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Bayesian Point Prediction for Rayleigh distribution when observations are censored to left and right

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Abstract

In this paper we utilize Bayesian approach to obtain predictors of the future observation from Rayleigh distribution when observations are censored to left as well as to the right. Bayesian predictor is obtained using natural conjugate prior under asymmetric loss function. Bayesian predictor is also obtained under the squared error loss function. For each loss predictive risks are calculated. Lastly, predictors are compared for the smallest future ordered observation on the basis of 1000 randomly generated sample using Monte Carlo simulation technique.

Key words : Rayleigh Model, prior, conjugate prior, posterior, Bayes prediction, Multiply type II censoring.

Mathematics Subject Classification : 62F15, 62N01, 00A72

1. Introduction

The use of predictive inference got its appearance in recent past in which one wishes to infer about future sample on the basis of results obtained from the past sample of the same population. For example, a factory owner wishes to predict about lifetimes of certain type of machine tools to know about best inspection and replacement policy on the basis of recorded life time of machine tools of similar type. Such type of inference is known as predictive inference. A good deal of literature is available on the predictive inference for life time models using both classical and Bayesian approach (see for example, Lawless^{7,8}, Lingappaiah⁹, Howlader & Hossain⁶, Fernandez⁵, Ahmadi *et.al.*¹ among others. Aitchison and Dunsmore² is a text exclusively devoted on this topic.

When point prediction is under discussion, the consequence of being wrong must be viewed. Most of

the above literature has assumed that loss due to consequence of being wrong is proportional to the square of error i.e. equal weightage has given for positive error or negative error. But this seems unjustified in the case if positive error is more serious than negative error or vice versa. In the example mentioned above if actual lifetime of machine exceeds inspection time the overhead scrapping loss is incurred for unused productive capacity. In contrary to it if inspection time exceeds the actual time of machine there is a loss of production time. So under prediction and over prediction are not of equal importance in many practical situations, hence use of symmetric loss function is not justified.

The simplest asymmetric loss function for the prediction problems is the linear loss function suggested by Aitchison and Dunsmore² which associates unequal weights to under prediction and over prediction errors of equal magnitude. The loss function should be such that if we predict (y) correctly, the loss incurred must be zero, otherwise it should be proportional to the difference between predicted value (a) and the actual value (y). The constant of proportionalities are chosen according to relative importance of under-prediction and over prediction. The asymmetric linear loss function can be given as

$$L(a, y) = \begin{cases} \xi(a - y) & \text{if } y \leq a \\ \eta(y - a) & \text{if } y > a \end{cases} \quad (1)$$

where ξ is the loss per unit time for under prediction and η is the loss per unit time for over prediction. If η and ξ both are equal, then above loss function reduces to a symmetric loss function.

Point prediction was discussed by Zellner¹² using asymmetric LINEX loss. Ahmadi¹ also considered prediction in exponential distribution based on k-record values under LINEX loss function. Basak and Balakrishnan³ obtained predictors of failure times of censored units in progressively censored samples from normal distribution. Fernandez⁵ discussed inferences from type II doubly censored Rayleigh data in Bayesian perspective. Shastri and Pamari¹¹ obtained prediction limits for smallest future ordered observation when Rayleigh data is compounded with multiply type II censoring. Recently MirMostafaei *et. al.*¹⁰ obtained Bayesian prediction of minimal repair times of components' lifetimes under Rayleigh distribution. But nothing has appeared in literature regarding point prediction under asymmetric linear loss for Rayleigh model.

To illustrate the use of linear loss function, we have considered the problem of point prediction for the future ordered observation from one parameter Rayleigh distribution when the data available is censored to the left as well as on right.

2. Point Prediction :

The time to failure x , of a Rayleigh component has probability density function (pdf)

$$f(x|\sigma) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right) \quad x \geq 0, \sigma > 0 \quad (2)$$

with cumulative distribution function

$$F(x) = 1 - \exp\left(-\frac{x^2}{2\sigma^2}\right) \quad (3)$$

Let us assume that x_1, x_2, \dots, x_n be a random sample of size n put on test for which the failure time is distributed according to Rayleigh distribution. Suppose the experimenter was not able to observe first $(l-1)$ observations, hence $x_1 < \dots < x_{l-1}$ are censored to the left and due to large lifetimes he doesn't wish to observe the rest $(n-r)$ observations, which make observations $x_{r+1} < \dots < x_n$ censored to the right. Thus, $x_l < \dots < x_r$ are the observed life times.

