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New Characterizations of Urysohn Spaces Using Regularly Open Sets and Subspaces

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Abstract

Within this paper, Urysohn spaces are further characterized using regularly open sets, and open and regularly open subspaces.

Key words: Urysohn, regularly open sets, and subspaces.

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1. Introduction

Within this paper all spaces are topological spaces.

Urysohn spaces were introduced by P. Urysohn in 1925[10].

Definition 1.1. A space (X,T) is Urysohn iff for distinct elements x and y in X, there exist open sets U and V such that x is in U, y is in V, and Cl(U) and Cl(V) are disjoint.

The Urysohn separation axiom has been further investigated and characterized through the years and is, today, included within the study of classical topology. As a result of its inclusion and importance in classical topology, Urysohn spaces continue to be investigated and the search for additional characterizations and properties of Urysohn spaces continues. Within this paper Urysohn spaces are further investigated and characterized using regularly open sets, related sets and processes, and subspaces.

The study of subspaces is also included within classical studies of topology. Until recently, the question concerning subspaces for a property P within a topological space has been "Does a space have property P if and only if every subspace has property P? Within this paper, properties for which the above statement is true are called subspace properties. The proofs of the converse statement for the subspace theorem cited above are all the same with the property itself only mentioned: "Since the space is a subspace of itself and every subspace has the property, then the space has the property." As a result proper subspace inherited properties were introduced and investigated, giving the properties themselves a new, meaningful role in subspace questions[1].

Definition 1.1. Let (X,T) be a space and let P be a property of topological spaces. If (X,T) has property P when every proper subspace of (X,T) has property P, then P is said to be a proper subspace inherited property[1].

Since singleton set spaces satisfy many topological properties, within the recent paper[1] only spaces with three or more elements were considered. Each of the subspace properties T_0, T_1, T_2, Urysohn, regular, and T_3 proved to be proper subspace inherited properties giving new characterizations for each of those properties.

Theorem 1.1. A space (X,T) has property P iff every proper subspace of (X,T) has property P, where P can be each of the properties cited above[1].

The results above raised the question of whether or not topological properties could be further characterized using only certain types of sets within the space as subspaces. With the important role of open and closed sets in the study of topology, a natural place to start such an investigation would be with open set and closed set subspaces, which led to new characterizations of T_0 spaces[2], T_1 spaces[3], and T_2 spaces[4].

The successes using open set and closed set subspaces for the T_0 and T_1 separation axioms raised questions about other types of sets in a topological space that could be considered for subspace questions leading to the consideration of regularly open sets.

Regularly open sets were introduced in 1937[9].

Definition 1.2. Let (X,T) be a space and let A be a subset of X. Then A is regularly open if and only if A= Int(Cl(A)).

Within the 1937 paper[9] it was shown that the set of regularly open sets of a space (X,T) forms a base for a topology Ts on X coarser than T and the space (X,Ts) was called the semiregularization space of (X,T). The space (X,T) is semiregular if and only if the set of regularly open sets of (X,T) is a base for T[9].

The introduction of regularly open sets led to the introduction of regularly closed sets.

Definition 1.3. Let (X,T) be a space and let C be a subset of X. Then C is regularly closed iff one of the following equivalent conditions is satisfied: (1) X\C is regularly open and (2) C= Cl(Int(C))[11].

Since their introduction, each of regularly open and regularly closed sets have been further investigated giving additional knowledge and insights in the study of topology. The investigation of subspace questions for regularly open sets[5] led to the following discoveries, which are used in this paper. For a space (X,T), the regularly open sets of (X,T) equal the regularly open sets of (X,Ts). The semiregularization process generates at most one new topology. Thus a space (X,T) is semiregular if and only if T = Ts. For an open set O in a space (X,T), $Cl_T(O)= Cl_{Ts}(O)$ and $(Ts)_O = (T_O)s$, which led to the discovery that semiregular is an open set subspace property, but not a proper open set inherited property.

