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3-Minimally Nonouterplanar Graphs of Semitotal – Block Graphs and Total – Block Graphs

[1]M. H. MUDDEBINAL, [2]JAYASHREE B. SHETTY and [3]SHABBIR AHMED

[1]Department of Mathematics
Gulbarga University, Gulbarga – 585 106 (India)
mhmuddebihal@yahoo.co.in
[2]Department of Mathematics
Govt. First Grade College, Humnabad – 585 330 (India)
jayasherrbshetty@gmail.com
[3]Department of Mathematics
Gulbarga University, Gulbarga – 585 106 (India)
Glbhyb09@rediffmail.com

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Abstract

In this paper, we obtain characterizations of graphs whose semitotal-block graphs and total-block graphs are 3-minimally nonouterplanar.

1. Introduction

In[3], Kulli introduced the concepts of the semitotal-block graphs and total-block graphs. In[4] and [5], the planarity and outer planarity of these graph valued functions were discussed. In[6], one finds the minimally non-outer planarity of these graph valued functions. In[1], D.G. Akka and M.S. Patil finds the 2-minimally non-outer planarity of these graph valued functions. In this paper, we obtain the characterizations of graphs whose semitotal-block graphs and total-block graphs are 3-minimally nonouterplanar.

The following definitions will be noted for later use. A graph G is called a block if it has more than one vertex, is connected and has no cutertices.

A block of a graph G is a maximal subgraph of G which itself a block. If B = {u_1, u_2,...............u_r $r\geq2$} is a block of G, then we say that vertex u_1 and block B are incident with each other as are u_2 and B so on. If two distinct blocks B_1 and B_2 are incident with a common cutvertex, then they are adjacent blocks. The vertices and blocks of a graph are called the members.

The following will be useful in the proof of our results.

Lemma 1[6]. For the graph $K_{1,3}$, $i[T(K_{1,3})]=2$.

Theorem A[1]. The total block graph $T_B(G)$ of a connected outer planar graph G is 2-minimally nonouterplanar if and only if

1) G is a path P_n, $n\geq 3$ together with an end edge adjoined at some non-end vertex
   or
2) G is a path P_n, $n\geq 2$ together with two vertices each adjoined to a pair of adjacent vertices of P_n
   or
3) G is a cycle of length 4 together with two paths P_m and P_n ($m\geq 1$, $n\geq 2$) adjoined at two consecutive vertices
   or
4) G is a cycle C_n, $n\geq 4$ with a diagonal edge joining a pair of vertices of length exactly 2.

Theorem B[4]. The total block graph $T_B(G)$ of a graph G is planar if and only if G is outer planar and every cutvertex of G lies on at most 3 blocks.

Theorem C[5]. The total block graph $T_B(G)$ of a graph G is outer planar if and only if each component of G is a path.

Theorem D[6]. A graph G is a cycle if and only if the semitotal – block graph and total-block graph are isomorphic to a wheel.

Theorem E[2]. The total graph T (G) of a graph G is planar if and only if the maximum degree among the vertices of G is atmost 3 and every vertex of degree 3 is a cutvertex.

Theorem F[3]. A connected graph G is a tree if and only if the graph T(G) and $T_B(G)$ are isomorphic.

2. Main Results:

A criterion for the semitotal block graph of a connected graph G to be 3-minimally nonouterplanar is given in the following theorem.

Theorem 1. The semitotal block graph $T_b(G)$ of a connected graph G is 3-minimally nonouterplanar if and only if (1) or (2) holds.

1) G has exactly three cycles and each cycle is a block
   or
2) G is a cycle C_n ($n\geq 6$) together with a diagonal edge joining a pair of vertices of length (n-3).

Proof. Suppose $T_b(G)$ is 3-minimally nonouterplanar. Then $T_b(G)$ is planar.

We now consider the following cases.

Case 1. Assume G is a tree. Then every block of $T_b(G)$ is a triangle. Hence $T_b(G)$, is outer planar, a contradiction.

Case 2. Assume G is not a tree.

We consider the following subcases of case 2.

Subcase 2.1. Suppose G has four

cycles. Then we have following subcases of subcase 2.1.

Subcase 2.1.1. Assume each cycle is a block. Then each cycle in $T_b(G)$ gives a wheel. Hence, i $[T_b(G)]>3$, a contradiction.

