous structures have also been underlined in the previous chapter.

This chapter is addressed to the inverse images of fuzzy serni-topogenous order & some interesting results on product fuzzy syntopogenous spaces.

INVERSE IMAGE OF A FUZZY SEMI-TOPOGENOUS ORDER

Let f: $X \to Y$ be a function and let ζ be a fuzzy semi-topogenous order on Y. The mapping.
$\zeta' : I^X \times I^X \to I$, $\zeta'(\mu,\rho) = \zeta(f(\mu), 1-f(1-\rho))$
is a fuzzy semi-topogenous order on X.
We will call ζ' the inverse image of ζ by the mapping f and we will denote if by $f^{-1}(\zeta)$.

Proposition 1: Let f be a function from X to Y and let ζ, ζ' be fuzzy semi-topogenous orders on Y. Then :

(1) For $\mu,\rho \in I^Y$ we have
$f^{-1}(\zeta)(f^{-1}(\mu), f^{-1}(\rho)) \geq \zeta(\mu,\rho)$ (*)
In case f is onto, we have equality in (*)

(2) $f^{-1}(\zeta)$ is the coarsest fuzzy semi-topogeneus order ζ_1 on X for which
$\zeta_1(f^{-1}(\mu), f^{-1}(\rho)) \geq \zeta(\mu,\rho)$
for all $\mu,\rho \in I^Y$

3) If $\zeta \leq \zeta'$, then $f^{-1}(\zeta) \leq f^{-1}(\zeta')$. In case f is onto, the converse is also true.

4) If $\{\zeta_\lambda : \lambda \in \wedge\}$ is a family of fuzzy semi-topogenous orders on Y, then
$f^{-1}(\operatorname{Sup}_\lambda \zeta_\lambda) = \operatorname{Sup}_\lambda f^{-1}(\zeta_\lambda)$

(5) $[f^{-1}(\zeta)]^q = f^{-1}(\zeta^q)$

(6) $[f^{-1}(\zeta)]^\rho = f^{-1}(\zeta^\rho)$ and ana $[f^{-1}(\zeta)]^b = f^{-1}(\zeta^b)$.

(7) If ζ is topogenous, then $f^{-1}(\zeta)$ is topogenous.

(8) If ζ is perfect (resp. biperfect), then $f^{-1}(\zeta)$ is perfect (resp. biperfect).

(9) $[f^{-1}(\zeta)]^C = f^{-1}(\zeta^C)$

(10) If ζ is symmetrrical, then $f^{-1}(\zeta)$ is also symmetrical.

(11) If $\zeta_1 = f^{-1}(\zeta)$, $\zeta_2 = f^{-1}(\zeta')$ and $\zeta_3 = f^{-1}(\zeta o \zeta')$, then ζ_3 is a coarser than $\zeta_1 o \zeta_2$. If f is onto, then $\zeta_3 = \zeta_1 o \zeta_2$.

Proof :

(1) Since $f(f^{-1}(\mu)) \leq \mu$ and $f(1-f^{-1}(\rho)) \leq 1-\rho$,(*) follows from the fact that ζ is a fuzzy semi-

topogenous order. If f is onto then $f(f^{-1}(\mu)) = \mu$ and $f(1 - f^{-1}(\rho)) = 1 - \rho$ and so we have equality in (*),

(2) If follows from (1) and from the fact for $\mu,\rho \in I^X$ we have $f^{-1}(f(\mu)) \geq \mu$ and $f^{-1}(1-f(1-\rho)) \leq \rho$.

(3) It follows from (1).

(4) By (3), we have

$$f^{-1}(\vee_{\lambda} \zeta_\lambda) \geq \vee_{\lambda} f^{-1}(\zeta_\lambda)$$

On the other hand, if $\zeta_1 = \vee_{\lambda} f^{-1}(\zeta_\lambda)$, then for all $\mu,\rho \in I^Y$ we have

$$\zeta_1(f^{-1}(\mu), f^{-1}(\rho)) \geq f^{-1}(\zeta_\lambda)(f^{-1}(\mu), f^{-1}(\rho)) \geq \zeta_\lambda(\mu,\rho).$$

This together with (2) implies that $\zeta_1 \geq f^{-1}(\vee_{\lambda} \zeta_\lambda)$.

(7) Let ζ be topogenous. Then

$f^{-1}(\zeta)(\mu_1 \vee \mu_2, \rho) = \zeta(f(\mu_1) \vee f(\mu_2), 1-f(1-\rho))$

$=\zeta(f(\mu_1), 1 - f(1-\rho)) \wedge \zeta(f(\mu_2), 1 - f(1-\rho))$

$=f^{-1}(\zeta)(\mu_1,\rho) \wedge f^{-1}(\zeta)(\mu_2,\rho)$

Analogously, we get that $f^{-1}(\zeta)(\mu,\rho_1 \wedge \rho_2) = f^{-1}(\zeta)(\mu,\rho_1) \wedge f^{-1}(\zeta)(\mu,\rho_2)$.

(8) The proof is analogous to that of (7).

(5) By (3) and (7) we get that

$[f^{-1}(\zeta)]^q \leq f^{-1}(\zeta^q)$

On the other hand, let $\mu,\rho \in I^X$ and let (σ_i), (φ_j) be finite families of fuzzy sets in Ywith $f(\mu) = \vee \sigma_i$, $1-f(1-\rho) = \wedge \varphi_j$. Then $\mu \leq f^{-1}(f(\mu)) \leq \vee f^{-1}(\sigma_i)$ and $\rho \geq \wedge f^{-1}(\varphi_j)$, and hence

$f^{-1}(\zeta)^q(\mu,\rho) \geq f^{-1}(\zeta)^q(\vee_i f^{-1}(\sigma_i), \wedge_j f^{-1}(\varphi_j))$

$\geq \inf_{i,j} f^{-1}(\zeta)(f^{-1}(\sigma_i), f^{-1}(\varphi_j))$

$\geq \inf_{i,j} \zeta(\sigma_i, \varphi_j)$

If follows from this that

(6) The proof is analogous to that of (5).

(9) & (10) follow directly from the difinitions.

(11) Let μ,ρ be fuzzy sets in X, $A \subset Y$ and $B=f^{-1}(A)$. Then

$\zeta_2(\mu,B) = \zeta'(f(\mu), [f(B^C)]^C) \geq \xi'(f(\mu), A)$,

$\zeta_1(B,\rho) = \zeta(f(B), 1-f(1-\rho)) \geq \zeta(A, 1-f(1-\rho))$

Hence

$\zeta o \zeta'(f(\mu), 1 - f(1 - \rho)) = \sup_{A \subset Y} \zeta'(f(\mu), A) \wedge \zeta(A, 1 - f(1 - \rho)$

$\leq \sup_{B \subset X} \zeta_2(\mu, B) \wedge \zeta_1(B,\rho)$

$= \zeta_1 o \zeta_2(\mu, \rho)$

and so $\zeta_3 \leq \zeta_1 o \zeta_2$

Finally, let f be onto. For $A \subset X$ and $B = f(A)$, we have $[f(A^C)]^C \subset B$ and so $\tau(f^{-1}(S)$

$= \zeta'(f(\mu),[f(A^C)]^C \wedge \zeta(f(A), 1 - f(1 - \rho))$

$\leq \zeta'(f(\mu), B) \wedge \zeta(B, 1 - f(1-\rho))$

$= f^{-1}(\zeta o \zeta')(\mu,\rho)$

Thus, for f onto, we have $\zeta_1 o \zeta_2 \leq \zeta_3$ and the result follows.

Proposition 2 :

Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be functions and let ζ be a fuzzy semi-topogenous order on Z. Then $(gof)^{-1}(\zeta) = f^{-1}(g^{-1}(\zeta))$

Proof:

It follows directly from the previous definitions.

INVERSE IMAGE OF A FUZZY SYNTOPOGENOUS STRUCTURE

Proposition 3 :

Let $f : X \rightarrow Y$ be a function and let S be a fuzzy syntopogenous structure on Y. Then :

(1) $f^{-1}(S) = \{f^{-1}(\zeta) : \zeta \in S\}$ is a fuzzy syntopogenous structure on X.