Examples are known showing that semiregular is not a closed set subspace property and that for a closed set C in a space (X,T), $(Ts)_C$ need not be (T_C)s.

In a follow-up paper[6], regularly open T_i spaces; i= 0, 1, 2, and regularly open Urysohn spaces were defined by replacing the word open in the definition of T_i; i= 0,1,2, and Urysohn by regularly open, respectively. Within the paper[7], regularly open T_0 spaces were further investigated using subspaces and in the paper[8], regularly open T_1 spaces were further investigated using subspaces. In the paper[4], T_2 spaces were further investigated using convergence and subspaces. Results in that paper prove to be extremely useful in this paper.

Within this paper, Urysohn spaces are further investigated using regularly open and related sets and processes and subspaces. As in the cited investigations above, all spaces in this paper will have three or more elements.

2. Regularly Open Urysohn Spaces

As indicated above, in the paper[6], regularly open Urysohn spaces were defined.

Definition 2.1. A space (X,T) is regularly open Urysohn iff for distinct elements x and y in X, there exist disjoint regularly open sets U and V such that x is in U, y is in V, and Cl(U) and Cl(V) are disjoint.

Within that paper[6], it was proven that for a space (X,T) the following are equivalent: (a) (X,T) is Urysohn, (b) (X,Ts) is Urysohn, and (c) (X,T) is regularly open Urysohn.

Combining this result with the fact that (Ts)s= Ts, gives the next result.

Corollary 2.1. A space (X,T) is regularly open Urysohn iff (X,Ts) is regularly open Urysohn.

Hence both Urysohn and regularly open Urysohn are semiregularization properties, i.e., properties simultaneously shared by both (X,T) and (X,Ts).

The results above raised the following question: "If (X,T) is Urysohn, and thus regularly open Urysohn, must T= Ts?" The following example shows the answer is "no."

Example 2.2. Let X denote the real numbers, let Q denote the set of rational numbers, let T be the usual absolute metric topology on X, and let B be T unioned with {(a,b) intersection Q: a and b are real numbers, a < b}. Then B is a base for a topology W of X, (X,W) is Urysohn, and W is not Ws.

3. Urysohn Spaces and Subspaces.

Theorem 3.1. Let (X,T) be a space. Then the following are equivalent: (a) (X,T) is Urysohn, (b) for each subset Y of X, $(Y,(Ts)_Y)$ is Urysohn, and (c) for each proper subset Y of X, $(Y,(Ts)_Y)$ is Urysohn.

The proof is straightforward using the results above and is omitted.

The following example shows that Urysohn is not a proper T-open set property or a proper Ts-open set property or proper regularly open set property or proper T-closed set property or proper regularly closed set property.

Example 3.1. Let X= {a,b,c} and let T= {φ,X,{a},{c},{a,c}}. Then (X,T)= (X,Ts) is not Urysohn but it is proper T-open set Urysohn, proper regularly open set Urysohn, and proper regularly closed set Urysohn.

Theorem 3.2. Let (X,T) be a space. Then the following are equivalent: (a) (X,T) is Urysohn, (b) for each T-open set O, (O,T_O) is Urysohn, (c) for each T-open set O, $(O,(T_O)s)$ is Urysohn, (d) for each T-open set O, $(O,(Ts)_O)$ is Urysohn, (e) for each proper T-open set O and C= $Cl_T(O)$, both (O,T_O) and (C,T_C) are Urysohn, and for each z in X, there exists a proper T-open set containing z, (f) for each proper T-open set O, (O,T_O) is Urysohn, for each T-open set O with C= $Cl_T(O)$ not X, $(C,(T_C)s))$ is Urysohn , and for each z in X, there exists a proper T-open set containing z, (g) for each proper T-open set O, $(O,(T_O)s)$ is Urysohn, for each T-open set O with C= $Cl_T(O)$ not X, $(C,(T_C)s)$ is Urysohn, and for each z in X, there exists a proper T-open set containing z, and (h) for each proper T-open set O and C=$Cl_T(O)$, both $(O,(T_O)s)$ and $(C,(T_C)s)$ are Urysohn, and for each z in X, there exists a proper T-open set containing[10-11] z.