Subcase 2.1.2. Assume G has two cycles C_1 and C_2 as blocks. Then the remaining block is isomorphic to K_4-x In $T_b(G)$, C_1 and C_2 gives wheels as W_1 and W_2, where as $i(K_4\text{-}x)=2$. Thus, $i[T_b(G)]>3$, a contradiction.

Subcase 2.1.3. Assume G has two cycles C_1 and C_2 as blocks, which are isomorphic to $(K_4\text{-}x)$. Then in $T_b(G)$ $i(K_4\text{-}x)=2$. Hence, i $[T_b(G)]>3$, a contradiction.

Subcase 2.1.4. Assume G has four cycles as a block B, and remaining blocks are edges of G. Thus, G is a maximal outer planar graph with 6 vertices. In $T_b(G)$ the block vertex b is adjacent with each vertex of B. Thus $i[T_b(G)]>3$, a contradiction.

Subcase 2.2. Suppose G has three cycles. Then there exists two blocks B_1 and B_2 in which one block B_1 is a cycle and B_2 is isomorphic to K_4-x such that atleast three vertices of K_4-x are adjacent to atleast one block. In embedding of $T_b(G)$, $i[T_b(B_1)]=1$ and $i[T_b(B_2)]>2$. Hence $i[T_b(G)]>3$, a contradiction.

Subcase 2.3. Suppose G has two cycles. Then we have subcases of subcase 2.3.

Subcase 2.3.1. Assume each cycle is a block. Then each block and corresponding block vertices forms wheel in $T_b(G)$. Hence, $i[T_b(G)]<3$, a contradiction.

Subcase 2.3.2. Assume G has two cycle as a block. Then we consider the following subcases of subcase 2.3.2.

Subcase 2.3.2.1. Suppose G is isomorphic to K_4-x. Then $i[T_b(G)]<3$, a contradiction.

Subcase 2.3.2.2. Suppose a vertex of K_4-x is adjacent to some blocks. Then the block vertex b corresponds to K_4-x is adjacent to all vertices of K_4-x. In embedding $T_b(G)$ in any plane, we have $i[T_b(G)]<2$, a contradiction.

Subcase 2.3.2.3. Suppose two vertices of K_4-x are adjacent to some blocks. Then the block vertex b corresponds to $(K_4\text{-}x)$ is adjacent to all vertices of K_4-x. In embedding $T_b(G)$ in any plane, we have $i[T_b(G)]<3$, a contradiction.

Subcase 2.3.2.4. Suppose three vertices of $(K_4\text{-}x)$ are adjacent to atleast one block. Then in $T_b(G)$ the edges joining the block vertex of K_4-x and all vertices of K_4-x generates the planar representation such that the block vertices of blocks which are adjacent to three vertices of K_4-x lies in the interior region of $T_b(G)$ with $i[T_b(G)] >3$, a contradiction.

Subcase 2.3.2.5. Suppose each vertex of K_4-x is adjacent to atleast one block. Then the block vertex b corresponds to $(K_4\text{-}x)$ is adjacent to all vertices of $(K_4\text{-}x)$. In plane embedding of $T_b(G)$. We have $i[T_b(G)]>4$, a contradiction.

Subcase 2.4. Suppose G is unicyclic graph. Then $i[T_b(G)]<3$, a contradiction.

Case 3. Assume G is a cycles C_n ($n\geq6$). Then we have following subcases of case 3.

Subcase 3.1. Suppose G is a cycle C_n ($n\geq6$) as a block, with diagonal edge joining a pair of vertices of length (n-3). Then G contains one more cycle C'_n as a block, clearly $i[T_b(G)]>3$, a contradiction.

Subcase 3.2. Suppose G is a cycle C_n ($n\geq6$) as a block, together with diagonal edge joining a pair of vertices of length (n-4). Then $i[T_b(G)]>3$, a contradiction.

Subcase 3.3. Suppose G is a cycle C_n($n\geq6$) as a block, together with diagonal edge joining a pair of vertices of length (n-2). Then $i[T_b(G)]<3$, a contradiction.

Conversely, suppose (1) holds. Then G has exactly 3 cycles and each cycle is a block. By Theorem D, $T_b(G)$ has exactly three wheels as blocks. We know that every wheel is a minimally nonouterplanar. Thus $i[T_b(G)]=3$.