The likelihood function (LF) for this situation can be written as

$$L = \frac{n!}{(l-1)!(n-r)!} \prod_{i=1}^{l-1} F(x_i, \sigma) \prod_{i=l}^r f(x_i | \sigma) \prod_{i=r+1}^N R(x_i, \sigma) \quad (4)$$

Using (2) and (3), and on simplification, it reduces to

$$L = \frac{n!}{(l-1)!(n-r)!} \left(\frac{1}{\sigma^2}\right)^A \sum_{p=0}^{l-1} \Omega_p \exp\left[-\frac{px_l^2 + S^2}{2\sigma^2}\right] \prod_{i=l}^r x_i \quad (5)$$

where

$$S^2 = \sum_{i=l}^r x_i^2 + (n-r)x_r^2$$

$$\Omega_p = (-1)^{l-1} \binom{l-1}{p}, \quad A = r - l + 1$$

Consider a conjugate family of prior for the parameter σ

$$g(\sigma/a, b) = \frac{a^b}{\Gamma(b)2^{b-1}} \sigma^{-2b-1} \exp\left[-\frac{a}{2\sigma^2}\right]; \quad \sigma > 0; a, b > 0 \quad (6)$$

Combining LF(5) with prior (6) via Bayes theorem, the posterior distribution is defined and obtained as

$$p(\sigma|\underline{x}) = \frac{L(\underline{x}, \sigma)g(\sigma)}{\int L(\underline{x}, \sigma)g(\sigma)d\sigma}$$

$$p(\sigma|\underline{x}) = \frac{C_x^{-1}}{\Gamma(A+b)2^{A+b-1}} \sum_{p=0}^{l-1} \Omega_p \exp\left[\frac{px_l^2 + S^2 + a}{2\sigma^2}\right] \quad (7)$$

where

$$C_x = \sum_{p=0}^{l-1} \Omega_p \left[\frac{px_l^2 + S^2 + a}{2\sigma^2}\right]^{-(A+b)}$$

Let y_1, y_2, \dots, y_m be the second independent random sample of size m of future observation from the model (1), then the density of k^{th} ordered future observation ($1 \leq k \leq m$) will be obtained by

$$f(y|\sigma) = \frac{m!}{(k-1)!(m-k)!} [F(y)]^{k-1} f(y) [1 - F(y)]^{m-k}$$

Substituting and solving

$$f(y|\sigma) = \beta^{-1}(k, m-k+1) \sum_{i=0}^{k-1} \Omega_i \exp\left[-\frac{1}{2\sigma^2}(M+i)y^2\right] \left(\frac{y}{\sigma^2}\right) \quad (8)$$

where

$$\beta^{-1}(k, m - k + 1) = \frac{m!}{(k - 1)! (m - k)!}$$

and

$$M = m - k + 1$$

Then the Bayes Predictive density for future k^{th} ordered observation will be

$$h(y|\underline{x}) = \int f(y|\sigma)p(\sigma|\underline{x}) d\sigma$$

substituting the values

$$h(y|\underline{x}) = \frac{2(A + b)\beta^{-1}(k, M)}{C_x} \sum_{p=0}^{l-1} \sum_{i=0}^{k-1} \Omega_p \Omega_i y [px_l^2 + S^2 + a + (M + i)y^2]^{-(A+b+1)} \quad (9)$$

The optimal value of point predictor may be obtained by differentiating the expected loss w.r.t. a . The expected loss can be written as

$$\begin{aligned} L(a) &= E(L(a, y_{(k)})) \\ &= \xi \int_0^a (a - y_{(k)}) h(y_{(k)}|S) dy_{(k)} + \eta \int_a^\infty (y_{(k)} - a) h(y_{(k)}|S) dy_{(k)} \end{aligned} \quad (10)$$