(2) If S is perfect, biperfect or symmetrical, then $f^{-1}(S)$ is perfect, biperfect or symmetrical, respectively,

(3) $f^{-1}(\tau(S)) = \tau(f^{-1}(S))$

(4) If $(S_\lambda)_{\lambda \in \wedge}$ is a family of fuzzy syntopogenous structures on Y, then

$f^{-1}(\underset{\lambda}{\vee} S_\lambda) \leq \underset{\lambda}{\vee} f^{-1}(S_\lambda)$

Proof:

(1) and (2) follows directly from the definitions and from proposition (1).

(3) Let $\zeta = \zeta_s$, $\zeta_1 = \zeta_{f^{-1}(S)}$, $\tau = \tau(S)$, $\tau_1 = \tau(f^{-1}(S))$. For a fuzzy subset μ of Y we have $\mu^o(y) = \zeta_s(y,\mu)$.

Let now $\mu \in \tau$ and let ρ be the τ_1 – interior of $f^{-1}(\mu)$,

Then,

$\rho(x) = \zeta_1(x, f^{-1}(\mu)) = \underset{\zeta \in S}{\mathrm{Sup}}\, f^{-1}((\zeta)(x, f^{-1}(\mu)))$

$= \underset{\zeta \in S}{\mathrm{Sup}}\, \zeta(f(x), 1\text{-}f(1\text{-}f^{-1}(\mu)))$

$\geq \underset{\zeta \in S}{\mathrm{Sup}}\, \zeta(f(x),\mu)$

$= \mu^o(f(x)) = \mu(f(x)) = f^{-1}(\mu)(x)$

and so $\rho \geq f^{-1}(\mu)$, which implies that $f^{-1}(\mu) \in \tau_1$. Conversely, let $\sigma \in \tau_1$ and $\mu = 1 - f(1-\sigma)$. If μ° is the τ- interior of μ, then

$\sigma(x) = \zeta_1(x,\sigma) = \zeta_s(f(x), 1 - f(1-\sigma))$

$= \zeta_s(f(x), \mu) = \mu^o (f(x)) = f^{-1} (\mu^o) (x)$.
Thus $\sigma = f^{-1}(\mu^o) \in f^{-1} (\tau)$.
Hence $\tau_1 = f^{-1}(\tau)$.

(4) Let $\lambda_1,\lambda_2, ... ,\lambda_n \in \wedge$, $\zeta_K \in S_{\lambda_K}$. Then
$f^{-1}[((\zeta_1 \vee ..... \vee \zeta_n)^q] = [f^{-1}(\zeta_1) \vee ..... \vee f^{-1} (\zeta_n)]^q$.
It follows from this that
$f^{-1}(\vee S_\lambda) = \vee f^{-1} (S_n)$

CONTINUITY:

Definition 1 :

Let (S,S') be fuzzy syntopogenous structures on X, Y respectively, and let f be a function from X to Y. Then f is said to be (S,S') - continuous if $f^{-1} (S')$ is coarser than S.

Proposition 4 :

Let (X,S_1), (Y, S_2) and (Z, S_3) be fuzzy syntopogenous spaces and let $f : X \to Y$, $g : Y \to Z$ be functions. Then :

(1) If f is $S_1, S_2)$ - continuous, then f is $(\tau(S_1), \tau (S_2))$ - continuous.

(2) If f is (S_1,S_2) - continuous and g is (S_2, S_3) - continuous, then g o f is (S_1,S_3) - continuous.

Proof :

(1) Since $f^{-1} (S_2)$ is coarser than S_1, we have $\tau(f^{-1} (S_2)) \subset \tau(S_1)$.
Since $\tau(f^{-1} (S_2)) = f^{-1}(\tau(S_2))$, the result follows.

(2) It follows from the equality $(gof)^{-1}(S_3) = f^{-1}(g^{-1}(S_3))$.

We have also the following propositions without proof.

Proposition 5:

Let $\{(Y_\lambda,S_\lambda): \lambda \in \wedge\}$ be a family of fuzzy syntopgenous spaces, X a set and, for each $\lambda \in \wedge$, $f_\lambda:X \to Y_\lambda$ a function. If $S = \vee_{\lambda\in\wedge} f_\lambda^{-1}(S_\lambda)$, then each f_λ is (S,S_λ)-continuous. Moreover, S is coarser than any fuzzy syntopogenous structure S' on X for which each f_λ is (S',S_λ) - continuous.

Propositian 6:

Let $(Y_\lambda,S_\lambda)_{\lambda\in\wedge}$, f_λ and S be as in the preceding proposition and let (Y,S') be a fuzzy syntopogenous space. Then a function $g:Y\to X$ is (S,S')- continuous iff each f_λ og is (S',S_λ) - continuous.

Proof :

The necessity follows from propositions (4) and (5). Conversely, let us assume that each f_λ og is (S',S_λ) - continuous and let $S'_\lambda= f_\lambda^{-1} (S_\lambda)$.
Then $g^{-1} (S'_\lambda) = (f_\lambda og)^{-1} (S_\lambda)$ is coarser than S'.
Since $g^{-1}(S) = \vee_{\lambda\in\wedge} g^{-1}(S'_\lambda)$,

we get that $g^{-1}(S)$ is coarser than S' and so g is (S',S) - continuous.

Definition 2:

Let Y be a subset of X and let ζ be a fuzzy semi-topogenous order on X. The restriction ζ / Y of ζ to Y is defined to be the fuzzy semi-topogenous order $f^{-1}(\zeta)$ on Y, where f: $Y \rightarrow X$ is the inclusion map. If S is a fuzzy syntopogenous structure on X, then $S/Y = \{\zeta/Y:\zeta\in S)$ is defined as the fuzzy syntopogenous structure induced by S on Y. If f : $Y \rightarrow X$ is the inclusion map, then $\tau(S/Y)=f^{-1}(\tau(S)) = \tau(S) / Y$.

Proposition 7 :

let (X, S), (Y,S') be fuzzy syntopogenous spaces and let f be a function from X to Y. Then f is (S,S')-continuous iff $f:X\rightarrow f(X)$ is $(S,S' / f(X))$ - continuous.

Proof:

The proof follow from proposition (6),

PRODUCT FUZZY SYNTOPOGENOUS SPACES

Definition 3 :

Let $(X_\lambda ,S_\lambda)_{\lambda\in\wedge}$ be a family of fuzzy syntopogenous spaces and let $X = \Pi_{\lambda\in\wedge} X_\lambda$. If π_λ denotes the cononical projection of X onto X_λ, then the fuzzy syntopogenous structure $\vee_{\lambda\in\wedge} \pi_x^{-1} (S_\lambda) = S$ is called the product of the family $(S_\lambda)_{X\pounds A}$ and it will be denoted by $\pi_{\lambda\in\wedge} S_\lambda$. The set X equipped with the product fuzzy syntopogenous structure is called the product of the family $(X_\lambda,S_\lambda)_{\lambda\in\wedge}$.

we have now the following theorem with the help of theorem (6).

Proposition. 8:

Let $(X_\lambda,S_\lambda)_{\lambda\in\wedge}$, X and $S = \Pi_{\lambda\in\wedge} S_\lambda$ be as in definition (3). Then : (1) the fuzzy topology $\tau(S)$ coincides with the product of the fuzzy topologies $\tau(S_\lambda)$, $\lambda \in \wedge$.

(2) If g is a function from a fuzzy syntopogenous space (Y,S') to X, then g is (S',S) -continuous iff each $\pi_\lambda o g$ is (S',S_λ) - continuous.

Proposition 9 :

Let $(X_\lambda,S_\lambda)_{\lambda\in\wedge}$ be a family of biperfect fuzzy syntopogenous spaces, $X = \Pi X_\lambda$ and $S = \Pi S_\lambda$. Then S^b is coarser than any biperfect fuzzy syntopogenous structure S' on X for which each π_λ is (S',S_λ) - continuous.

Moreover $\tau(S^b) = \tau(S) = \Pi\tau(S_\lambda)$.

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Chapter-4

Section-I

Fuzzy quasi-proxirnities

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Section-II

Fuzzy m-n syntopogenous space

Chapter - 4

Section - I of this chapter deals with fuzzy quasi-proximity corresponding to the fuzzy syntopogenous structure.

For a fuzzy quasi-proximity δ on X, it is always possible to construct a fuzzy proximity δ^* which is the coarsest of all fuzzy proximities which are finer than δ.