Suppose (2) holds, now we can make use of mathematical induction on n of cycle C_n Suppose n=6. Then G is a cycle C_6 with the vertices $\{v_1, v_2.........v_6\}$, together with diagonal edge x joining a pair of vertices v_1 and v_4 of length 3. So that G has two cycles C'_4 and C''_4 with the vertices v_1, v_2, v_3, v_4, v_1 and v_1, v_4, v_5, v_6, v_1 respectively. Since C_P, $P\geq6$ is a block, let b be a block vertex in $T_b(G)$ which is adjacent to all the vertices of C_P, $P\geq6$. In planar embedding of $T_b[C_6]$. It is easy to see that the planar embedding of $T_b(G)$, either v_2, v_3 of cycle C'_4 or v_5, v_6 of cycle C''_4 together with a block vertex b lie in the interiorregion of planar embedding. Hence, $T_b(G)$ is 3-minimally nonouterplanar. Assume that result is true for n=k. Then G is a cycle of length C_k, clearly $T_b(G)$ is (K-3) – minimally nonouterplanar.

Suppose n=k+1. Then G is a cycle of length C_{k+1}. Then we have to prove that $T_b(G)$ is (k-2) – minimally nonouterplanar.

Let v_{k+1} be vertex on a cycle C_{k+1}. If we delete a vertex v_{k+1} from a cycle C_{k+1} by deleting the edges $e_k=(v_{k+1}, v_k)$ and $e_{k+1} = (v_{k+1}, v_1)$ which are incident with a vertex v_{k+1}, resulting a cycle of length C_k. By inductive hypothesis $T_b(C_k)$ is (k-3) – minimally nonouterplanar. Now rejoining a vertex v_{k+1} to a cycle C_k by joining the edges e_{k+1} and e_k, resulting a cycle of length C_{k+1}. It has two cycles C_4 with the vertices v_1, v_2, v_3, v_4, v_1, and C'_k with the vertices $v_1, v_4, v_5,..............v_k, v_{k+1}, v_1$. In $T_b[C_{k+1}]$ the block vertex b corresponds to C_{k+1} is adjacent to all the vertices of C_{k+1}. Such that v_2, v_3, b lies in the interior region of planar embedding.

Hence, $T_b(G)$ has $[(k+1)-3]=(k-2)$ minimally nonouterplanar.

Hence the proof.

In the following theorem, we establish a criterion for the total – block graph of a connected graph to be 3-minimally nonouterplanar.

Theorem 2. The total – block graph $T_B(G)$ of a connected outer planar graph G is 3-minimally nonouterplanar if and only if.

1) G has exactly three triangles as blocks, such that atmost two blocks lie on a common cut vertex,

   or

2) G has exactly two cycles C_3 and C_4 as blocks,

   or

3) G is a triangle together with two paths P_m and P_n $(m\geq 2, n\geq 2)$ incident at a same vertex,

   or

4) G is a triangle together with paths P_m, P_n $(m\geq 2, n\geq 2)$ and P_2 incident at different vertices,

   or

5) G is a cycle C_5 together with a path P_n, $(n\geq 2)$ incident to a vertex of C_5,

   or

6) G is a cycle C_5 together with two paths P_m and P_n $(m\geq 1, n\geq 1)$ adjoined at two consecutive vertices,

   or

7) G is a cycle of length $C_n(n\geq 5)$ together with two diagonal edges each joining a pair of vertices of length exactly two or together with two diagonal edges each joining a pair of vertices of length two and three which are adjacent,

   or

8) G is a cycle of length $C_n(n\geq 6)$ together with a diagonal edge joining a pair of vertices of length exactly 3.

Proof. Suppose $T_B(G)$ is 3-minimally nonouterplanar. Then $T_B(G)$ is planar.

We now consider the following cases.

Case 1. Assume G is a tree. Then by Theorem F, T(G) and $T_B(G)$ are isomorphic and hence by Theorem E, G has maximum degree atmost 3 and every vertex of degree 3 is a cut vertex.

We consider subcases of case 1.

Subcase 1.1. Suppose G has atleast two cut vertices of degree 3. Then G has two subgraphs which are isomorphic to $K_{1,3}$. Then by Lemma 1, $i[T(K_{1,3})]=2$. Since $T(K_{1,3}) = T_B(K_{1,3})$, $i[T_B(K_{1,3})]=2$, Since $T_B(K_{1,3}) \subset T_B(G)$, $i[T_B(G)]>3$, a contradiction.

Subcase 1.2. Suppose G has a vertex v lies on 3 blocks and each block has no end vertex. Then G has a subgraph isomorphic to $S(K_{1,3})$. On planar embedding of $T_B(G)$, $i[T_B(S(K_{1,3}))]\geq 4$. Since $S(K_{1,3})$ is a subgraph of G, $i[T_B(G)]\geq 4$, a contradiction.

