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On Pairwise Minimal Continuous Maps In Bitopological Spaces

BASAVARAJ M. ITTANAGI

Department of Mathematics, Siddaganga Institute of Technology, Affiliated to VTU, Belagavi, Tumakuru-572 103, Karnataka State, (India)
Corresponding author Email: dr.basavarajsit@gmail.com
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Abstract

The aim of this paper is to define and study of new class of maps called pairwise minimal continuous, pairwise maximal continuous, pairwise minimal irresolute and pairwise maximal irresolute maps in bitopological spaces and investigate the relations between these kinds of continuity.

Keywords and Phrases: Pairwise minimal open sets, pairwise maximal open sets.
2010 Mathematics classification: 54E55

1. Introduction

In the years 2001 and 2003, F. Nakaoka and N. Oda[5,6,7] in- troduced and studied minimal open (resp. minimal closed) sets and maximal open (resp. maximal closed) sets, which are subclasses of open (resp. closed) sets. The complements of minimal open sets and maximal open sets are called maximal closed sets and minimal closed sets respectively. Also in the years 2011 and 2012 [1,2], S. S. Benchalli, Basavaraj M. Ittanagi and R. S. Wali, introduced and studied minimal open sets and maps in topological spaces and pairwise minimal open and pairwise maximal open sets in bitopological spaces.

J. C. Kelly[3], in the year 1963, first initiated the concept of bitopological spaces. He defined a bitopological space (X, τ_1, τ_2) to be a set X equipped with two topologies τ_1 and τ_2 on X and initiated the systematic study of bitopological space. He extended the notions of separation axioms of single topological space to bitopological space. Also in the year 1999, Maki, Sundaram and Balachandran[4] have introduced the concept of $\tau_j - \sigma_k$ continuous, bi-continuous and strongly bi-continuous maps in bitopological spaces. Here we present some of the definitions, which are used in our study.

In this paper, we introduce and investigate a new class of maps called $(\sigma_k, \sigma_l) - M_i\, O(Y) - \tau_i$-continuous, $(\sigma_k, \sigma_l) - M_a\, O(Y) - \tau_i$-continuous, $(\sigma_k, \sigma_l) - M_i\, O(Y) - (\tau_i, \tau_j)$-irresolute and $(\sigma_k, \sigma_l) - M_a\, O(Y) - (\tau_i, \tau_j)$-irresolute maps in bitopological spaces.

2. Preliminaries :

2.1 Definition[5]: A proper nonempty open subset U of a topological space X is said to be a minimal open set if any open set which is contained in U is ϕ or U.

2.2 Definition[6]: A proper nonempty open subset U of a topological space X is said to be maximal open set if any open set which contains U is X or U.

2.3 Definition[7]: A proper nonempty closed subset F of a topological space X is said to be a minimal closed set if any closed set which is contained in F is ϕ or F.

2.4 Definition[7]: A proper nonempty closed subset F of a topological space X is said to be maximal closed set if any closed set which contains F is X or F.

The family of all minimal open (resp. minimal closed) sets in a topological space X is denoted by $M_i\, O(X)$ (resp. $M_i\, C(X)$). The family of all maximal open (resp. maximal closed) sets in a topological space X is denoted by $M_a\, O(X)$ (resp. $M_a\, C(X)$).

2.5 Definition[1]:

i) A topological space (X, τ) is said to be T_{min} space if every nonempty proper open subset of X is minimal open set.

ii) A topological space (X, τ) is said to be T_{max} space if every nonempty proper open subset of X is maximal open set.

2.6 Definition[1]: Let X and Y be the topological spaces. A map $f : X \to Y$ is called

i) minimal continuous (briefly min-continuous) if $f^{-1}(M)$ is an open set in X for every minimal open set M in Y.

ii) maximal continuous (briefly max-continuous) if $f^{-1}(M)$ is an open set in X for every maximal open set M in Y.

iii) minimal irresolute (briefly min-irresolute) if $f^{-1}(M)$ is minimal open set in X for every minimal open set M in Y.

iv) maximal irresolute (briefly max-irresolute) if $f^{-1}(M)$ is maximal open set in X for every maximal open set M in Y.

2.7 Definition[3]: Let X be a set and τ_1 and τ_2 be two different topologies on X. Then (X, τ_1, τ_2) is called a bitopological space.