Section - I

FUZZY QUASI-PROXIMITIES

Definition. 1 :

A fuzzy quasi-proximity on X is a function $\delta : I^X \times I^X \rightarrow I$ satisfying the following axioms :

(1) $\delta(0,1) = \delta(1,0) = 0$

(2) $\delta(\mu_1 \vee \mu_2,\rho) = \delta(\mu_1,\rho) \vee \delta(\mu_2,\rho)$,
$\delta(\mu,\rho_1 \vee \rho_2) = \delta(\mu,\rho_1) \vee \delta(\mu,\rho_2)$,

(3) $\delta(\mu, \rho) \geq (\mu \wedge \rho)(x)$ for all $x \in X$.

(4) If $\delta(\mu,\rho) < \theta$, then there exists $A \subset X$ such that $\delta(\mu,A) \vee \delta(A^C,\rho) < \theta$.

(5) $|\delta(\mu,\rho) - \delta(\mu', \rho')| \leq \|\mu - \mu'\| + \|\rho - \rho'\|$

Now we have the following theorem :

Theorem 1 :

If δ is a fuzzy quasi-proximity on X and if $\zeta = \zeta_\delta$ is defined by

$\zeta(\mu,\rho) = 1 - \delta(\mu, 1-\rho)$

then $S_\delta = \{\zeta_\delta\}$ is a fuzzy topogenous structure on X. Moreover the mapping $\delta \rightarrow S_\delta$, from the collection of all fuzzy quasi-proximities on X to the collection of all fuzzy topogenous structures on X, is one-to-one and onto.

Now let δ be a fuzzy quasi-proximity on X and let S_δ be the corresponding fuzzy topogenous structure.

The mapping

$\mu \rightarrow \bar{\mu}$, $\bar{\mu}(x) = \delta(x,\mu) = 1 - \zeta_\delta(x, 1 - \mu)$

is a fuzzy closure operator on X,

such that

$\bar{\mu} = \bar{\mu}^{\tau(S_\delta)}$.

We will denote by $\tau(\delta)$ the corresponding fuzzy topology. Thus

$\tau(\delta) = \tau(S_\delta)$.

Given a fuzzy quasi-proximity δ on X, the function

$\delta^{-1}: I^X \times I^X \to I$, $\delta^{-1}(\mu,\rho) = \delta(\rho,\mu)$

is a fuzzy quasi-proximity on X,

This corresponds to the fuzzy topogenous structure

$S_\delta^C = \{\zeta_\delta^C\}$.

We say that a fuzzy quasi-proximity δ_1 is finer than another one δ_2 if $\delta_1 \le \delta_2$. In this case we also say that δ_2 is coarser than δ_1.

Theorem 2 :

Let $(\delta_\alpha)_{\alpha\in\wedge}$ be a family of fuzzy quasi-proxirnities on X and let

$\delta: I^X \times I^X \to I$,

$$\delta(\mu,\rho) = \inf \{ \bigvee_{i,j} \bigwedge_{\alpha\in\wedge} \delta_\alpha(\mu_i,\rho_j)\}$$

Where the infimum is taken over all finite families $(\mu_i),(\rho_j)$ of fuzzy sets with $\mu = \vee\, \mu_i$, $\rho = \vee\, \rho_j$. Then δ is the coarsest of all fuzzy quasi-proxomities on x which are finer than each δ_α . If each δ_α is a fuzzy proximity, so is δ. Moreover $\tau(\delta) = \text{Sup}\ \tau(\delta_\alpha)$.

Proof :

First we show that the coarsest fuzzy quasi-proximity on X finer than each δ_α exists. In fact, let $S_\alpha = (\zeta_\alpha), \zeta_\alpha(\mu,\rho) = 1 - \delta_\alpha(\mu, 1 - \rho)$.

Then S_α is a fuzzy topogenous structure by theorem (1).

Let $S = \vee_{\alpha\in\wedge} S_\alpha$ and $S^t = (\zeta_s)$.

We defme δ_o by $\delta_o(\mu,\rho) = 1 - \zeta_s(\mu, 1 - \rho)$.

Then, δ_o is a fuzzy quasi-proximity on X. Since S^t is finer than each S_α , it follows that δ_o is finer than each δ_α. On the other hand, if δ' is a fuzzy quasi-proximity on X finer than each δ_α and if $S' = \{\zeta_{\delta'}\}$ is the corresponding fuzzy topogenous structure, then S' is finer than S_α, which implies that S' is finer that S^t and hence δ_o is coarser than δ' . Thus δ_o is the coarset of all fuzzy quasi-proximities on X which are finer than each δ_α. If $(\mu_i),(\rho_j)$ are finite families with $\mu = \vee\, \mu_i$ and $\rho = \vee\, \rho_j$, then $\delta_o(\mu,\rho) = \bigvee_{i,j} \delta_o(\mu_i,\rho_j) \le \bigvee_{i,j} \bigwedge_{\alpha} \delta_\alpha(\mu_i,\rho_j)$ which implies that $\delta_o(\mu,\rho) \le \delta(\mu,\rho)$.

On the other hand, let $\theta > \delta_o(\mu,\rho)$.

Then $\zeta_s(\mu, 1 - \rho) > 1 - \theta$ and hence there are $\alpha_1, \ldots\ldots\ldots\ldots, \alpha_n \in A$ such that $(\zeta_{\alpha_1}, \vee \ldots\ldots \vee \zeta_{\alpha_n})^q (\mu, 1 - \rho) > 1 - \theta$ where, for $\alpha \in A$, $\zeta_\alpha(\sigma,\phi) = 1 - \delta_\alpha(\sigma, 1 - \phi)$.

There are finite families $(\mu_i),(\sigma_j)$ of fuzzy sets with $\mu = \vee\, \mu_i$, $1 - \rho = \wedge\, \sigma_j$ and, for each pair (i,j) there exists k, $1 \le k \le n$, such that $\zeta_{\alpha K}(\mu_i, \sigma_j) > 1 - \theta$ and so $\delta_{\alpha K}(\mu_i, 1 - \sigma_j) < \theta$. Setting $\rho_j = 1 - \sigma_j$, we have $\rho = \vee\, \rho_j$. Thus, for each pair (i j) we have $\wedge_\alpha \delta_\alpha(\mu_i,\rho_j) < \theta$ and so $\delta(\mu,\rho) \le \bigvee_{i,j} \bigwedge_{\alpha} \delta_\alpha(\mu_i,\rho_j) \le \theta$.

This proves that $\delta \le \delta_o$ and so $\delta = \delta_o$. If each δ_α is a fuzzy proximity, then $\delta(\mu,\rho) = \delta(\rho,\mu)$

and so δ is a fuzzy proximity. Finally,

$$\tau(\delta) = \tau(\delta_o) = \tau(S') = \tau(S)$$

$$= \operatorname{Sup}_{\alpha\in A} \tau(\delta_\alpha) = \operatorname{Sup}_{\alpha} \tau(\delta_\alpha)$$

Definition 2 :

If $(\delta_\alpha)_{\alpha\in A}$ is a family of fuzzy quasi-proximities (resp, fuzzy proximities) on X, then we will call the fuzzy quasi-proximity δ, (defined in previous theorem) the supremum of the family $(\delta_\alpha)_{\alpha\in A}$. We shall denote it by $\operatorname{Sup}_{\alpha} \delta_\alpha$.

For a fuzzy quasi-proximity δ on X, we will denote and δ^* the $\delta \vee \delta^{-1} = \operatorname{Sup}\{\delta,\delta^{-1}\}$. It is obvious that δ^* is a fuzzy proximity and it is the coarsest of all fuzzy proximities which are finer than δ.

We have the following theorem :

Theorem 3: Let μ,ρ be fuzzy sets in a fuzzy quasi-proximity space (X, δ).

Then,

$$\delta(\mu,\rho) = \delta(\bar{\mu}^{\tau(\delta^*)}, \bar{\rho}^{\tau(\delta^*)})$$

$$= \delta(\bar{\mu}^{\tau(\delta^{-1})}, \bar{\rho}^{\tau(\delta)}).$$

Proof:

Since $\sigma \le \bar{\sigma}^{\tau(\delta^*)} \le \bar{\sigma}^{\tau(\delta^{-1})} \wedge \bar{\sigma}^{\tau(\delta)}$ for each fuzzy set σ in X, it suffices to show that

$$\delta(\bar{\mu}^{\tau(\delta^{-1})}, \bar{\rho}^{\tau(\delta)}) \le \delta(\mu,\rho).$$

Let $\theta > \delta(\bar{\mu}, \rho)$. We choose $\in > o$ such that $\theta - \in > \delta(\mu,\rho)$.