Differentiating w.r.t. a and simplifying, we get

$$L'(a) = (\eta + \xi) \int_0^a h(y_{(k)}|S) dy_{(k)} - \eta \quad (11)$$

$$L''(a) = (\eta + \xi) h(a|S) \quad , \quad (> 0)$$

which implies that the solution of (11) when equated to zero provides the optimal value of a for which expected loss is minimum. Hence point predictor, say $y_{(k)L^*}$, under linear loss is the solution of

$$\int_0^{y_{(k)L^*}} h(y_{(k)}|S) dy_{(k)} = \frac{\eta}{(\eta + \xi)} \quad (12)$$

On substituting value of $h(y_{(k)}|S)$ from (8) in (12) and simplifying, we have

$$\beta^{-1}(k, M) C_x^{-1} \sum_{p=0}^{l-1} \sum_{i=0}^{k-1} \Omega_p \Omega_i \frac{1}{(M + i)} \left[(px_l^2 + S^2 + a)^{-(A+b)} - (px_l^2 + S^2 + a + (M + i)y_{(k)L^*}^2)^{-(A+b)} \right] = \frac{\eta}{(\eta + \xi)} \quad (13)$$

Above equation is solved for $y_{(k)L^*}$ by using Bisection method.

It is well known that point predictor under quadratic loss is the mean of predictive pdf. Thus for k^{th} ordered future observation, the predictor is given by

$$y_{(k)Q^*} = E[y_{(k)}|S] = \int_0^\infty y_{(k)} \cdot h(y_{(k)}|S) dy_{(k)} \quad (14)$$

On solving, we get

$$y_{(k)Q^*} = \frac{\beta^{-1}(k, M) C_x^{-1} (A + b) \sqrt{\pi} \sqrt{A + b - \frac{1}{2}}}{2\sqrt{A + b + 1}} \sum_{p=0}^{l-1} \sum_{i=0}^{n-1} \Omega_p \Omega_i \frac{1}{(M + i)^{3/2}} (px_l^2 + S^2 + a)^{-(A+b-\frac{1}{2})} \quad (16)$$

Thus the point predictor $\mathcal{Y}_{(k)}Q^*$ is available in a nice closed form but its usage is justified only if under-prediction and over-prediction are of equal importance. Contrary to it if over-prediction and under-prediction are of unequal importance, the use of $\mathcal{Y}_{(k)}Q^*$ may not be appropriate and one might consider predictor under linear loss. Naturally, a question arises whether we lose enough due to use of $\mathcal{Y}_{(k)}Q^*$ if the appropriate loss is linear. Similarly, it would be also worthwhile to investigate whether we lose enough due to the use of $\mathcal{Y}_{(k)}L^*$ instead of $\mathcal{Y}_{(k)}Q^*$ if over-prediction and under prediction are of equal importance. To get an answer to these queries, we propose to compare $\mathcal{Y}_{(k)}Q^*$ and $\mathcal{Y}_{(k)}L^*$ under both linear and quadratic loss function. The comparison can be carried out on the basis of predictive risk which may be defined as the average loss incurred by the use of a particular predictor for a specified loss function. The predictor corresponding to which the predictive risk is minimum, may then be recommended for use. The predictive risk may be defined as

$$PR(\mathcal{Y}_{(k)}^*) = E[L\{\mathcal{Y}_{(k)}^*, \mathcal{Y}_{(k)}\}]$$

where $\mathcal{Y}_{(k)}^*$ is the predictor of $\mathcal{Y}_{(k)}$ and $L\{\mathcal{Y}_{(k)}^*, \mathcal{Y}_{(k)}\}$ denotes the specified loss. Naturally, the expectation E is to be taken over whole informative as well as future sample space. Beaulieu⁴ obtained approximate pdf of sum of Rayleigh random variables

$$p(S|\sigma^2) = \frac{S^{2L-1}}{2^{L-1} b_1^L (L-1)!} \exp\left[-\frac{S^2}{2b_1}\right] \quad (16)$$