2.8 Definition[2]: Let $i, j \in \{1, 2\}$ be the fixed integers and (X, τ_1, τ_2) be a bitopological space.

i) A proper nonempty τ_i-open subset M in X is said to be a (τ_i, τ_j)- minimal open (briefly (τ_i, τ_j)-min open) set if any τ_j-open set which is contained in M is either ϕ or M itself.

ii) A proper nonempty τ_i-open subset M in X is said to be a (τ_i, τ_j)-maximal open (briefly (τ_i, τ_j)-maximal open) set if any τ_j-open set which contains M is either X or M itself.

iii) A proper nonempty τ_i-closed subset F in X is said to be a (τ_i, τ_j)-minimal closed (briefly (τ_i, τ_j)-min closed) set if any τ_j-closed set which is contained in F is either ϕ or F itself.

iv) A proper nonempty τ_i-closed subset F in X is said to be a (τ_i, τ_j)- maximal closed (briefly (τ_i, τ_i)-maximal closed) set if any τ_j -closed set which contains F is either X or F itself.

The family of all (τ_i, τ_i)-minimal open (resp. (τ_i, τ_j)-minimal closed) sets in a bitopological space (X, τ_1, τ_2) is denoted by $(\tau_i, \tau_j) - M_i O(X)$ (resp. $(\tau_i, \tau_j) - M_i C(X)$). The family of all (τ_i, τ_j)-maximal open (resp. (τ_i, τ_j)-maximal closed) sets in a bitopological space (X, τ_1, τ_2) is denoted by $(\tau_i, \tau_j) - M_a O(X)$ (resp. $(\tau_i, \tau_j) - M_a C(X)$).

2.9 Definition[2]: Let $i, j \in \{1, 2\}$ be the fixed integers.

i) A bitopological space (X, τ_1, τ_2) is said to be pairwise-T_{min} space if every nonempty proper τ_i -open set is (τ_i, τ_j)-minimal open set.

ii) A bitopological space (X, τ_1, τ_2) is said to be pairwise-Tmax space if every nonempty proper τi -open set is (τ_i, τ_j)-maximal open set.

2.10 Definition[4]: A map f : (X, τ_1, τ_2) '! (Y, σ_1, σ_2) is called

i) $\tau_j - \sigma_k$ -continuous if $f^{-1}(V) \in \tau_j$ for every $V \in \sigma_k$,

ii) bi-continuous if f is $\tau_1 - \sigma_1$ -continuous and $\tau_2 - \sigma_2$ -continuous,

iii) strongly-bi-continuous (briefly, s-bi-continuous) if f is bi-continuous, $\tau_1 - \sigma_2$ -continuous and $\tau_2 - \sigma_1$ - continuous,

Throughout this chapter (X, τ_1, τ_2), (Y, σ_1, σ_2) and (Z, η_1, η_2) denote nonempty bitopological spaces on which no separation axioms are assumed, unless other- wise mentioned and the fixed integers $i, j, k, l, m, n \in 1, 2$.

3. Pairwise minimal continuous and pairwise maximal continuous maps :

3.1 Definition: Let $i, j, k, l \in \{1, 2\}$ be fixed integers. Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces. A map $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called

i) $(\sigma_k, \sigma_l) - M_i O(Y) - \tau_i$ -continuous (pairwise minimal continuous) if $f^{-1}(M) \in \tau_i$ -open set in X for every M $\in (\sigma_k, \sigma_l) - M_i O(Y)$.

ii) (σ_k, σ_l)–$M_a O(Y)$– τ_i -continuous (pairwise maximal continuous) if f^{-1}(M) $\in \tau_i$ -open set in X for every $M \in (\sigma_k, \sigma_l) - M_a O(Y)$.

iii) (σ_k, σ_l)–$M_i O(Y)$–(τ_i, τ_j)-irresolute (pairwise minimal irresolute) if f^{-1}(M) $\in (\tau_i, \tau_j) - M_i O(X)$ for every $M \in (\sigma_k, \sigma_l) - M_i O(Y)$.

iv) (σ_k, σ_l)–$M_a O(Y)$–(τ_i, τ_j)-irresolute (pairwise maximal irresolute) if $f^{-1}(M) \in (\tau_i, \tau_j) - M_a O(X)$ for every $M \in (\sigma_k, \sigma_l) - M_a O(Y)$.

3.2. Remark : If $\tau_1 = \tau_2 = \tau$ and $\sigma_1 = \sigma_2 = \sigma$ in the Definition 3.1, then the $(\sigma_k, \sigma_l) - M_i O(Y) - \tau_i$ -continuous, $(\sigma_k, \sigma_l) - M_a O(Y) - \tau_i$ -continuous, $(\sigma_k, \sigma_l) - M_i O(Y) - (\tau_i, \tau_j)$-irresolute and $(\sigma_k, \sigma_l) - M_a O(Y) - (\tau_i, \tau_j)$-irresolute maps coincide with minimal continuous, maximal continuous, minimal irresolute and maximal irresolute maps respectively.