There exists a subset A of X such that $\delta(\mu,A) \vee \delta(A^c,\rho) < \theta - \in$. If $x \notin A$, then

$\bar{\rho}^{\tau(\delta)}(x) = \delta(x,\rho) < \delta(A^c,\rho) < \theta - \in$ and so $\bar{\rho}^{\tau(\delta)} \le A \vee (\theta - \in)$.

Thus $\delta(\mu, \bar{\rho}^{\tau(\delta)}) \le \delta(\mu, A \vee (\delta - \in))$

$$= \delta(\mu,A) \vee \delta(\mu, \theta - \in) \le \theta - \in,$$

Since $\delta(\delta,A) < \theta - \in$ and $\delta(\mu,\theta-\in) = \delta(\mu,\theta - \in) - \delta(\mu,A) < \theta - \in$

Again, since $\delta(\mu, \bar{\rho}^{\tau(\delta)}) < \theta$, there exists a subset B of X such that

$\delta(\mu,B) \vee \delta(B^c, \bar{\rho}^{\tau(\delta)}) < \theta$. If $x\in B$, then

$$\bar{\mu}^{\tau(\delta^{-1})}(x) = \delta^{-1}(x,\mu) = \delta(\mu,x) \le \delta(\mu, B) < \theta.$$

Therefore $\bar{\mu}^{\tau(\delta^{-1})} \le \theta \vee B^c$

and so

$\delta(\bar{\mu}^{\tau(\delta^{-1})}, \bar{\rho}^{\tau(\delta)}) \le \delta(\theta \vee B^c, \bar{\rho}^{\tau(\delta)}) = \delta(\theta, \bar{\rho}^{\tau(\delta)}) \vee \delta(B^c, \bar{\rho}^{\tau(\delta)}) \le \theta.$

Thus for each $\theta>\delta(\mu,\rho)$, we have $\theta\geq\delta(\bar{\mu}^{\tau(\delta^{-1})},\bar{\rho}^{\tau(\delta)})$

& thus $\delta(\mu,\rho)\geq\delta(\bar{\mu}^{\tau(\delta^{-1})},\bar{\rho}^{\tau(\delta)})$ and this completes the proof.

INITIAL FUZZY QUASI-PROXIMITIES

***Definition 3* :**

A function f, from a fuzzy quasi-proximity space (X, δ) to another one (Y,δ′) , is said to be proximally continuous or a proximity mapping if $\delta(\mu,\rho)\leq\delta'(f(\mu), f(\rho))$ for all fuzzy sets μ , ρ in X.

Now let f be a function from X to Y and let δ be a fuzzy quasi-proximity on Y. Let $S_\delta = \{\zeta_\delta\}$ be the corresponding fuzzy topogenous structure. Let δ_1 be the fuzzy quasi-proximity on X corresponding to the fuzzy topogenous structure $S = f^{-1}(S_\delta)$. We will call δ_1 the inverse image of δ by f and will be denoted by $f^{-1}(\delta)$.

For μ, ρ fuzzy sets in X, we have

$$f^{-1}(\delta)(\mu,\rho) = 1-f^{-1}(\zeta_\delta)(\mu, 1-\rho)$$
$$= 1-\zeta_\delta(f(\mu), 1-f(\rho))$$
$$= \delta(f(\mu), f(\rho))$$

clearly $f^{-1}(\delta)$ in the coarsest fuzzy quasi-proximity δ′ on X for which f is (δ′,δ)-proximally continuous. Also it is clear that $f^{-1}(\delta)$ is a fuzzy pxoximity when δ is a fuzzy proximity.

Moreover,

$$\tau(f^{-1}(\delta)) = \tau(f^{-1}(S_\delta)) = f^{-1}(\tau(S_\delta))$$
$$= f^{-1}(\tau(\delta))$$

Thus we have the following theorem without proof :

Proposition.4 :

If f is a function from X to Y and if δ is a fuzzy quasi-proximity on Y, then the mapping

$f^{-1}(\delta):I^X\times I^X\to I$

$f^{-1}(\delta)(\mu,\rho) = \delta(f(\mu), f(\mu))$

is the coarsest of all fuzzy quasi-proximities δ′ on X for which f is (δ′,δ) -proximally continuous.

If δ in a fuzzy proximity, so is $f^{-1}(\delta)$.

Moreover $\tau(f^{-1}(\delta)) = f^{-1}(\tau(\delta))$

Proposition .5 :

Let $(X_\alpha ,\delta_\alpha)_{\delta\in A}$ be a family of fuzzy quasi-proximity spaces, X a set and, for each $\alpha \in A$, $f_\alpha:X\to X_\alpha$ a function.

We define $\delta:I^X \times I^X \to I$ by

$$\delta(\mu,\rho) = \inf (\bigvee_{i,j}\bigwedge_{\alpha\in A} \delta_\alpha(f_\alpha(\mu_i), f_\alpha(\rho_j)),$$

where the infimum is taken over all finite families $(\mu_i),(\rho_j)$ of fuzzy sets in X for which $\mu = \vee \mu_i$, $\rho = \vee \rho_j$. Then :

(1) δ is the coarsert of all fuzzy quasi-proximities δ′ on X for which each f_α

is (δ',δ_α) - proximally continuous.

(2) If each δ_α is a fuzzy proximity, then δ in a fuzzy proximity.

(3) A mapping f, from a fuzzy quasi-proximity space (Y,δ_1) to (X,δ) is proximally continuous iff f_α of is $(\delta_1, \delta_\alpha)$- proximally continuous.

(4) $\tau(\delta)$ coincides with the weakest fuzzy topology z on X for which each f_α is $(\tau,\tau(\delta_\alpha))$ - continuous.

Proof :

By proposition (4), the coarsest fuzzy quasi-proximity on X with respect to which each f_α is proximally continuous coincides with $\text{Sup}_\alpha f_\alpha^{-1}(\delta_\alpha)$. Thus (1) and (2) follows from theorem (2) and proposition (4).

(3) Let (Y,δ_1) be a fuzzy quasi-proximity space and let f be a function from Y to X. Iff is (δ_1,δ)-proximally continuous, then each f_α of is (δ_1,δ_α)- proximally continuous. Since $(f_\alpha \circ f)^{-1}(\delta_\alpha)= f^{-1}(f_\alpha^{-1}(\delta_\alpha))$ and δ is finer than $(f_\alpha^{-1}(\delta_\alpha)$. Conversely, let us suppose that each $f_\alpha \circ f$ is $(\delta_1, \delta_\alpha)$-proximally continuous. Let μ,ρ be fuzzy sets in Y and let $(\mu_i),(\rho_j)$ be finite families of fuzzy sets in X with $f(\mu) = \vee \mu_i$, $f(\rho) = \vee \rho_j$.

We have

$\mu \leq f^{-1}(f(\mu)) = \vee f^{-1}(\mu) = \mu_o$

$\rho \leq \vee f^{-1}(\rho_j) = \rho_o$

For each α we have,

$\delta_1(f^{-1}(\mu_i)\ f^{-1}(\rho_j))$

$\leq (f_\alpha \text{ o} f)^{-1}(\delta_\alpha)(f^{-1}(\mu_i),\ f^{-1}(\rho_j))$

$\leq \delta_\alpha(f_\alpha(\mu_i),\ f_\alpha(\rho_j))$

since $f(f^{-1}(\sigma)) \leq \sigma$ for each fuzzy set σ in X. Hence

$$\delta_1(\mu,\rho) \leq \delta_1(\mu_o,\rho_o) = \bigvee_{i,j} \delta_1(f^{-1}(\mu_i),f^{-1}(\rho_j))$$

$$\leq \bigvee_{i,j} \bigwedge_{\alpha} \delta_\alpha(f_\alpha(\mu_i),\ f_\alpha(\rho_j))$$

Therefore $\delta_1(\mu,\rho) \leq \delta(f(\mu),\ f(\rho))$

$= f^{-1}(\delta)(\mu,\rho)$

and so f is (δ_1, δ) - proximally continuous.

(4) The coarsest of all fuzzy topogies τ on X for which each f_α is $(\tau,\tau(\delta_\alpha))$-continuous coincides with $\vee_\alpha f_\alpha^{-1}(\tau(\delta_\alpha))$.