where

$$b_1 = \frac{\sigma^2}{L} [(2L-1)!!]^{-1}$$

and $(2L-1)!! = (2L-1)(2L-3) \dots \dots 3.1$

Incorporating above results, we obtain predictive risk as

$$\begin{aligned} PR(\mathcal{Y}_{(k)}^*) &= \int_0^\infty \int_0^\infty L\{\mathcal{Y}_{(k)}^*, \mathcal{Y}_{(k)}\} h(\mathcal{Y}_{(k)}|S) p(S|\sigma^2) dS d\mathcal{Y}_{(k)} \\ PR(\mathcal{Y}_{(k)}^*) &= \frac{\beta^{-1}(n, M) C_x^{-1}(A+b)}{2^{L-2} b_1^L (L-1)!} \int_0^\infty \int_0^\infty L\{\mathcal{Y}_{(k)}^*, \mathcal{Y}_{(k)}\} \sum_{p=0}^{l-1} \sum_{i=0}^{n-1} \Omega_p \Omega_i \\ &\quad \cdot \mathcal{Y} [px_l^2 + S^2 + a + (M+i)y^2]^{-(A+b+1)} S^{2L-1} \exp\left(-\frac{S^2}{2b_1}\right) dS d\mathcal{Y}_{(k)} \quad (17) \end{aligned}$$

Assuming $L\{\mathcal{Y}_{(k)}^*, \mathcal{Y}_{(k)}\}$ to be linear, the predictive risks for $\mathcal{Y}_{(k)}L^*$ and $\mathcal{Y}_{(k)}Q^*$ can be obtained as

$$\begin{aligned} PR_L(\mathcal{Y}_{(k)}L^*) &= \frac{\beta^{-1}(k, M) C_x^{-1}(A+b)}{2^{L-2} b_1^L (L-1)!} \int_0^\infty \left[\xi \int_0^{\mathcal{Y}_{(k)}L^*} (\mathcal{Y}_{(k)}L^* - \mathcal{Y}_{(k)}) + \eta \int_{\mathcal{Y}_{(k)}L^*}^\infty (\mathcal{Y}_{(k)} - \mathcal{Y}_{(k)}L^*) \right] \\ &\quad \cdot \sum_{p=0}^{l-1} \sum_{i=0}^{k-1} \Omega_p \Omega_i \mathcal{Y}_{(k)} [px_l^2 + S^2 + a + (M+i)\mathcal{Y}_{(k)}^2]^{-(A+b+1)} S^{2L-1} \exp\left(-\frac{S^2}{2b_1}\right) dS d\mathcal{Y}_{(k)} \quad (18) \end{aligned}$$

and

$$PR_L(y_{(k)Q^*}) = \frac{\beta^{-1}(k, M)C_x^{-1}(A + b)}{2^{L-2}b_1^L(L-1)!} \int_0^\infty \left[\xi \int_0^{y_{(k)Q^*}} (y_{(k)Q^*} - y_{(k)}) + \eta \int_{y_{(k)Q^*}}^\infty (y_{(k)} - y_{(k)Q^*}) \right] \\ \cdot \sum_{p=0}^{l-1} \sum_{i=0}^{k-1} \Omega_p \Omega_i y_{(k)} [px_l^2 + S^2 + a + (M + i)y_{(k)}^2]^{-(A+b+1)} S^{2L-1} \exp\left(-\frac{S^2}{2b_1}\right) dS dy_{(k)} \quad (19)$$

Similarly, the predictive risks of the predictors $\mathcal{Y}_{(n)L^*}$ and $\mathcal{Y}_{(n)Q^*}$ under quadratic loss are

$$PR_Q(y_{(k)L^*}) = \frac{\beta^{-1}(k, M)C_x^{-1}(A + b)}{2^{L-2}b_1^L(L-1)!} \int_0^\infty \int_0^\infty (y_{(k)L^*} - y_{(k)})^2 \\ \cdot \sum_{p=0}^{l-1} \sum_{i=0}^{k-1} \Omega_p \Omega_i y_{(k)} [px_l^2 + S^2 + a + (M + i)y_{(k)}^2]^{-(A+b+1)} S^{2L-1} \exp\left(-\frac{S^2}{2b_1}\right) dS dy_{(k)} \quad (20)$$

and

$$PR_Q(y_{(k)Q^*}) = \frac{\beta^{-1}(k, M)C_x^{-1}(A + b)}{2^{L-2}b_1^L(L-1)!} \int_0^\infty \int_0^\infty (y_{(k)Q^*} - y_{(k)})^2 \\ \cdot \sum_{p=0}^{l-1} \sum_{i=0}^{k-1} \Omega_p \Omega_i y_{(k)} [px_l^2 + S^2 + a + (M + i)y_{(k)}^2]^{-(A+b+1)} S^{2L-1} \exp\left(-\frac{S^2}{2b_1}\right) dS dy_{(k)} \quad (21)$$