3.3 Theorem: Every $\tau_i - \sigma_k$ continuous map is $(\sigma_k, \sigma_l) - M_i O(Y) - \tau_i$ - continuous but not conversely.

Proof: Let f : $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a $\tau_i - \sigma_k$ continuous map. To prove that f is $(\sigma_k, \sigma_l) - M_i O(Y)$–

τ_i -continuous. Let N be any (σ_k, σ_l)-minimal open set in Y. Since every (σ_k, σ_l)-minimal open set is an σ_k -open set, N is an σ_k -open set in Y. Since f is $\tau_i - \sigma_k$ continuous, $f^{-1}(N)$ is an τ_i -open set in X. Hence f is a $(\sigma_k, \sigma_l) - M_i\, O(Y) - \tau_i$ -continuous.

3.4 Example: Let X = Y = {a, b, c, d} be with topologies $\tau_1 = \{\phi, \{a\}, \{a, b\}, X\}$, $\tau_2 = \{\phi, \{a\}, \{a, c\}, X\}$, $\sigma_1 = \{\phi, \{a\}, \{a, b, c\}, Y\}$ and $\sigma_2 = \{\phi, \{a\}, \{a, b\}, \{a, b, c\}, Y\}$. Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be an identity map. Then f is a $(\sigma_1, \sigma_2) - M_i\, O(Y) - \tau_1$ -continuous but it is not a $\tau_1 - \sigma_1$ -continuous map, since for the σ_1 -open set $\{a, b, c\} \in Y, f^{-1}(\{a, b, c\}) = \{a, b, c\}$ which is not an τ_1 -open set in X.

3.5 Theorem: Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a $(\sigma_k, \sigma_l) - M_i\, O(Y) - \tau_i$ - continuous, onto map and (Y, σ_1, σ_2) be a pairwise-T_{min} space. Then f is a $\tau_i - \sigma_k$ continuous.

Proof: Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a $(\sigma_k, \sigma_l) - M_i\, O(Y) - \tau_i$ -continuous, onto map. Note that the inverse image of ϕ and Y are always τ_i -open sets in a bitopological space X. Let N be any nonempty proper τ_k -open set in Y. By hypothesis, (Y, σ_1, σ_2) is pairwise-T_{min} space, it follows that N is a (σ_k, σ_l)-minimal open set in Y. Since f is $(\sigma_k, \sigma_l) - M_i\, O(Y) - \tau_i$ -continuous, $f^{-1}(N)$ is an τ_i -open set in X. Therefore f is a $\tau_i - \sigma_k$ continuous.

3.6 Theorem : Every $\tau_i - \sigma_k$ continuous map is $(\sigma_k, \sigma_l) - M_a\, O(Y) - \tau_i$ - continuous but not conversely.

Proof: Similar to that of Theorem 3.3.

3.7 Example: Let Let $X = Y = \{a, b, c, d, e\}$ be with topologies $\tau_1 = \{\phi, \{a\}, \{a, b, c, d\}, X\}$, $\tau_2 = \{\phi, \{b\}, \{a, b, c, d\}, X\}$, $\sigma_1 = \{\phi, \{a, b\}, \{a, b, c, d\}, Y\}$ and $\sigma_2 = \{\phi, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}, Y\}$. Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be an identity map. Then f is a $(\sigma_1, \sigma_2) - Ma\, O(Y) - \tau_1$ -continuous but it is not a $\tau_1 - \sigma_1$ continuous map, since for the σ_1 -open set $\{a, b\} \in Y, f^{-1}(\{a, b\}) = \{a, b\}$ which is not an τ_1 -open set in X.

3.8 Theorem: Let $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a $(\sigma_k, \sigma_l) - M_a\, O(Y) - \tau_i$ -continuous, onto map and let (Y, σ_1, σ_2) be a pairwise-T_{max} space. Then f is a $\tau_i - \sigma_k$ continuous.

Proof: Similar to that of Theorem 3.5.

3.9 Remark: $(\sigma_k, \sigma_l) - M_i\, O(Y) - \tau_i$ -continuous and $(\sigma_k, \sigma_l) - M_a\, O(Y) - \tau_i$ - continuous maps are independent of each other.