Now,

$$\tau(\delta) = \tau(\sup_\alpha f_\alpha^{-1}(\delta_\alpha)) = \sup_\alpha \tau(f_\alpha^{-1}(\delta_\alpha)$$

$$= \sup_\alpha f_\alpha^{-1}(\tau(\delta_\alpha))$$

PRODUCT OF FUZZY QUASI-PROXIMITY SPACES

Definition .4 :

Let $(X_\alpha , \delta_\alpha)_{\alpha\in A}$ be a family of fuzzy quasi-proxirnity spaces and let $X = \Pi_{\alpha\in A} X_\alpha$. Then, the product fuzzy quasi-proximity $\Pi_{\alpha\in A}\,\delta_\alpha$ on X is defined to be the coarsest of all fuzzy quasi-proximities on X for which each projection $\pi_\alpha : X \to X_\alpha$ is proximally continuous.

Using proposition 5, we therefore have :

Proposition .6:

Let $(X_\alpha,\delta_\alpha)_{\alpha\in A}$ and X be defined in (4),

If $\delta = \Pi_{\alpha\in A}\,\delta_\alpha$, then

(1) $\delta(\mu,\rho) = \inf\,(\vee_{i,j}\ \wedge_\alpha \delta_\alpha\,(\pi_\alpha(\mu_i),\pi_\alpha(\rho_j)))$ where the infimum is taken over all finite families (μ_i) , (ρ_j) of fuzzy sets in X with $\mu = \vee\ \mu_i, \rho = \vee\ \rho_j$

(2) If each δ_α is a fuzzy proximity, then δ is a fuzzy proximity.

(3) A function f, from a fuzzy quasi-proximity space (Y,δ_1) to (X,δ) is proximally continuous iff each $\pi_\alpha \circ f : (Y,\delta_1) \to (X_\alpha ,\delta_\alpha)$ is proximally continuous.

(4) $\tau(\delta) = \Pi_\alpha\ \tau(\delta_\alpha)$

Section - II

Unified theory of spatial structures have already been studied by Csā szā r, Doicinov and Lal & Lal.

This section is concerned with three new concepts enriching classical fuzzy spatial structures based on fuzzy n-metroid lattice and semi n-umformity.

A : Fuzzy m-n Syntopogenous space

B : n-Uniform space

C : m-n Proximity space

A : Fuzzy m-n Syntopogenous Space :

1. With the introduction of a fuzzy m-n Syntopogenous structure on a set P in terms of m-n tuple of fuzzy set relations coarser than super fuzzy set relation, a new approach to a Syntopogenous structure can be established.

A fuzzy symmetrical m-n topogenous structure characterises a fuzzy m-n proximity $\delta_{m,n}$ generalising fuzzy proximity. A perfect fuzzy n-topogenous structure characterizes fuzzy n-ary closure, n-ary interior and fuzzy n neighbourhood (nhd) structure, while a biperfect fuzzy n syntopogenous structure characterizes fuzzy n-uniforrnity generalising Huttonian fuzzy uniformity. Throughout, fuzzy sets will be denoted by small Roman letters excepting pointic (elemental) letters p,q,r an m,n tuple of which by , $< a_i; b_j > i \in m, j \in n$ and in particular an n tuple $< b_j >$ in which jth component is replaced by a fixed x by $< b_j \to x >$, real numbers by small Greek letters.

Let P be a set, elements of which be called points denoted by the letters p, q, r. Then a mapping $x : P \to I$ is called a fuzzy set in P with x(p) as x-ness of p, which is a fuzzy point if it has a singleton support p of strength (say α), denoted by α p or by p^{α}. The set of Fuzzy subsets IP is a pseudo complemented lattice.

2-Fuzzy m-n Topogenous Ordering

Definition 2.1

A mapping $>>_{m,n} : IP^m \times IP^n \rightarrow (0,1)$ is said to be an m,n a fuzzy semi topogenous ordering provided :

1. $a_i = b_j = 0$, or $= 1$ for each $i \in m$, $j \in n = < a_i >>_{m,n} b_j >$;
2. $< a_1 >>_{m,n} b_j > \Rightarrow a_i \geq b_j$ each i,j
3. $a_i > b_j$, $b_j >>_{m,n} c_j > d_j \Rightarrow >< a_i >>_{m,n} d_j >$;

   The dual << m,n and complement $>>^c$ of which are defined by :
4. $< a_i <<_{m,n} b_j > \Leftrightarrow < 1 - a_{m-1-i} >>_{m,n} 1 - b_{n-1-j}$
5. $< a_1 >>^c b_j, > \Leftrightarrow < 1 - b_{n-1-j} >>_{n,m} > 1 - a_{m-1-i} >$

Remarks 2.2 : $< a_i >>_{m,n} b_j >$ can be expanded to $< a_i >>_{m,n} a_i >$ by inserting universal element 1 in the terminal n - m terms of $< a_i >$ if $m < n$ and null element 0 in the terminal m-n terms of $< b_j >$ if $m > n$.

Theorem 2.3 :

(a) $< a_i <<^c_{m,n} b_j > \Leftrightarrow < b_j >>_{n,m} a_i >$

(b) $<<_{m,n}$ satisfies properties dual to those of $>>_{m,n}$

(c) Each of $>>_{m,n}$ and $<<_{m,n}$ is transitive

Proof I. $< a_i <<^c_{m,n} b_j > \rightarrow < 1 - b_{n-1-j} <<_{n,m} 1 - a_{m-1-i} > \Leftrightarrow < b_j >>_{n,m} a_i >$

II $< a_i >>_{m,n} b_j >$, $< b_j >>_{n,n} c_j > \Rightarrow a_i > b_j$

$< b_j >>_{n,n} c_j >$, $c_j \geq c_j$, $j \in n \Rightarrow < a_i >>_{m,n} c_j >$

Definition 2.4 A fuzzy semi topogenous ordering $>>_{m,n}$ is said to be :

topogenous iff it is invariant w.r.t. $\wedge$ & $\vee$:

perfect iff it is invariant w.r.t. arbitrary $\wedge$ only :

biperfect iff it is invariant w.r.t. arbitrary $\wedge$ & $\vee$

Remarks 2.5 $<<_{m,n}$ is perfect it is invariant w.r.t. arbitrary V which is biperfect if $<<^c_{m,n}$ is also perfect.

3. Fuzzy m-n Syntopogenous structures :

Definition 3.1

A non empty set S = { $>>_{m,n}$ } of fuzzy topogenous orderings on a set P is called an m-n fuzzy syntopogenous structure (fst) provided S is updirected w.r.t the finer relation with an interpolation property ;

$< a_i >>_{m,n} b_j > \Rightarrow \exists\ x \in I^p$ and $>>'_{m,n} \in S$ such that $>>_{m+1,n}$

is finer than $>>_{m,n}$ i.e. $< a_i >>_{m,n} b_j > \Rightarrow \exists\ x \in I^p, >>' \in S$

Such that $< a_i >>'_{m,m} < a_i \to x > >>'_{m,n} b_j >, i \in m, j \in n$

Theorem 3.2 :

$S^d = \{<<_{m,n}\}$ of fuzzy dual topogenous orderings is a fst iff it is updirected with an interpolation and conversely.

Proof. I. $<<'_{m,n} <<''_{m,n} \in S^d$ and $< a_i <<'_{m,n} b_j >, < a_i <<''_{m,n} b_j >$

$\Rightarrow < 1 - q_m - 1 - i >>'_{m,n} 1 - b_{n-1-j}$

and $< 1 - a_{m-1-i} >>'_{m,n} 1 - b_{n-1-j} > \Rightarrow \exists >> \in S$ such that

$< 1 - a_{m-1-i} >>_{m,n} 1 - b_{n-1-j} > \Rightarrow < a_i <<_{m,n} b_j >$

II. $< a_i << b_j > \Rightarrow < 1 - q_{m-1-i} >>_{m,n} 1 - b_{n-1-j} > \Rightarrow \exists >>'_{m,n} \in S$ &

$1 - x \in I^{p0}$ such that $< 1 - a_{m-1-i} >>_{m,m} >> l - a_{m-1-i} \to$

$1 - x >>_{m,n} 1 - b_{n-1-j}$ for $i \in m, \Rightarrow \exists << \in S^d\ \exists\ x \in I^{p0}$

such that $< a_i <<_{m,n}, < a_i \to x <<_{m,n} b_j >, i \in m$

Definition 3.3 A set P with an m-n fst S is called an m-n fst space which is :

topogenous iff S consists of a singleton topogenous ordering ;

perfect (biperfect) iff every member of S is perfect (biperfect)

Definition 3.4 S_1 is finer than S_2 iff for every $>>_{m,n}$ in S_2

there exist a member of S_1 finer than $>>_{m,n}$, which is equivalent

provided S_2 is also finer than S_1.