Subcase 1.3. Suppose G has a vertex v lies on 3-blocks in which atleast one block has an end vertex of G. Then by condition (1) of Theorem A, $T_B(G)$ is a 2-minimally nonouterplanar, a contradiction.

Case 2. Assume G is not a tree.

We consider the following subcases of case (2).

Subcase 2.1. Suppose G has three cycles. Then we have the following subcases of subcase 2.1.

Subcase 2.1.1. Assume G has three cycles, in which two cycles are C_3 and other cycle is $C_n(n\geq 4)$. In $T_B(G)$, each C_3 gives K_4. Then $i(K_4) = 1$. For the cycle $C_n(n\geq 4)$, $i(C_n) \geq 2$. Hence, $i[T_B(G)]>3$, a contradiction.

Subcase 2.1.2. Assume G has three cycles C_3 as blocks and these three blocks lie on a common cut vertex. Then in $T_B(G)$, each cycle C_3 and corresponding block vertex forms K_4 as a subgraph. But in $T_B(G)$ the three block vertices of cycles are mutually adjacent. Further the edges joining the block vertices of C_3 increases the inner vertex number in a planar embedding. Hence, $i[T_B(G)]>3$., a contradiction.

Subcase 2.2. Suppose G has four cycles C_3 as blocks. Then we have subcases of subcase 2.2.

Subcase 2.2.1. Assume G has four cycles C_3 as blocks, such that each two C_3 lie on a common cut vertex. The block vertices corresponds to cycles C_3 and corresponding vertices of cycles C_3 are adjacent in $T_B(G)$. Then each cycle C_3 forms K_4 as subgraphs in $T_B(G)$. Since, the block vertices are adjacent in $T_B(G)$. Then the edges joining these three vertices generates the increase in the inner vertex number. Thus, $i[T_B(G)]>3$, a contradiction.

Subcase 2.2.2. Assume there exists a bridge between the cycles C_3. In $T_B(G)$ each cycle C_3 forms K_4 and bridges form triangles as subgraphs. In $T_B(G)$ the block vertices are also adjacent. Thus, $i[T_B(G)]>3$, a contradiction.

Subcase 2.3. Suppose G has two cycles, $C_n(n\geq 3)$. Then we consider following subcases of subcase 2.3.

Subcase 2.3.1. Assume G has cycles C_3, as blocks B_1 and B_2. Then in $T_B(G)$ the block vertices b_1 and b_2 corresponds to B_1 and B_2, which are adjacent to every vertex of B_1 and B_2. Also block vertices b_1 and b_2 are adjacent in $T_B(G)$. Hence, in K_4 is an induced subgraph in $T_B(G)$. Clearly $i[T_B(G)]<3$, a contradiction.

Subcase 2.3.2. Assume G has exactly two cycles C_4 as a block. Then in $T_B(G)$, each C_4 has $i[T_B(C_4)]=2$. Since each $T_B(C_4)$ is an induced subgraph of $T_B(G)$. Then $i[T_B(G)]>3$, a contradiction.

Subcase 2.3.3. Assume G has C_3 and C_5 as blocks. Then in planar embedding of $T_B(G)$, $i[T_B(C_3)]=1$ and $i[T_B(C_5)]=3$. Thus $i[T_B(G)]>3$, a contradiction.

Subcase 2.3.4. Assume G has C_4 and C_5 as blocks. Then by subcase 2.3.2 and subcase 2.3.3, $i[T_B(G)]>3$, a contradiction.

Subcase 2.4. Suppose G has a triangle

together with 3 paths $P_n(n\geq2)$ incident at a unique vertex. Suppose each path is of length atmost two. Then in depicting the $T_B(G)$ in any plane, $i[T_B(G)]>3$, a contradiction.

Subcase 2.5. Suppose G has a triangle together with paths $P_n(n\geq2)$ incident at different vertices. Suppose three paths as P_3 which are incident at different vertices. Then in $T_B(G)$ each P_3 forms triangle and a triangle of G forms a subgraph as K_4. The edges $ei \in E[T_B(G)]$ which are incident to the blocks vertices of paths of G generates the inner vertex number of $T_B(G)$ as $i[T_B(G)]>3$, a contradiction.