3.10 Example: In Example 3.4, f is a $(\sigma_1, \sigma_2) - M_i\, O(Y) - \tau_1$ -continuous but it is not a $(\sigma_1, \sigma_2) - M_a\, O(Y) - \tau_1$ -continuous. In Example 3.7, f is a $(\sigma_1, \sigma_2) - M_a\, O(Y) - \tau_1$ -continuous but it is not a $(\sigma_1, \sigma_2) - M_i\, O(Y) - \tau_1$ - continuous.

3.11 Theorem: Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces. A map $f : X \to Y$ is a $(\sigma_k, \sigma_l) - M_i\, O(Y) - \tau_i$ -continuous if and only if the inverse image of each (σ_k, σ_l)-maximal closed set in Y is a τ_i -closed set in X.

Proof: The proof follows from the definition and fact that the complement of (σ_k, σ_l)-minimal open set is (σ_k, σ_l)-maximal closed set.

3.12 Theorem: Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces. A map $f : X \to Y$ is a $(\sigma_k, \sigma_l) - M_a\, O(Y) - \tau_i$ -continuous if and only if the inverse image of each (σ_k, σ_l)-minimal closed set in Y is a τ_i -closed set in X.

Proof: The proof follows from the definition and fact that the complement of (σ_k, σ_l)-maximal open set is (σ_k, σ_l)-minimal closed set.

3.13 Theorem: Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces and A be a nonempty subset of X. If $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is a $(\tau_k, \tau_l) - M_i\, O(Y) - \tau_i$-continuous then the restriction map $f_A : (A, \tau_{1A}, \tau_{2A}) \rightarrow (Y, \sigma_1, \sigma_2)$ is a $(\sigma_k, \sigma_l) - M_i\, O(Y) - \tau_{iA}$-continuous map.

Proof: Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is a $(\sigma_k, \sigma_l) - M_i\, O(Y) - \tau_i$-continuous. To prove $f_A : (A, \tau_{1A}, \tau_{2A}) \rightarrow (Y, \sigma_1, \sigma_2)$ is a $(\sigma_k, \sigma_l) - M_i\, O(Y) - \tau_{iA}$-continuous map. Let N be any (σ_k, σ_l)-minimal open set in Y. Since f is $(\sigma_k, \sigma_l) - M_i\, O(Y) - \tau_i$-continuous, $f^{-1}(N)$ is an τ_i-open set in X. By definition of relative topology, $f_A^{-1}(N) = A \cap f^{-1}(N)$. Therefore $A \cap f^{-1}(N)$ is an τ_{iA}-open set in A. Therefore f_A is a $(\sigma_k, \sigma_l) - M_i\, O(Y) - \tau_{iA}$-continuous map.

3.14 Theorem: Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces and let A be a nonempty subset of X. If $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is a $(\sigma_k, \sigma_l) - M_a\, O(Y) - \tau_i$-continuous then the restriction map $f_A : (A, \tau_1 A, \tau_2 A) \rightarrow (Y, \sigma_1, \sigma_2)$ is a $(\sigma_k, \sigma_l) - M_a\, O(Y) - \tau_{iA}$-continuous map.

Proof : Similar to that of Theorem 3.13.

3.15 Remark : The composition of $(\sigma_k, \sigma_l) - M_i\, O(Y) - \tau_i$-continuous maps need not be a $(\sigma_k, \sigma_l) - M_i\, O(Y) - \tau_i$-continuous map.

3.16 Example: Let $X = Y = Z = \{a, b, c, d\}$ be with topologies $\tau_1 = \{\phi, \{a\}, \{a, c\}, X\}$, $\tau_2 = \{\phi, \{a\}, \{a, d\}, X\}$, $\sigma_1 = \{\phi, \{a\}, \{a, b\}, \{a, b, c\}, Y\}$, $\sigma_2 = \{\phi, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, Y\}$, $\eta_1 = \{\phi, \{a, b\}, \{a, b, c\}, Z\}$ and $\eta_2 = \{\varphi, \{a, b\}, \{a, c, d\}, Z\}$. Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ and $g : (Y, \sigma_1, \sigma_2) \rightarrow (Z, \eta_1, \eta_2)$ be the identity maps. Then clearly f is a $(\sigma_1, \sigma_2) - M_i\, O(Y) - \tau_1$-continuous and g is a $(\eta_1, \eta_2) - M_i\, O(Y) - \sigma_1$-continuous but $go\, f : (X, \tau_1, \tau_2) \rightarrow (Z, \eta_1, \eta_2)$ is not a $(\eta_1, \eta_2) - M_i\, O(Y) - \tau_1$-continuous map, since for the (η_1, η_2)-minimal open set $\{a, b\}$ in Z, $(gof)^{-1}(\{a, b\}) = \{a, b\}$ which is not a τ_1-open set in X.