To every fuzzy 1,n st. on P, there corresponds a fuzzy n-ary closure

(-) and n-ary interior (0) operation defined by

Definition 3.5 $\overline{< a_i >} = \wedge\ b : < b >>_{1,n} a_i >$

$< a_i >^0 = \vee\ b : < b <<_{i,n} a_i >$

Theorem 3.5(0): $\overline{I^{p^n}} \to I^{p^0}$; for which $< x_i >^0$ is contained in each x_i; $()^0$ is order preserving and distributed over $\wedge$ and is indempotent with

$\overline{< x_i >} = 1 - < 1 - x_{n-1-i} >^0$ having the dual properties.

Proof. I. $y < < x_i >^0 \Rightarrow < y < <_{1,n} x_i > \Rightarrow y <$ each x_i

Hence $< x_i >^0$ is contained in each x_i.

II $y << a_i >^0 \wedge < b_i >^0 \Rightarrow < y <<_{1,n} a_i >$

& $< y <<_{1,n} b_i > \Rightarrow < y <<_{1,n} a_i \wedge b_i >$

$\Rightarrow < y < (a_i \wedge b_i)^0 >$

Hence $<< a_i >^0 \wedge < b_i >^0 < a_i \wedge b_i >^0 >$

which with order preservation of (0)

$\Rightarrow$ equality of both sides.

III $< b(a_i)^0 > \Rightarrow < b <<_{1,n} a_i >$

$\Rightarrow \exists$ cs. t. $< b <<_{1,1} c <<_{1,n} a_i >$

$\Rightarrow < b < c^0 < c < a_i^0 >$
$\Rightarrow < b < c^{00} < c^0 < a_i^{00} >$
$\Rightarrow < a_i^0 < a_i^{00} >$

IV $< b > \bar{a}_i > \Rightarrow < b >>_{1,n} a_i >$
$\Rightarrow < 1 - b <<_{1,n} 1\text{-}a_{n-1-i} >$
$\Leftrightarrow < 1 - b < (1\text{-}a_{n-1-i})^0 >$
$\Leftrightarrow b > 1 - < 1 - a_{n-1-i} >^0$
Hence $< \bar{a}_i > = 1 - < 1\text{-}a_{n-1-i}>^0$

Remark - A f-n interior, whence n-closure operation does not define a fuzzy n-syntopogenous structure.

(B) n- Uniform Space

0 INTRODUCTION

Introduction of an n-uniformity with its n-neighbourhood (nhd); n-ary closure and n-ary Interior operations, m,n uniform proximity $P_{m,n}$, leads to an extension of the notions of classical spatial structures on a set.

1. The structure of an n-uniformity is based on some basic properties of an (n+1)- ary relation on a set P.

Let $< R_i \mid i \in n+1 >$ be the set of (n+1), ary relation on a set P.

Then $<p_i> \leftarrow 0^{n+1} < R_i > \Leftrightarrow \exists\, p \in P$ s.t $< p_i \to p > \in R_i \mid i \in n + 1$ is the composition of the set R_i of (n+l)-ary relations, where $<p_i \to p>$ is an (n+1) tuple in which p_i th term is replaced by an element p.

Remarks

An m,n tuple $<<a_i>; <b_j>>$ $i \in m$, $j \in n$ is shortly denoted by
$<a_i ; b_j > \mid i \in m; j \in n$.

Definition 1.1:

The image of an n tuple $<A_i>$ $i \in n$ of sets under an (n+l)-ary relation R is defined by setting:
$R<A_i> = \{p \mid <p_i>\in <A_i> \mid <p_i ; p > \in R\}$
which can be extended under the composition $O^{n+1} <R_j>$, $j \in n+1$.

Theorem 1.2:

$P_n \in O^{n+1} <R_j> <A_i>$, $j \in n+1$, $i \in n \Leftrightarrow \exists\, p \in R_n <A_i>$ s.t. $P_n \in R_i <p_i \to p >$
for each $i \in n$, $j \in n+1$.

Proof:

$P_n \in O^{n+1} <R_j> <A_i> \Rightarrow \exists < p_i > \in <A_i>$ s.t.$<p_i;$

$p_n > = < p_j > \in O^{n+1} < R_j >$, $i \in n$, $j \in n+1 \Rightarrow \exists\, p \in P$ s.t.
$< p_j \to P > \in R_j$ for $j \in n+1 \Rightarrow p_n \in R_i < p_i \to p >$ for $i \in n$.

2. Quasi n-Uniformity

Definition 2.1:

A quasi (Q) n-uniformity on a set P is a set $v \subset p^{n+l}$ of (n+l)-ary relation on P satisfying the properties :

V1. $\Delta \subset U \in v$, where Δ is the set of n+1 tuples of the form $<x_i;$ $x_m > i \in n$; and at least one of the $x_i = x_m$.

V2. v is closed w.r.t. $\cap$ & super relation;

V3. For each $U \in v$; $\exists V \in v$ with $V^{n+1} \subset U$, where
$\langle p_i \rangle \in V^{n+l} \Leftrightarrow \exists p \in P$s.t. $< p_i \to p > \in V$ for each $i \in n+1$;
which is symmetric provided:
$< P_i > \in U \Rightarrow \sigma < p_i > \in U$ where $\sigma < p_i >$ is a permutation of $\langle p_i \rangle$
for each $U \in V$.

A set P with a Q n-unifonnity v is called a Q n-uniform space $<p, v>$.
A mapping $f : < p_1, v_1, > \to < p_2, v_2 >$ is said to be uniform provided
for every $V_2 \in v_2 \ni V_1 \in v_1$ for which $< p_i > \in V_1 \Rightarrow < f\, p_i > \in V_2$.
We have now the following obvious theorem :

Theorem 2.2 :

Let g: $< P_2, V_2 > \to < P_3, v_3 >$ be another uniform mappings then so is
$g \circ f : < p_1, v_1 > \to < p_3, v_3 >$.
In a Q n-uniform space, n-spatial structures are to be introduced.

Definition 2.3:

An n-nhd of an a tuple $<p_i>$ denoted by $U <p_i> = \{p \mid < p_i; p > \in U\}$.

Theorem 2.4:

The set $N < p_i > = \{U < p_i > \mid U \in v\}$ is an nhd filter with an interpolation property :

N1. $U < p_i >$ contains each of the point p_i;

N2. $N <p_i>$ is closed w.r.t. $\cap$ & super set relation;

N3. U be a nhd $<p_i > \Rightarrow \exists\, V \in N<p_i>$ s.t
whenever $p \in V <p_i>$, U is a nhd of each n tuple $< p_i \to p >$ for $i \in n$.

Proof :

It suffices to prove an interpolation property N3 only.
Let U be a nhd of $<p_i>$ and $p_n \in U < p_i >$ for $i \in n$.
Then $<p_i; p_n >=<p_j > \in U$ for $i \in n$, $j \in n + 1 \Rightarrow \exists$ Vs.t. $V^{n+1} \subset U$.
Hence there exists an
$p \in P$ s.t.$< p_j \to p > \in V$ for each $j \in n +1 \Rightarrow < p_j > \in U$.
Whence $p_n \in V^{n+l} < p_i >$ and $p \in V < p_i >$ for

$i \in n \Rightarrow p_n \in V < p_i \rightarrow p > \Rightarrow p_n \in U < p_i >$ for such $i \in n$.
Clearly $V < p_i \rightarrow p > \subset U < p_i >$ for each n tuple $< p_i \rightarrow p >$,
whence $U<p_i>$ is nhd of each n tuple containing p.

In a Q n-uniform space, the notion of an n ary closure (-) and an n-ary interior $(\ )^0$ operations are introduced:

Definition 2.5

$<A_i>^0 = \{p|\exists\ U \in V \text{ s.t. } U(p) \subset \cup A_i,\ i \in n\}$

$< A_i > = \{p|U(p) \cap A_i \neq \phi$ for each $U \in V$ and A_i in $< A_i >$

Theorem 2.6

(a) Each of $(\ )^0$ & (–) is an n-ary set operation satisfying the properties :

I1. $<A_i>^0 \subset UA_i$

I2. $<A_i>^0 \cap < B_i >^0 = <A_i \cap B_i$

I3. $<A_j>^{00} = <A_i>^0$

(b) $(^-)$ has dual properties with $< \overline{A} > = 1 - < 1 - A_i >^0$.