3. Comparison of Predictors for the smallest observation :

In this section, comparison of the predictors for the smallest observation from a future sample has been made. The predictors and their corresponding risks, for this particular case, may be obtained by putting $n=1$ in (13), (15). The predictor under linear loss is obtained as

$$C_x^{-1} \sum_{p=0}^{l-1} \Omega_p [(px_l^2 + S^2 + a)^{-(A+b)} - (px_l^2 + S^2 + a + my_{(1)L}^*)^{-(A+b)}] = \frac{\eta}{(\eta + \xi)} \quad (22)$$

Similarly, the predictor under quadratic loss comes out to be

$$y_{(1)Q^*} = \frac{C_x^{-1}(A + B)\sqrt{\pi} \Gamma\left(A + b - \frac{1}{2}\right)}{2\sqrt{m} \Gamma(A + b + 1)} \sum_{p=0}^{l-1} \Omega_p (px_l^2 + S^2 + a)^{-(A+b-\frac{1}{2})} \quad (23)$$

Putting $n=1$ in (18), (19) the predictive risks of the predictors $\mathcal{Y}_{(1)L^*}$ and $\mathcal{Y}_{(1)Q^*}$ under linear loss function are

$$PR_L(y_{(1)L^*}) = \frac{mC_x^{-1}(A + b)}{2^{L-2}b_1^L(L-1)!} \int_0^\infty \left[\xi \int_0^{y_{(1)L^*}} (y_{(1)L^*} - y_{(1)}) + \eta \int_{y_{(1)L^*}}^\infty (y_{(1)} - y_{(1)L^*}) \right] \\ \cdot \sum_{p=0}^{l-1} \Omega_p y_{(1)} [px_l^2 + S^2 + a + (m + i)y_{(1)}^2]^{-(A+b+1)} S^{2L-1} \exp\left(-\frac{S^2}{2b_1}\right) dS dy_{(1)} \quad (24)$$

and

$$PR_L(y_{(1)Q^*}) = \frac{mC_x^{-1}(A+b)}{2^{L-2}b_1^L(L-1)!} \int_0^\infty \left[\xi \int_0^{y_{(1)Q^*}} (y_{(1)Q^*} - y_{(1)}) + \eta \int_{y_{(1)Q^*}}^\infty (y_{(1)} - y_{(1)Q^*}) \right] \\ \cdot \sum_{p=0}^{l-1} \Omega_p y_{(1)} [px_l^2 + S^2 + a + (m+i)y_{(1)}^2]^{-(A+b+1)} S^{2L-1} \exp\left(-\frac{S^2}{2b_1}\right) dS dy_{(1)} \quad (25)$$

Putting $k=1$ in (20) and (21) the predictive risks of the predictors $y_{(1)L^*}$ and $y_{(1)Q^*}$ under quadratic loss function are

$$PR_Q(y_{(1)L^*}) = \frac{mC_x^{-1}(A+b)}{2^{L-2}b_1^L(L-1)!} \int_0^\infty \int_0^\infty (y_{(1)L^*} - y_{(1)})^2 \\ \cdot \sum_{p=0}^{l-1} \Omega_p y_{(1)} [px_l^2 + S^2 + a + (m+i)y_{(1)}^2]^{-(A+b+1)} S^{2L-1} \exp\left(-\frac{S^2}{2b_1}\right) dS dy_{(1)} \quad (26)$$

and

$$PR_Q(y_{(1)Q^*}) = \frac{mC_x^{-1}(A+b)}{2^{L-2}b_1^L(L-1)!} \int_0^\infty \int_0^\infty (y_{(1)Q^*} - y_{(1)})^2 \\ \cdot \sum_{p=0}^{l-1} \Omega_p y_{(1)} [px_l^2 + S^2 + a + (m+i)y_{(1)}^2]^{-(A+b+1)} S^{2L-1} \exp\left(-\frac{S^2}{2b_1}\right) dS dy_{(1)} \quad (27)$$

respectively.