3.17 Theorem: Let (X, τ_1, τ_2), (Y, σ_1, σ_2) and (Z, η_1, η_2) be three bitopological spaces. If $f : X \rightarrow Y$ is $\tau_i - \sigma_k$-continuous and $g : Y \rightarrow Z$ is $(\eta_m, \eta_n) - M_l\, O(Y) - \sigma_k$-continuous maps, then $go\, f : X \rightarrow Z$ is a $(\eta_m, \eta_n) - M_i\, O(Y) - \tau_i$-continuous.

Proof: Let N be any (η_m, η_n)-minimal open set in Z. Since g is $(\eta_m, \eta_n) - Mi\, O(Y) - \sigma_k$-continuous, $g^{-1}(N)$ is an σ_k-open set in Y. Again since f is $\tau_i - \sigma_k$ continuous, $f^{-1}(g^{-1}(N)) = (go\, f)^{-1}(N)$ is an τ_i-open set in X. Hence $go\, f$ is a $(\eta_m, \eta_n) - M_i\, O(Y) - \tau_i$-continuous.

3.18 Remark : The composition of $(\sigma_k, \sigma_l) - M_a\, O(Y) - \tau_i$-continuous maps need not be a $(\sigma_k, \sigma_l) - M_a\, O(Y) - \tau_i$-continuous map.

3.19 Example: Let $X = Y = Z = \{a, b, c, d\}$ be with topologies $\tau_1 = \{\phi, \{a\}, \{a, b, c\}, X\}$, $\tau_2 = \{\phi, \{b\}, \{a, b, c\}, X\}$, $\sigma_1 = \{\phi, \{a, b\}, \{a, b, c\}, Y\}$, $\sigma_2 = \{\phi, \{a\}, \{a, b\}, \{a, b, c\}, Y\}$, $\eta_1 = \{\phi, \{b\}, \{a, b\}, Z\}$ and $\eta_2 = \{\phi, \{a\}, \{a, b\}, Z\}$. Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ and $g : (Y, \sigma_1, \sigma_2) \rightarrow (Z, \eta_1, \eta_2)$ be the identity maps. Then clearly f is a $(\sigma_1, \sigma_2) - M_a\, O(Y) - \tau_1$-continuous and g is a $(\eta_1, \eta_2) - M_a\, O(Y) - \sigma_1$-continuous but $go\, f : (X, \tau_1, \tau_2) \rightarrow (Z, \eta_1, \eta_2)$ is not a $(\eta_1, \eta_2) - M_a\, O(Y) - \tau_1$-continuous map, since for the (η_1, η_2)-maximal open set $\{a, b\} \in Z$, $(go\, f)^{-1}(\{a, b\}) = \{a, b\}$ which is not a τ_1-open set in X.

3.20 Theorem: Let (X, τ_1, τ_2), (Y, σ_1, σ_2) and (Z, η_1, η_2) be three bitopological spaces. If $f : X \rightarrow$

Y is $\tau_i - \sigma_k$-continuous and $g : Y \to Z$ is $(\eta_m, \eta_n) - M_a\, O(Y) - \sigma_k$-continuous maps, then $g : X \to Z$ is a $(\eta_m, \eta_n) - M_a\, O(Y) - \tau_i$- continuous.

Proof : Similar to that of Theorem 3.17.

3.21 Theorem : Every $(\sigma_k, \sigma_l) - M_i\, O(Y) - (\tau_i, \tau_j)$-irresolute map is $(\sigma_k, \sigma_l) - M_i\, O(Y) - \tau_i$-continuous but not conversely.

Proof : Let f: $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a $(\sigma_k, \sigma_l) - M_i\, O(Y) - (\tau_i, \tau_j)$-irresolute map. To prove $(\sigma_k, \sigma_l) - M_i\, O(Y) - \tau_i$-continuous. Let N be any (σ_k, σ_l)- minimal open set in Y. Since f is $(\sigma_k, \sigma_l) - M_i\, O(Y) - (\tau_i, \tau_j)$-irresolute, $f^{-1}(N)$ is a (τ_i, τ_j)-minimal open set in X. Since every (τ_i, τ_j)-minimal open set is an τ_i-open set, $f^{-1}(N)$ is an τ_i-open set in X. Hence f is a $(\sigma_k, \sigma_l) - M_i\, O(Y) - \tau_i$-continuous.