Proof:

If suffices to prove idempotency of each of the operations

I. $p \in <A_i>^0 \Rightarrow \exists\ U \in v$ s.t $U(p) \subset \cup A_i \Rightarrow \exists\ V \in v$ s.t $VV(p) \subset$
$U(p) \subset UA_{ij}\ q \in V(p) \Rightarrow V(q) \subset VV(p) \subset U(p) \subset U\ A_i \Rightarrow q \in < A_i>^0$
Hence $V(p) \subset < A_i >^0 \Rightarrow p \in < A_i >^{00}$

II. $p \in < \overline{A}_i > \Rightarrow \exists\ U \in V, \exists A_i$ in $<A_i>$ s.t. $U(p) \cap A_i$
$= \phi \Rightarrow \exists\ V \in v$ s.t. $VV(p) \subset 1 - A_i$.

Hence $q \in V(p) \Rightarrow V(q) \subset 1 - A_i \Rightarrow q \not\subset < \overline{A}_i > \Rightarrow V(p) \cap < \overline{A}_i > = \phi \Rightarrow p \not\subset < \overline{\overline{A}}_i >$

3. Uniform m.n. Proximity :

An m uniformity on a set P defines an m proximity P_m, which can be extended to an m,n proximity $P_{m,n}$.

Definition 3.1

$U < A_i > = \{p|\exists < p_i > \in < A_i >$ s.t. $< p_j; p > \in U,\ i \in m$ is called m- adic set nhd of $<A_i>$.

Theorem 3.2

The set V of m-adic set nhds on P satisfying the properties listed below is called a quasi m proximity P_m;

P1. $U<A_i>$ contains each A_i;

P2. v is closed w.r.t. $\cap$ and super set relation;

P3. $U \in v \Rightarrow \exists\ V \in v$ for which $V^{n=1} \subset U$;

Proof

It suffices to prove P_3 only i.e. $V^{n+1} <A_i> \subset U < A_i >$.

Let $p_n \in U < A_{i;} > \Rightarrow \exists <p_i> \in <A_i>$ s.t. $< p_i;p_n > = < p_j >$, $i \in n$. $j \in n + 1 \in U$

$\Rightarrow \exists\ V \in v$ s.t. $< p_j > \in V^{n+1} \Rightarrow < p_j > \in U \Rightarrow \exists\ p \in P$ s.t. $< p_j \to p > \in V$

$\Rightarrow <p_i> \in U$ for $j \in n+1$.

Hence $p \in V < A_i >$, $P_n \in V < A_i \to p > \Rightarrow p_n \in U < A_i >$ for $i \in n$ & V

Definition 3.3

$<A_i,B_j> \in P_{m,n} \Leftrightarrow U<A_i> \cap B_j \neq \phi$ for every $U \in V$ & B_j in $< B_j >$.

Theorem 3.4

A quasi proximity $P_{m,n}$ satisfies the properties :

P1. $<A_i;B_j> \in P_{m,n} \Rightarrow$ each of A_i & $B_j \neq \phi$;

P2. $<A_i;B_j> \notin P_{m,n} \Leftarrow \exists\ A_i$ in $<A_i>$,

$\exists B_j$ in $< B_j >$ & $A_i = B_j$ in $< B_j >$ in $A_i = B_j = \phi$,

P3. $< A_i;\ B_j\ U\ C_j > \in P_{m,n} \Rightarrow < A_i;\ B_j > \in P_{m,n}$ or, $< A_i,\ C_j > \in P_{m,n}$;

P4. $<A_i;B_j> \notin P_{m,n} \Rightarrow \exists$ disjoint sets C & D st $< A_i;c > \notin P_{m,n}$ &

$< Ai \to D;\ B_j > \notin P_{m,n}$

Proof : It suffices to prove P_4 only.

$< A_iB_j > \notin P_{m,n} \Rightarrow \exists\ U \in v \exists B_j$ in $< B_j >$ st $U < A_i > \subset p - B_j \Rightarrow \exists\ V$

$\in v$st. $V^{n+l} < A_i > \subset U < A_i > \subset p - B_j$.

$V < A_2 > \subset U < A_i > \Rightarrow <A_i;\ P\text{-}U<A_i> \notin P_{m,l}$

Let $C=P-U<A_i>$ and $D \subset <A_i>$.

Then $V < A_i \to D > \subset P - B_j$ for each m-tuple $< A_i \to D >$ and

$C \cap D = \phi \Rightarrow < A_i \to D\text{: } B_j > \notin P_{m,n}$.

Definition 3.5

The conjugate U* of U in a Q n-uniform space in defined by setting :

$U^* < A_i > \cap B_j = \phi$ for some B_j in $<B_j> \Rightarrow U < B_j > \cap A_i = \phi$

for some A_i in $<A_i>$.

Theorem 3.6

The conjugate $P^*_{m,n} = P_{n,m}$.

Proof :

$< A_i;\ B_j > \notin P^*_{m,n} \Rightarrow \exists\ U^* \in V^*$, $\exists\ B_j$ in $< B_j >$ s.t. $U^* < A_i > \cap B_j = \phi \Rightarrow U < B_j > \cap A_i$ $= \phi$ for some A_i in $< A_i > \Rightarrow < B_j;\ A_i > \notin P_{n,m}$.

Remarks 3.7

An 1 uniformity called simply a uniformity defines 1-1 proximity P which is extensible to P_{ln} for which $< \overline{A_i} >= \{p | < p; A_i > \in P_{l,m}\}$ is called proximal n-ary closure.

(C) m-n Proximity Space

Introduction of an m,n proximity $\delta_{m,n}$, its m,n ordoform <<m,n, characterization with an n-ary proximal closure and interior operation enriches classical spatial structures. For clarity, an m,n tuple of subsets of a set P is denoted by $<<A_i>; <B_j>>$ or shortly $<A_i;B_j>>$ $i \in m$, $j \in n$ and in particular an n tuple $<B_j>$ in which j th term B, is replaced by fixed C is denoted by $< B_j \to C >$.

1. Quasi Semi m,n Proximity :

Definition 1.1

An m,n set relation $\delta_{m,n} \subset p^m \times p^n$ in a set P is called a quasi semi (qs) m-n proximity provided:

$\delta 1.$ $<A_i; B_j> \in \delta_{m,n} \Rightarrow$ each of A_i in $<A_i>$ and B_j in $<B_j>$ is non void;

$\delta 2.$ $<A_i; B_j> \notin \delta_{m,n} \Rightarrow \exists A_i$ in $< A_i > \exists B_j$ in $<B_j>$ either of which is void ;

$\delta 3.$ $<A_i; B_j \text{ U } C_j > \in \delta_{m,n} \Leftrightarrow$

$<A_i; B_j> \in \delta_{m,n}$ or $<A_i; C_j > \in \delta_{m,n}$

$< A_i \bigcup B_i ; C_j > \in \delta_{m,n} \Leftrightarrow < A_i; C_j > \notin$

$\in \delta_{m,n}$ or $< B_j; C_j > \in \delta_{m,n}$

(both sided distribution over U)

Theorem 1.2

$\delta_{m,n}$ is both sided order preserving i.e.

$<A_i; B_j> \in \delta_{m,n}$ with $B_j \subset C_j \Rightarrow < A_i ; C_j > \in \delta_{m,n}$

Proof follows from $\delta 3$.

Definition 1.3

A set P with a Qs $\delta_{m,n}$ is said to be an Qs m-n proximity space, $< P, \delta_{m,n} >$ in which congugate $\delta^*_{m,n}$ of $\delta_{m,n}$ is defined by :

$<A_i, B_j > \in \delta^*_{m,n} \Rightarrow <B_j; A_i> \in \delta_{n,m}$

A qs $\delta_{n,m}$ is a semi proximity iff $\delta^*_{m,n} = \delta_{n,m}$.

Remarks 1.4

A semi $\delta_{1,n}$ and $\delta_{1,1}$, are simply denoted by δ_n and δ respectively.

Definition 1.5

A mapping $f:< P_i\, \delta'_{m,n} > \to < Q, \delta''_{m,n} >$

is said to be an m,n proximal mapping provided $<A_i, B_j> \in \delta'_{m,n} \Rightarrow < f A_i , fB_j > \in \delta''_{m,n}$.