It may be noted here that though the predictors and predictive risks are not in closed form, therefore they are evaluated using integrate function in R.

Now the of $y_{(1)L^*}$ w.r.t $y_{(1)Q^*}$ may be defined as

$$PRE_{LIN} = \frac{PR_L(y_{(1)Q^*})}{PR_L(y_{(1)L^*})} \quad (28)$$

Similarly, the PRE_{QRD} of $y_{(1)L^*}$ w.r.t $y_{(1)Q^*}$ may be defined as

$$PRE_{QRD} = \frac{PR_Q(y_{(1)Q^*})}{PR_Q(y_{(1)L^*})} \quad (29)$$

4. Discussion

Predictive risk under linear loss for smallest order future observation with predictive risk under quadratic loss for predictors obtained under different losses is compared on the basis of 1000 randomly generated samples, each of size 10 using Monte Carlo simulation technique.

A number of values for the different constants involved in predictive risks were considered, but the results have been reported for only some of the considered values, because of a number of reasons. We have considered the value of $L=3$ as mentioned in Beaulieu⁴ for sum of Rayleigh variables. Three different sample sizes namely 6, 10 and 25 were considered for both informative and future samples but $n=m=10$ is only reported because no significant change observed in the results with variation in sample sizes. Similarly, we have considered three different values of σ , namely 0.5, 1.0 and 2.0 and it was found that although risks differs by varying σ , the

Table 1 : Predictive risk efficiencies of $\mathcal{Y}_{(1)L}^*$ w.r.t $\mathcal{Y}_{(1)Q}^*$ under linear loss

η/ξ	b	$(r-l+1)/n$			
		0.2	0.4	0.6	0.8
0.20	0.25	2.782	2.456	2.359	2.312
0.25		2.367	2.091	2.014	1.973
0.50		1.51	1.361	1.321	1.299
1.00		1.232	1.163	1.141	1.134
1.50		1.111	1.074	1.062	1.06
2.00		1.065	1.049	1.047	1.047
2.50		1.048	1.053	1.059	1.064
3.00		1.048	1.071	1.082	1.088
3.50		1.056	1.092	1.11	1.118
4.00		1.069	1.118	1.141	1.153
0.20	0.50	2.713	2.441	2.01	2.308
0.25		2.299	2.075	1.905	1.969
0.50		1.474	1.352	1.318	1.301
1.00		1.217	1.156	1.142	1.133
1.50		1.103	1.071	1.065	1.088
2.00		1.061	1.049	1.048	1.049
2.50		1.049	1.055	1.065	1.067
3.00		1.052	1.073	1.083	1.089
3.50		1.063	1.096	1.112	1.12
4.00		1.079	1.123	1.143	1.155
0.20	1.00	2.598	2.408	2.337	2.302
0.25		2.207	2.052	1.993	1.964
0.50		1.421	1.34	1.311	1.297
1.00		1.19	1.153	1.139	1.133
1.50		1.088	1.068	1.063	1.059
2.00		1.054	1.048	1.047	1.049
2.50		1.048	1.056	1.061	1.064
3.00		1.058	1.076	1.085	1.091
3.50		1.073	1.1	1.116	1.122
4.00		1.094	1.128	1.149	1.156

Table 1(a) : Predictive risk efficiencies of $\mathcal{Y}_{(1)L}^*$ w.r.t $\mathcal{Y}_{(1)Q}^*$ under linear loss (Contd.)