Subcase 2.6. Suppose G is exactly one cycle $C_n(n\geq3)$ together with a path $P_n(n\geq2)$ incident at a vertex of a cycle. Then we consider following subcases of subcase 2.6.

Subcase 2.6.1. Assume G is a C_3 as a block B, together with a path $P_n(n\geq2)$ incident at a vertex. Then in $T_B(G)$ block vertex b corresponds to C_3 is adjacent to every vertex of B. Thus $i[T_B(C_3)]=1$. The remaining blocks of G gives a triangle as subgraph in $T_B(G)$. Thus $i[T_B(P_n)] = 0$. Hence $i[T_B(G)]<3$, a contradiction.

Subcase 2.6.2. Assume G is a cycle C_4 together with a path $P_n(n\geq2)$ incident at a vertex. Then in planar embedding of $T_B(G)$, it is easy to see that $i[T_B(G)]=2$. Hence $i[T_B(G)]<3$, a contradiction.

Subcase 2.6.3. Assume G has a cycle C_6 as a block B, together with a path $P_n(n\geq2)$ incident at a vertex. Then in $T_B(G)$ block vertex b which corresponds to C_6 is adjacent to every vertex of B. Thus $i[T_B(C_6)]>3$. The remaining blocks of G forms in $T_B(G)$ gives as outer planar induced subgraphs. Thus $i[T_B(P_n)]=0$. Hence, $i[T_B(G)]>3$, a contradiction.

Subcase 2.7. Suppose G is a cycle $C_n(n\geq4)$ together with two paths P_m and P_n $(m\geq1, n\geq1)$ adjoined at two vertices of a C_n $(n\geq4)$. Then we consider the following subcases of subcase 2.7.

Subcase 2.7.1. Assume G is a C_4 together with two paths P_m and P_n $(m\geq1, n\geq1)$ adjoined at two consecutive vertices. Then by Theorem A of condition (3), $i[T_B(G)]=2$. Hence, $i[T_B(G)]<3$, a contradiction.

Subcase 2.7.2. Assume G is a C_4 as a block B together with two paths P_m and P_n $(m\geq1, n\geq1)$ adjoined at two alternate vertices. Then in $T_B(G)$ the block vertex b which corresponds to B is adjacent to every vertex of B and adjacent block vertices of the paths in $T_B(G)$. In any planar embedding $i[T_B(G)]>3$, a contradiction.

Subcase 2.7.3. Assume G has a cycle C_5 together with two paths P_m and P_n $(m\geq1, n\geq1)$ adjoined at two alternate vertices of C_5. Then it is easy to see that $i[T_B(G)]>3$ in any planar embedding of $T_B(G)$, a contradiction.

Subcase 2.8. Suppose G is a cycle of length $C_n(n\geq5)$ together with two diagonal edges joining a pair of vertices. Then we consider the following subcases of subcase 2.8.

Subcase 2.8.1. Assume two diagonal edges joining a pair of vertices of length exactly three. Let B be a block of a C_n ($n \geq 5$). Then in $T_B(G)$ block vertex b corresponds to C_n is adjacent to every vertex of B. In planar embedding of $T_B(G)$ in any plane we have $i[T_B(G)]=5$. Hence, $i[T_B(G)]>3$, a contradiction.

Subcase 2.8.2. Assume two diagonal edges joining a pair of vertices of length two and three from same vertex to two alternate vertices. Then in planar embedding of $T_B(G)$ it is easy to see that $i[T_B(G)]=4$. Hence, $i[T_B(G)]>3$, a contradiction.

Subcase 2.8.3. Assume two diagonal edges joining a pair of vertices of length exactly three from same vertex to two consecutive vertices. Then in planar embedding of $T_B(G)$, the four vertices of a cycle C_n and block vertex b corresponding to cycle C_n are the only five inner vertices in $T_B(G)$. Thus $i[T_B(G)]=5$. Hence, $i[T_B(G)]>3$, a contradiction.

Subcase 2.9. Suppose G is a cycle C_n ($n \geq 6$) as a block together with a diagonal edge joining a pair of vertices. Then we have following subcases of subcase 2.9.

Subcase 2.9.1. Assume a diagonal edge joining a pair of vertices of length exactly three. Then G contains one more cycle C'_n as a block. Clearly $i[T_B(G)]>3$, a contradiction.

Subcase 2.9.2. Assume a diagonal edge joining a pair of vertices of length four. Then $i[T_B(G)]>3$, a contradiction.

Subcase 2.9.3. Assume a diagonal edge joining a pair of vertices of length two. Then $i[T_B(G)]<3$, a contradiction.