3.22 Example: Let $X = Y = \{a, b, c, d\}$ be with topologies $\tau_1 = \{\phi, \{a\}, \{a, b\}, X\}$, $\tau_2 = \{\phi, \{b\}, \{a, b\}, X\}$, $\sigma_1 = \{\phi, \{a, b\}, \{a, b, c\}, Y\}$ and $\sigma_2 = \{\phi, \{a, b\}, \{a, b, d\}, Y\}$. Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be an identity map. Then f is a $(\sigma_1, \sigma_2) - M_i\, O(Y) - \tau_1$-continuous but it is not a $(\sigma_1, \sigma_2) - M_i\, O(Y) - (\tau_1, \tau_2)$- irresolute map, since for the (σ_1, σ_2)-minimal open set $\{a, b\}$ in Y, $f^{-1}\{a, b\}) = \{a, b\}$ which is not a (τ_1, τ_2)-minimal open set in X.

3.23 Theorem: Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a $(\sigma_k, \sigma_l) - M_i\, O(Y) – (\tau_i, \tau_j)$-irresolute, onto map and let (Y, σ_1, σ_2) be a pairwise-T_{min} space. Then f is a $\tau_i - \sigma_k$-continuous.

Proof : Proof follows from the Theorems 3.21 and 3.5.

3.24 Theorem: Every $(\sigma_k, \sigma_l) - M_a\, O(Y) - (\tau_i, \tau_j)$-irresolute map is $(\sigma_k, \sigma_l) - M_a\, O(Y) - \tau_i$-continuous but not conversely.

Proof : Similar to that of Theorem 3.21.

3.25 Example: Let $X = Y = \{a, b, c, d\}$ be with topologies $\tau_1 = \{\phi, \{a, b\}, \{a, b, c\}, X\}$, $\tau_2 = \{\phi, \{a, b\}, \{a, b, d\}, X\}$, $\sigma_1 = \{\phi, \{a\}, \{a, b\}, Y\}$ and $\sigma_2 = \{\phi, \{b\}, \{a, b\}, Y\}$. Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be an identity map. Then f is a $(\sigma_1, \sigma_2) - M_a\, O(Y) - \tau_1$-continuous but it is not a $(\sigma_1, \sigma_2) - M_a\, O(Y) - (\tau_1, \tau_2)$-irresolute map, since for the (σ_1, σ_2)-maximal open set $\{a, b\} \in$ Y, $f^{-1}(\{a, b\}) = \{a, b\}$ which is not a (τ_1, τ_2)-maximal open set in X.

3.26 Theorem: Let f: $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a $(\sigma_k, \sigma_l) - M_a\, O(Y) - (\tau_i, \tau_j)$-irresolute, onto map and let (Y, σ_1, σ_2) be a pairwise-T_{max} space. Then f is a $\tau_i - \sigma_k$-continuous.

Proof : Proof follows from the Theorems 3.24 and 3.8.

3.27 Remark: $(\sigma_k, \sigma_l) - M_i\, O(Y) - (\tau_i, \tau_j)$-irresolute and $\tau_i - \sigma_k$ continuous maps are independent of each other.

3.28 Example: Let $X = Y = \{a, b, c, d\}$ be with topologies $\tau_1 = \{\phi, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, X\}$, $\tau_2 = \{\phi, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}, X\}$, $\sigma_1 = \{\phi, \{a, b\}, Y\}$ and $\sigma_2 = \{\phi, \{a, c\}, Y\}$. Let f: $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be an identity map. Then f is a $\tau_1 - \sigma_1$ continuous but it is not a $(\sigma_1, \sigma_2) - M_i\, O(Y) - (\tau_1, \tau_2)$-irresolute map, since for the (σ_1, σ_2)-minimal open set $\{a, b\} \in$ Y, $f^{-1}(\{a, b\}) = \{a, b\}$ which is not a (τ_1, τ_2)-minimal open set in X. In Example 3.4, f is a $(\sigma_1, \sigma_2) - M_i\, O(Y) - (\tau_1, \tau_2)$-irresolute but it is not a $\tau_1 - \sigma_1$ continuous map, since for the σ_1-open set $\{a, b, c\} \in Y$, $f^{-1}(\{a, b, c\}) = \{a, b, c\}$ which is not an τ_1-open set in X.