η/ξ	b	$(r-l+1)/n$			
		0.2	0.4	0.6	0.8
0.20	2.0	2.466	2.359	2.309	2.281
0.25		2.099	2.01	1.969	1.948
0.50		1.363	1.318	1.294	1.284
1.00		1.16	1.139	1.13	1.122
1.50		1.067	1.058	1.056	1.054
2.00		1.048	1.047	1.047	1.048
2.50		1.05	1.056	1.058	1.061
3.00		1.063	1.075	1.082	1.086
3.50		1.085	1.103	1.112	1.117
4.00		1.11	1.133	1.147	1.153
0.20	4.0	2.359	2.308	2.278	2.259
0.25		2.009	1.968	1.944	1.93
0.50		1.317	1.294	1.283	1.276
1.00		1.139	1.13	1.124	1.121
1.50		1.06	1.056	1.055	1.054
2.00		1.047	1.047	1.048	1.048
2.50		1.055	1.058	1.06	1.063
3.00		1.076	1.082	1.087	1.09
3.50		1.1	1.111	1.117	1.122
4.00		1.129	1.142	1.149	1.156
0.20	6.0	2.306	2.278	2.256	2.245
0.25		1.969	1.945	1.929	1.919
0.50		1.294	1.281	1.275	1.27
1.00		1.129	1.124	1.121	1.119
1.50		1.056	1.055	1.054	1.053
2.00		1.047	1.048	1.048	1.049
2.50		1.057	1.06	1.063	1.064
3.00		1.082	1.086	1.089	1.092
3.50		1.112	1.118	1.121	1.125
4.00		1.145	1.151	1.156	1.162

Table 2 : Predictive risk efficiencies of $\mathcal{Y}_{(1)L}^*$ w.r.t $\mathcal{Y}_{(1)Q}^*$ under quadratic loss

η/ξ	b	$(r-l+1)/n,$			
		0.2	0.4	0.6	0.8
0.20	0.25	0.944	0.748	0.697	0.679
0.25		0.959	0.764	0.722	0.702
0.50		0.972	0.833	0.806	0.791
1.00		0.989	0.939	0.931	0.933
1.50		0.985	0.978	0.984	0.99
2.00		0.998	1.005	1.009	1.007
2.50		1.004	0.998	0.989	0.981
3.00		1.008	0.972	0.948	0.93
3.50		1.004	0.933	0.895	0.869
4.00		0.999	0.889	0.84	0.81
0.20	0.50	0.891	0.739	0.695	0.676
0.25		0.902	0.757	0.717	0.7
0.50		0.925	0.828	0.803	0.794
1.00		0.973	0.936	0.93	0.933
1.50		0.983	0.981	0.99	0.993
2.00		1.001	1.007	1.012	1.01
2.50		1.011	1.001	0.991	0.982
3.00		1.005	0.972	0.948	0.933
3.50		0.997	0.931	0.894	0.874
4.00		0.984	0.829	0.836	0.814
0.20	1.00	0.823	0.722	0.693	0.679
0.25		0.832	0.743	0.711	0.697
0.50		0.884	0.821	0.8	0.793
1.00		0.959	0.936	0.935	0.935
1.50		0.981	0.984	0.989	0.993
2.00		1.005	1.012	1.013	1.012
2.50		1.012	0.998	0.988	0.982
3.00		1.001	0.963	0.942	0.933
3.50		0.98	0.92	0.888	0.871
4.00		0.953	0.869	0.833	0.809

Table 2(a) : Predictive risk efficiencies of $\mathcal{Y}_{(1)L}^*$ w.r.t $\mathcal{Y}_{(1)Q}^*$ under quadratic loss (Contd.)

η/ξ	b	$(r-l+1)/n,$			
		0.2	0.4	0.6	0.8
0.20	2.0	0.759	0.704	0.979	0.666
0.25		0.775	0.732	0.722	0.657
0.50		0.842	0.81	0.792	0.785
1.00		0.946	0.937	0.935	0.933
1.50		0.1017	0.989	0.993	0.999
2.00		1.003	1.007	1.007	1.01
2.50		1	0.992	0.986	0.982
3.00		0.976	0.952	0.937	0.93
3.50		0.943	0.901	0.879	0.868
4.00		0.908	0.848	0.821	0.804
0.20	4.0	0.702	0.681	0.667	0.657
0.25		0.723	0.702	0.688	0.678
0.50		0.809	0.793	0.785	0.777
1.00		0.938	0.935	0.933	0.932
1.50		0.987	0.992	0.995	1.001
2.00		1.004	1.007	1.007	1.011
2.50		0.989	0.984	0.977	0.977
3.00		0.949	0.936	0.926	0.923
3.50		0.901	0.878	0.865	0.858
4.00		0.848	0.82	0.804	0.792
0.20	6.0	0.683	0.679	0.659	0.653
0.25		0.704	0.69	0.68	0.674
0.50		0.796	0.787	0.781	0.776
1.00		0.937	0.935	0.932	0.932
1.50		0.989	0.995	0.997	1.002
2.00		1.004	1.006	1.007	1.01
2.50		0.98	0.977	0.973	0.975
3.00		0.932	0.926	0.919	0.917
3.50		0.873	0.864	0.854	0.851
4.00		0.818	0.981	0.789	0.782