Conversely, suppose (1) holds. Then G has exactly 3 cycles and each cycle is a block, such that every cut vertex of G lies on atmost two blocks and each triangle has atleast one vertex of degree two. Then by Theorem D, $T_B(G)$ has exactly three wheels as blocks. We know that every wheel is a minimally nonouterplaner. Hence, $i[T_B(G)]=3$.

Suppose (2) holds. Then G has cycles C_4 and C_3 as blocks. We have following cases.

Case 1. Assume the cycle C_4 and C_3 have a vertex in common. Let cycle C_4 as vertices $\{v_1, v_2, v_3, v_4\}$ cycle C_3 as vertices $\{v_4, v_5, v_6\}$ in which v_4 is a cut vertex incident to both C_3 and C_4. Then in $T_B(G)$ the block vertex b_1 and corresponding vertices of cycle C_4 are mutually adjacent in the exterior region of a cycle C_4. Thus, vertices v_1 and v_2 of a cycle C_4 are only two inner vertices in $T_B(G)$. Similarly the block vertex b_2 and corresponding vertices of a cycle C_3 are mutually adjacent in the exterior region of a cycle C_3. Thus, vertex v_6 of a cycle C_3 is only one inner vertex in $T_B(G)$. Hence, $T_B(G)$ has three inner vertices. Thus $i[T_B(G)]=3$. This proves (2).

Case 2. Assume the path P_n ($n \geq 2$) in between the cycles C_4 and C_1. Let cycle C_4 as vertices $\{v_1, v_2, v_3, v_4\}$ and cycle C_3 as vertices $\{v'_4, v_5, v_6\}$ and path P_n as vertices $P_n = \{p_1, p_2, ...p_{n-1}, p_n\}$ with out lose of generality let us assume that vertex v_4 of cycle C_4 is coincide with vertex p_1 of path P_n and similarly vertex v'_4 of cycle C_3 coincide with vertex P_n of path P_n. Let b, b′ and $b_1, b_2, ... b_{n-1}$ are the block vertices corresponds to cycle C_4, C_3 and path P_n respectively. Then in $T_B(G)$ the block vertex b and corresponding vertices of cycle C_4 are embedded in a plane in such a way that they are mutually adjacent in the exterior region of a cycle C_4. Thus, vertices v_1, v_2 of a cycle C_4 are only two inner vertices in $T_B(G)$. Similarly block vertex b′ and corresponding vertices of cycle C_3 are mutually adjacent in the exterior region of a cycle C_3. This forms K_4 as a subgraph in $T_B(G)$. Thus, vertex v_6 is only one inner vertex in $T_B(G)$. The block vertices $b_1, b_2, ...b_{n-1}$ are also adjacent to the corresponding vertices of path P_n. This forms triangles in $T_B(G)$. Here $T_B(G)$ is outer planar. Thus $T_B(G)$ has exactly three inner vertices as v_1, v_2, from cycle C_4 and v_6 from cycle C_3. Hence, $i[T_B(G)]=3$. This proves (2).

Suppose (3) holds. Let G is a triangle with vertices $\{v_1, v_2, v_3\}$, path P_m with vertices $\{p_1, p_2, ... p_{m-1}, p_m\}$ and path P_n with vertices $\{p'_1, p'_2, ... p'_{n-1}, p'_n\}$, with out lose of generality let us assume that paths P_m and P_n are incident at vertex v_1 of a triangle. Then vertex v_1 of a triangle, vertex p_1 of path P_m and vertex p'_1 of path P_n are coincide. Clearly $(G-v_1)$ has disjoint paths. Then by Theorem C. $T_B(G-v_1)$ is outerplanar. Let b be the block vertex corresponding to a triangle, $b'_1, b'_2, ...b'_{m-1}$ are the block vertices corresponding to a path P_m and $b''_1, b''_2, ... b''_{n-1}$ are the block vertices corresponding to a path P_n. Then in $T_B(G)$ block vertex b corresponding vertices of a triangle are mutually adjacent. Then $T_B(C_3)$ is isomorphic to a wheel. The block vertices $b'_1, b'_2, ...b'_{m-1}$ and corresponding vertices $\{p_1, p_2, ... p_{m-1}, p_m\}$ of a path P_m forms $v_1p_2b'_1$, $p_2p_3b'_2$, $p_3p_4b'_3$, ... $p_{m-1}p_m b'_{m-1}$ as triangles as an induced subgraphs in $T_B[G]$. Similarly path P_n forms $v_1p'_2b''_1$, $p'_2p'_3b''_2$, ...$p'_{n-1}p'_nb''_{n-1}$ as triangles as an induced sub graphs in $T_B(G)$. The vertices, v_2, v_3 of a triangle and corresponding block vertex b are only three inner vertices in $T_B(G)$ in any planar embedding. Hence $i[T_B(G)]=3$. This proves (3).