3.29 Remark: $(\sigma_k, \sigma_l) - M_a\, O(Y) - (\tau_i, \tau_j)$-irresolute and $\tau_i - \sigma_k$ continuous maps are independent of each other.

3.30 Example: In Example 3.28, f is a $\tau_1 - \sigma_1$ continuous but it is not a $(\sigma_1, \sigma_2) - M_a\, O(Y) - (\tau_1, \tau_2)$-irresolute map, since for the (σ_1, σ_2)-maximal open set $\{a, b\} \in Y$, $f^{-1}(\{a, b\}) = \{a, b\}$ which is not a (τ_1, τ_2)-maximal open set in X. In Example 3.7, f is a $(\sigma_1, \sigma_2) - M_a\, O(Y) - (\tau_1, \tau_2)$-irresolute but it is not a $\tau_1 - \sigma_1$ continuous map, since for the σ_1-open set $\{a, b\} \in Y$, $f^{-1}(\{a, b\}) = \{a, b\}$ which is not an τ_1-open set in X.

3.31 Remark: $(\sigma_k, \sigma_l) - M_i\, O(Y) - (\tau_i, \tau_j)$-irresolute and $(\sigma_k, \sigma_l) - M_a\, O(Y) - (\tau_i, \tau_j)$-irresolute maps are independent of each other.

3.32 Example: In Example 3.4, f is a $(\sigma_1, \sigma_2) - M_i\, O(Y) - (\tau_1, \tau_2)$-irresolute but it is not a $(\sigma_1, \sigma_2) - M_a\, O(Y) - (\tau_1, \tau_2)$-irresolute map. In Example 3.7, f is a $(\sigma_1, \sigma_2) - M_a\, O(Y) - (\tau_1, \tau_2)$-irresolute but it is not a $(\sigma_1, \sigma_2) - M_i\, O(Y) - (\tau_1, \tau_2)$- irresolute map.

3.33 Theorem: Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces. A map $f: X \to Y$ is a $(\sigma_k, \sigma_l) - M_i\, O(Y) - (\tau_i, \tau_j)$-irresolute if and only if the inverse image of each (σ_k, σ_l)-maximal closed set in Y is a (τ_i, τ_j)-maximal closed set in X.

Proof: The proof follows from the definition and fact that the complement of (σ_k, σ_l)-minimal open set is (σ_k, σ_l)-maximal closed set.

3.34 Theorem: Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces. A map $f: X \to Y$ is a $(\sigma_k, \sigma_l) - M_a\, O(Y) - (\tau_i, \tau_j)$-irresolute if and only if the inverse image of each (σ_k, σ_l)-minimal closed set in Y is a (τ_i, τ_j)-minimal closed set in X.

Proof: The proof follows from the definition and fact that the complement of (σ_k, σ_l)-maximal open set is (σ_k, σ_l)-minimal closed set.

3.35 Theorem: Let (X, τ_1, τ_2), (Y, σ_1, σ_2) and (Z, η_1, η_2) be three bitopolog- ical spaces. If $f: X \to Y$ is $(\sigma_k, \sigma_l) - M_i\, O(Y) - (\tau_i, \tau_j)$-irresolute and $g: Y \to Z$ is $(\eta_m, \eta_n) - M_i\, O(Z) - (\sigma_k, \sigma_l)$-irresolute maps, then $g o f: X \to Z$ is a $(\eta_m, \eta_n) - M_i\, O(Z) - (\tau_i, \tau_j)$-irresolute.

Proof: Let N be any (η_m, η_n)-minimal open set in Z. Since g is $(\eta_m, \eta_n) - M_i\, O(Z) - (\sigma_k, \sigma_l)$-irresolute, $g^{-1}(N)$ is a (σ_k, σ_l)-minimal open set in Y. Again since f is $(\sigma_k, \sigma_l) - M_i\, O(Y) - (\tau_i, \tau_j)$-irresolute, $f^{-1}(g^{-1}(N)) = (g o f)^{-1}(N)$ is a (τ_i, τ_j)-minimal open set in X. Therefore $g o f$ is a $(\eta_m, \eta_n) - M_i\, O(Z) - (\tau_i, \tau_j)$-irresolute.

3.36 Theorem: Let (X, τ_1, τ_2), (Y, σ_1, σ_2) and (Z, η_1, η_2) be three bitopological spaces. If $f: X \to Y$ is $(\sigma_k, \sigma_l) - M_a\, O(Y) - (\tau_i, \tau_j)$-irresolute and $g: Y \to Z$ is $(\eta_m, \eta_n) - M_a\, O(Z) - (\sigma_k, \sigma_l)$-irresolute maps, then $g o f: X \to Z$ is a $(\eta_m, \eta_n) - M_a\, O(Z) - (\tau_i, \tau_j)$-irresolute.