risk efficiencies remains mostly unchanged and so $\sigma = 2.0$ has only been reported. Negligible change was found in risk efficiencies when hyperparameter a was varied for (2.0, 4.0, 6.0) whereas risk efficiencies was found significant in varying hyperparameter b . Six different values, namely 0.25, 0.50, 1.0, 2.0, 4.0 and 6.0 were considered for hyperparameter b . Different values 0.2, 0.4, 0.6 and 0.8 was considered for censoring fraction $(r - l + 1)/n$, whereas a number of values were assigned to linear loss parameter (η, ξ) so as to keep the ratio η / ξ fixed at 0.20, 0.25, 0.50, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5 and 4.0. The results are summarized in tables 1 and 2.

Table 1 present the relative efficiencies of $\mathcal{Y}_{(1)L^*}$ w.r.t $\mathcal{Y}_{(1)Q^*}$ under linear loss. It is evident from the table that, in most of the cases $\mathcal{Y}_{(1)L^*}$ performs better than that of $\mathcal{Y}_{(1)Q^*}$. It is observed that PRE_{LIN} decreases as the ratio η / ξ increases till 3.0, but PRE_{LIN} increases as the ratio η / ξ further increases beyond this limit. It may be concluded here that PRE_{LIN} attains its minimum in the neighbourhood in between 2.5 to 3.0 and increases on either side of this range.

It may be observed from tables that as we increase the value of hyperparameter b , PRE_{LIN} first decreases up to $(\eta / \xi \leq 3.0)$ then increases for smaller censoring fraction $(r - l + 1)/n = 0.2$. However for large values of censoring fraction ($=0.4, 0.6$ and 0.8), though a similar trend is noticed, but the value of η / ξ up to which PRE_{LIN} decreases, shifts towards smaller value of it. As we increase the censoring fraction $(r - l + 1)/n$, PRE_{LIN} decreases for $0.2 \leq \eta / \xi \leq 2.0$ and increases for other values, provided b is small. For large choice of $b \geq 4.0$, it decreases up to $(\eta / \xi \leq 1.5)$ and then increases.

Relative efficiencies of $\mathcal{Y}_{(1)L^*}$ w.r.t $\mathcal{Y}_{(1)Q^*}$ under squared error loss function is summarized in Table 2. In most of the cases PRE_{QRD} is observed to be less than unity. But PRE_{QRD} increases with the increase in ratio η / ξ , attains its maximum value around $(\eta / \xi = 2.0)$ and then start decreasing for censoring fraction $(r - l + 1)/n = 0.2$ and $c=0.25$, a situation is exceptional where maxima is at point $(\eta / \xi = 3.0)$. Thus predictor $\mathcal{Y}_{(1)L^*}$ with η / ξ around 2.0 can be taken without any significant loss even if quadratic loss seems to be more appropriate. It may further be deduced from the tables that for large choice of censoring fraction $(r - l + 1)/n \geq 0.4$, it may be noted that PRE_{QRD} decreases with increase in hyperparameter b except in the case when $\eta / \xi = 1.5$. However, for smaller choice of censoring fraction $(r - l + 1)/n = 0.2$, a slight increase in PRE_{QRD} is observed for η / ξ around 3.0 and when $b = 0.5, 2.0$. A decrease in PRE_{QRD} is noticed for $\eta / \xi \leq 1.0$ and $\eta / \xi > 2.5$ when censoring fraction is increased.

5. Conclusion

When data is doubly type II censored, predictor obtained under linear loss can be used safely because it is either more efficient (in case when asymmetric loss is actual loss) or almost equally efficient (in case when quadratic loss is actual loss) compared to the usual predictor obtained under quadratic loss. It needs to be mentioned here that the use of quadratic loss is advisable if one is quite sure about its sustainability. However, in all other cases proposed linear loss is recommended use for as it provides both symmetric and asymmetric loss functions.

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