Suppose (4) holds. Let G is a triangle with vertices $\{v_1, v_2, v_3\}$, path P_m with vertices $\{p_1, p_2, ... p_{m-1}, p_m\}$, path P_n with vertices $\{p'_1, p'_2, ... p'_{n-1}, p'_n\}$ and path P_2 with vertices $\{x_1, x_2\}$. Without lose of generality let us assume that paths P_m, P_n and P_2 are incident at v_1, v_2, v_3 respectively. Then the vertices $v_1=p_1$, $v_2=p'_1$ and $v_3=x_1$. Let b be the block vertex corresponding to a triangle, $b'_1, b'_2, ...b'_{m-1}$ are the block vertices corresponding to a path P_m, $b''_1, b''_2, ... b''_{n-1}$ are the block vertices corresponding to a path P_n and b′is a block vertex corresponding to a path P_2. Then in $T_B(G)$ block vertex b corresponding vertices of a triangle are mutually adjacent. Then

$T_B(C_3)$ is isomorphic to a wheel. The block vertices $b'_1, b'_2, \ldots b'_{m-1}$ and corresponding vertices $\{p_1, p_2, \ldots p_{m-1}, p_m\}$ of a path P_m forms $v_1p_2b'_1, p_2p_3b'_2, p_3p_4b'_3, \ldots\ldots p_{m-1}\, p_mb'_{m-1}$ as triangles as an induced subgraphs in $T_B(G)$. Similarly path P_n forms $v_2p'_2b''_1$, $p_2p'_3\, b''_2, \ldots\ldots p'_{n-1}p'_n\, b''_{n-1}$ as triangles as an induced subgraphs in $T_B(G)$ and also path P_2 and corresponding block vertex b' form v_3x_2b' as a triangle as an induced subgraphs in $T_B(G)$. The vertex v_3 of a triangle, vertex x_2 of a path P_2 and corresponding block vertex b' of a path P_2 are exactly three inner vertices in $T_B(G)$ in any planar embedding.

Hence, $i[T_B(G)]=3$.

This proves (4).

Suppose (5) holds. Let G is a cycle C_5 as a block B, with vertices $\{v_1, v_2, v_3, v_4, v_5\}$ and path P_n with vertices $\{p_1, p_2, \ldots\ldots p_n\}$. With out lose of generality let us assume that path P_n is incident at vertex v_1 of a cycle C_5. Thus vertex $v_1=p_1$. Clearly $(G\text{-}v_1)$ has disjoint paths. Then by Theorem C, $T_B(G\text{-}v_1)$ is outer planar. Let b be a block vertex corresponding to a cycle C_5 and $b_1, b_2, \ldots\ldots b_{n-1}$ are block vertices corresponding to a path P_n. Then in $T_B(G)$ block vertex b corresponding to a cycle C_5 mutually adjacent to every vertex of B. Then $T_B(C_5)$ is isomorphic to a wheel. The block vertices $b_1, b_2, \ldots b_{n-1}$ and corresponding vertices $\{p_1, p_2, \ldots p_n\}$ of a path P_n forms $v_1p_2b_1$, $p_2p_3b_2, p_3p_4b_3, \ldots\, p_{n-1}p_nb_{n-1}$ triangles in $T_B(G)$, as an induced subgraph in $T_B(G)$. The vertices v_2, v_3 and v_4 of a cycle C_5 are only three inner vertices in $T_B(G)$ in any planar embedding. Hence, $T_B(G)$ is a 3-minimally nonouterplanar. This proves (5).