Theorem 1.6

Let $g: < Q, \delta'''_{m,n} > \to < R, \delta'''_{m,n} >$ be another proximal, than so is

g of $: < P, \delta'_{m,n} > \to < R, \delta'''_{m,n} >$.

Proof follows from 1.5

2 Ludato $\delta_{m,n}$

Definition 2.1

A Q semi proximity $\delta_{m,n}$ is a Ludato (L) Q proximity

iff $\delta L < A_i ; B_j > \in \delta_{m,n}$ and for each $x \in \bigcap B$,

$< B_j \to x; C_j > \in \delta_{n,n} \Rightarrow < A_i; C_j > \in \delta_{m,n}$

where $< B_j \to x >$ is an n tuple $<B_j>$ in which B_jth term is replaced by x.

In a QL proximity space $< P, \delta_n >$ the notion of an n-ary closure operaton is introduced;

Definition 2.2

$$< \overline{A_i} > = \{x1<x; A_i > \in \delta_m$$

$$< A_i >^0 = 1 - < \overline{1 - A_i} >.$$

Theorem 2.3

The n-ary operation (-) & ( )^0 satisfy the properties.

C1. $\exists A_i = \phi$ in$< A_i > \Rightarrow < \overline{A_i} > = \phi$

C2. $\overline{A_i \bigcup B_i} = \overline{A_i} \cup \overline{B_i}$

C3. $< A_i > = < \overline{A_i} >$

I$_1$. $A_j = P$ for each $j \in n \Rightarrow < A_j >^0 \ 1$

I$_2$. $<A_j \bigcap B_j >^0 = <A_j >^0 \bigcap <B_j>^0$

I$_3$. $<A_i>^{00} = <A_i>^{00}$

Proof:

It suffices to prove idempotency of the operatons

I $\quad \times \in < \overline{\overline{A_i}} > \Rightarrow < x, \overline{A_i} > \in \delta_{1,n}$

Also for every $y \in < \overline{A_i} >, < y, A_i > \in \delta_n$

Hence $<x, < \overline{A_i} >> \in \delta_{11}$ and $<y, A_i > \in \delta_n$

for every $y \in < \overline{A_i} > \Rightarrow <x, A_i > \in \delta_n$

$\Rightarrow x \in < \overline{A_i} >$

Hence $< \overline{\overline{A_i}} > \subset < \overline{A_i} >$.

II $\quad < A_i >^{00} = <P - < \overline{P - A_i} >>^0$

$$= P - [< \overline{\overline{P - A_i}} > = < A_i >^0$$

Theorem 2.4

A proximal mapping is continuous w.r.t. induced closures for which

$f < \overline{A_i} > \subset < \overline{fA_i} >$.

Proof:

$x \in <\overline{A_i}> \Rightarrow <x, A_i> \in \delta_{1,n}$ which with proximality of f $\Rightarrow$

$< fx, fA_i > \in \delta_{1,n}$

$\rightarrow fx \in (\overline{fA_i})$

Hence $f < \overline{A_i} > \subset < f\overline{A_i} >$

3EF $\delta_{m,n}$.

Definition 3.1

An S. $\delta_{m,n}$ is an EF proximity provided

$< A_i ; B_j > \notin \delta_{m,n} \Rightarrow$ there exists a set of $C \subset P$ s.t.

$<A_i ; P - < A_i \rightarrow c > \notin \delta_{m,1}$ &

$< A_i \rightarrow C; Bj > \notin \delta_{m,n}$

A quasi EF $\delta_{m,n}$ defines an m,n ordoformity $<<_{m,n}$ as well as its dual $>>_{m,n}$ by setting :

Definition 3.2

$< A_i <<_{m,n} B_j \Leftrightarrow < A_i; P - B_j > \notin \delta_{m,n}$

$< A_i >>_{m,n} B_j > \Leftrightarrow < P - A_i <<_{m,n} P - B_j >$

$\Leftrightarrow < (P - A_i); B_j > \notin \delta_{m,n}$

Theorem 3.3

(a) The relation $>>_{m,n}$ satisfies the properties called an m,n ordoformity.

01. $< A_i <<_{m,n} B_j > \Rightarrow A_i \subset B_j$ for each $i \in m$, $j \in n$

02. $A_i \subset B_i$, $B_i <<_{m,n} C_j, C_j \subset D$, for $i \in m$, $j \in n \Rightarrow < A_i <<_{m,n} D_j >$

03. $A_i <_{m,n} B_j >, < A_i <<_{m,n} C_j > \Rightarrow < A_i \bigcap C_j >$

04. $< A_i <<_{m,n} B_j > \Rightarrow \exists$ C s. t. $< A_i <<_{m,n} << A_i \rightarrow C > <<_{m,n} B_j >$

(b) $>>_{m,n}$ satisfies dual properties.

Proof :

I. $<A_i <<_{m,n} B_j > \Rightarrow <A_i; P\text{-}B_j > \notin \delta_{m,n}$

$\Rightarrow A_i \bigcap P - B_j = \phi \Rightarrow A_i \subset B_j$ for $i \in m$, $j \in n$

II. $< B_i <<_{m,n} C_j > \Rightarrow < B_i \ \bar{\delta}_{m,n} ; P - C_j >$

which with order preservation of $\delta_{m,n} \Rightarrow$

$< A_i , P - D_j > \notin \delta_{m,n} \Rightarrow < A_i <<_{m,n} D_j >$

III. $<A_i, P - B_j > \notin \delta_{m,n}$, $<A_i, P - C_j > \notin \delta_{m,n} \Rightarrow < A_i << B_i \bigcap C_j >$

IV. $< A_i <<_{m,n} B_j > \Rightarrow <A_i , P - B_j > \notin \delta_{m,n} \Rightarrow \exists$ C s.t. $< A_i; P - < A_i \rightarrow C> \notin \delta_{m,n}$

and $<A_i \rightarrow C, P - B_j > \notin \delta_{m,n} \Rightarrow < A_i <<_{m,n} < A_i \rightarrow C <<_{m,n} B_j >$

Theorem 3.4

A quasi EF $\delta_{m,n}$ is m,n ordoformisable as well as dual ordoformisable.

Proof I

$\delta_{m,n} \Rightarrow <<_{m,n}$ by 3.3

It suffices to prove $<<_{m,n} \Rightarrow \delta_{m,n}$.

$< A_i \bar{\delta}_{m,n} B_j > \Rightarrow < A_i << P - B_j > \Rightarrow < A_i \subset P - B_j > = A_i \cap B_j = \phi$

for $i \in m, j \in n$

$< A_i \bar{\delta}_{m,n} B_j >, < A_j \bar{\delta}_{m,n} C_j > \Rightarrow < A_i << P - B_j \cup C_j >> \Rightarrow$

$<A_i \bar{\delta}_{m,n} B_j \cup C_j >$

$<A_i \bar{\delta}_{m,n} B_j > \Rightarrow <A_i << P - B_j > \Rightarrow \exists C \subset P$ s.t.

$< A_i <<_{m,m} < A_i \rightarrow C <<_{m,n} P - B_j > \Rightarrow < A_i \bar{\delta}_{m,n} P - < A_i \rightarrow C >>$ &

$< A_i \rightarrow C \bar{\delta}_{m,n} B_j >$

An ordoformity $<<_{m,n}$ defines an n-ary closure and an n-ary interior by setting *:*

Theorem 3.5

$<A_i>^0 = \bigcup B\{B <<_{in} A_i>\}$

$< \overline{A_i} = \bigcap B \{< B >> A_i >\}$

$= \bigcap B \{< A_i <<_{n,1} B >\}$ for symmetric δ

Proof

I. $B \subset < A_i >^0 \Rightarrow < B << A_i > \Rightarrow \exists C$ s.t. $< B << C << A_i > \Rightarrow B \subset C^0$ and

$C \subset < A_i >^0 \Rightarrow B \subset C^0 \subset < A_i >^{00}$

Hence $<A_i>^{00} \subset < A_i>^0$

II. $B \subset < \overline{A_i} > \Rightarrow < B >> A_i > \Rightarrow \exists D \subset P$

$\Rightarrow < B >>_{1,1} D >>_{1,n} A_i > \Rightarrow B \supset \overline{D}$

$< D \supset \overline{A_i} > \Rightarrow < B \supset \overline{D}, D \supset \overline{\overline{A_i}} >$

$\Rightarrow > \overline{\overline{A_i}} > \subset < \overline{A_i} >$

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