Proof: Similar to that of Theorem 3.35.

3.37 Definition: Let $i, j, k, l \in \{1, 2\}$ be fixed integers. Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces. A map $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called

i) minimal bi-continuous if $(\sigma_1, \sigma_2) - M_i\, O(Y) - \tau_1$-continuous and $(\sigma_2, \sigma_1) - M_i\, O(Y) - \tau_2$-continuous.

ii) maximal bi-continuous if $(\sigma_1, \sigma_2) - M_a\, O(Y) - \tau_1$-continuous and $(\sigma_2, \sigma_1) - M_a\, O(Y) - \tau_2$-continuous.

iii) minimal bi-irresolute if $(\sigma_1, \sigma_2) - M_i\, O(Y) - (\tau_1, \tau_2)$-irresolute and $(\sigma_2, \sigma_1) - M_i\, O(Y) - (\tau_2, \tau_1)$-irresolute.

iv) maximal bi-irresolute if $(\sigma_1, \sigma_2) - M_a\, O(Y) - (\tau_1, \tau_2)$-irresolute and $(\sigma_2, \sigma_1) - M_a\, O(Y) - (\tau_2, \tau_1)$-irresolute.

3.38 Theorem: Let $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a map.

i) If f is bi-continuous then f is minimal bi-continuous.

ii) If f is bi-continuous then f is maximal bi-continuous.
iii) If f is minimal bi-irresolute then f is minimal bi-continuous.
iv) If f is maximal bi-irresolute then f is maximal bi-continuous.

Proof: i) Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a bi-continuous map. Therefore by definition, f is $\tau_1 - \sigma_1$ continuous and $\tau_2 - \sigma_2$ continuous and so by Theorem 3.3, f is $(\sigma_1, \sigma_2) - M_i\, O(Y) - \tau_1$ -continuous and $(\sigma_2, \sigma_1) - M_i\, O(Y) - \tau_2$ -continuous. Thus, f is a minimal bi-continuous.

ii) Similar to (i), using Theorem 3.6.

iii) Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a minimal bi-irresolute map. Therefore by definition, f is $(\sigma_1, \sigma_2) - M_i\, O(Y) - (\tau_1, \tau_2)$-irresolute and $(\sigma_2, \sigma_1) - M_i\, O(Y) - (\tau_2, \tau_1)$-irresolute and so by Theorem 3.21, f is $(\sigma_1, \sigma_2) - M_i\, O(Y) - \tau_1$-continuous and $(\sigma_2, \sigma_1) - M_i\, O(Y) - \tau_2$ -continuous. Thus, f is a minimal bi-continuous.

iv) Similar to (iii), using Theorem 3.24.

3.39 Definition: Let $i, j, k, l \in \{1, 2\}$ be fixed integers. Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces. A map $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called

i) minimal-s-bi-continuous if minimal bi-continuous, $(\sigma_1, \sigma_2) - M_i\, O(Y) - \tau_2$ - continuous and $(\sigma_2, \sigma_1) - M_i\, O(Y) - \tau_1$ -continuous.

ii) maximal-s-bi-continuous if maximal bi-continuous, $(\sigma_1, \sigma_2) - M_a\, O(Y) - \tau_2$ - continuous and $(\sigma_2, \sigma_1) - M_a\, O(Y) - \tau_1$ – continuous.

iii) minimal-s-bi-irresolute if minimal bi-irresolute, $(\sigma_1, \sigma_2) - M_i\, O(Y) - (\tau_2, \tau_1)$- irresolute and $(\sigma_2, \sigma_1) - M_i\, O(Y) - (\tau_1, \tau_2)$-irresolute.

iv) maximal-s-bi-irresolute if maximal bi-irresolute, $(\sigma_1, \sigma_2) - M_a\, O(Y) - (\tau_2, \tau_1)$- irresolute and $(\sigma_2, \sigma_1) - M_a\, O(Y) - (\tau_1, \tau_2)$–irresolute.

3.40 Theorem: Let : $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a map.
i) If is minimal-s-bi-continuous then is minimal bi-continuous.
ii) If is maximal-s-bi-continuous then is maximal bi-continuous.
iii) If is minimal-s-bi-irresolute then is minimal bi-irresolute
iv) If is maximal-s-bi-irresolute then is maximal bi-irresolute

Proof: follows from definitions.

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