Suppose that (6) holds. Let G is a cycle C_5 with vertices $\{v_1, v_2, v_3, v_4, v_5\}$, path P_m with vertices $\{p_1\, p_2, \ldots\ldots p_{m-1}\, p_m\}$ and path P_n with vertices $\{p'_1\, p'_2, \ldots\ldots p'_{n-1}\, p'_n\}$, without lose of generality let us assume that paths P_m and P_n incident at vertex v_1 and v_5 respectively. Thus, vertex $p_1=v_1$ and vertex $p'_1 = v_5$. Clearly $(G\text{-}v_1)$ or $(G\text{-}v_5)$ has disjoint paths. Then by Theorem C, $T_B(G\text{-}v_1)$ or $T_B(G\text{-}v_5)$ is outer planar. Let b be the block vertex corresponding to a cycle C_5, $b'_1, b'_2, \ldots\ldots b'_{m-1}$ are the block vertices corresponding to a path P_m and $b''_1, b''_2, \ldots\ldots b''_{n-1}$ are the block vertices corresponding to a path P_n. Then in $T_B(G)$ block vertex b corresponding vertices of cycle C_5 are mutually adjacent. Then $T_B(C_5)$ is isomorphic to a wheel. The block vertices $b'_1, b'_2, \ldots b'_{m-1}$ and corresponding vertices $\{p_1, p_2, \ldots p_{m-1}, p_m\}$ of a path P_m forms $v_1p_2b'_1, p_2p_3b'_2, p_3p_4b'_3, \ldots\ldots\ldots p_{m-1}\, p_m\, b'_{m-1}$ as triangles as an induced subgraphs in $T_B(G)$. Similarly path P_n forms $v_5p'_2b''_1$, $p'_2p'_3b''_2, \ldots\ldots p'_{n-1}p'_nb''_{n-1}$ as triangles as an induced subgraphs in $T_B(G)$. The vertices v_2, v_3, v_4 of a cycle C_5 are only three inner vertices in $T_B(G)$ in any planar embedding. Hence $i[T_B(G)]=3$. This proves (6).

Assume that (7) holds G has a cycle

$C_n(n\geq5)$ as a block together with two diagonal edges.

We have the following cases.

Case 1. Assume a cycle C_6 with vertices $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ together with two diagonal edges adjoined at v_1, v_3, and v_4, v_6. Clearly $(G\text{-}v_1)$ is a path. Let b be a block vertex corresponding to a cycle C_6. Then in $T_B(G)$ block vertex b is adjacent to every vertex of a cycle C_6. Then v_1 and v_3 are joined in the outer region of a cycle C_6. Then vertex v_2 becomes inner vertex. Similarly vertices v_4 and v_6 are also joined in the outer region of a cycle C_6. Then vertex v_5 becomes inner vertex in $T_B(G)$. But block vertex b is adjacent to every vertex of a cycle C_6. This forms a wheel in $T_B(G)$. Thus, vertices v_2, v_5 of a cycle C_6 and block vertex b are only three inner vertices in $T_B(G)$. Hence, $T_B(G)$ is a 3-minimally nonouterplanar.

Case 2. Assume a cycle C_5 with vertices $\{v_1, v_2, v_3, v_4, v_5\}$ together with two diagonal edges adjoined at v_1, v_3, and v_1, v_4. Clearly $(G\text{-}v_1)$ is a path. Then by Theorem C. $T_B(G\text{-}v_1)$ is outer planar. Let b be the block vertex corresponding to a cycle C_5. Then in $T_B(G)$ block vertex b is adjacent to every vertex of a cycle C_5. Then vertices v_1 and v_3 are joined in the outer region of a cycle C_5. Then vertex v_2 becomes inner vertex. Similarly vertices v_1 and v_4 are joined in the outer region of a cycle C_5. Then vertex v_3 becomes inner vertex. In $T_B(G)$ the block vertex b is adjacent to every vertex of a cycle C_5. This forms a wheel in $T_B(G)$. Thus, vertices v_2, v_3 of a cycle C_5 and block vertex b are only three inner vertices in $T_B(G)$. Hence, $i[T_B(G)]=3$. This proves (7).

Suppose (8) holds. Then G has a cycle $C_n(n\geq6)$ with vertices $v_1, v_2, v_3, v_4, \ldots\ldots v_n$ such that $e=v_3v_n$ is an edge joining the two distinct vertices of C_n. Then G has two cycles v_1, v_2, v_3, v_n, v_1 and $v_3, v_4, \ldots\ldots v_n, v_3$. Since G is a block let b be a block vertex of G. In $T_B(G)$, the block vertex b and all vertices of C_n $(n\geq6)$ are adjacent. Further, in planar embedding of $T_B(G)$, the edge $e=v^3vn$ drawn in such a way that the vertices v_1, v_2 and b lie in the interior region of G, which gives $i[T_B(G)]=3$.

This completes the proof.

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