Proof: By the results above (a) implies (b), (b) implies (c), and (c) implies (d).

(d) implies (e): Since X is T-open, then $(X,(Ts)_X)$ = (X,Ts) is Urysohn, which implies (X,T) is Urysohn. Hence (X,T) is T_2 and for each z in X, there exists a proper T-open set containing z [ ]. Also, since (X,T) is Urysohn, then every subspace of (X,T) is Urysohn and for each proper T-open set O and C= $Cl_T(O)$, both (O,T_O) and (C,T_C) are Urysohn.

Clearly (e) implies (f) and, by the results above, (f) implies (g).

(g) implies (h): Since for each proper T-open set O, $(O,(T_O)s)$ is Urysohn, then for each proper T-open set O, $(O,(T_O)s)$ is T_2 , which implies (O,T_O) is T_2. Since for each z in X, there exists a proper T-open set containing z, then (X,T) is T_2 [4].

Let x and y be distinct elements in X. Since X has three or more elements, there exists a z in X distinct from each of x and y. Since (X,T) is T_2, let U, V, and W be mutually disjoint open sets such that x is in U, y is in V, and z is in W. Then Y, the union of U and V, is a proper T-open set, C= $Cl_T(Y)$ is a proper subset of X, and both x and y are in C. Since $(C,(T_C)s)$ is Urysohn, then (C,T_C) is Urysohn. Let A and B be T_C-open sets such that x is in A, y is in B, and the T_C-closures of A and B are disjoint. Let D and E be T-open sets such that A is the intersection of D and C and B is the intersection of E and C. Then x is in F, the intersection of U and D, y is in G, the intersection of V and E, F is a T-open subset of A, and G is T-open subset of B. Since the T_C-closures of A and B are closed in C, which is closed in X, then the T_C-closures of A and B are closed in X. Thus x is in the T-open set F, $Cl_T(F)$ is a subset of the T_C-closure of A, y is in the T-open set G, and $Cl_T(G)$ is a subset of the T_C-closure of B. Hence statement (e) above is true, which implies (h) is true. (h) implies (a): Clearly (h) implies (g) and, by the argument above, (X,T) is Urysohn.

Theorem 3.3. Let (X,T) be a space.

Then the following are equivalent: (a) (X,T) is Urysohn, (b) for each Ts-open set O, (O,T_O) is Urysohn, (c) for each Ts-open set O, $(O,(T_O)s)$ is Urysohn, (d) for each Ts-open set O, $(O,(Ts)_O)$ is Urysohn, (e) for each proper Ts-open set O and $C= Cl_T(O)$, both $(O,(T_O)s)$ and $(C,(Ts)_C)$ is Urysohn and for each z in X, there exists a proper Ts-open set containing z, and (f) for each proper Ts-open set O, $(O,(Ts)_O)$ is Urysohn, for each Ts-open set O with $Cl_T(O)$ not X, $(C,(Ts)_C)$ is Urysohn, and for each z in X, there exists a proper Ts-open set containing z.

Proof: (a) implies (b): Since (X,T) is Urysohn, then (X,Ts) is Urysohn and, by Theorem 3.2, (b) is true.

Clearly, since Ts is a subset of T, (b) implies (c) and (c) implies (d).

(d) implies (e): By Theorem 3.2, (X,Ts) is Urysohn. If O is Ts-open, then O is T-open and $Cl_T(O)= Cl_{Ts}(O)$. Thus, by Theorem 3.2, (d) implies (e).

Clearly (e) implies (f).

(f) implies (a): Since for each U in Ts, $Cl_T(U)= Cl_{Ts}(U)$, then, by Theorem 3.2, (X,Ts) is Urysohn, which implies (X,T) is Urysohn.

Thus Urysohn is a Ts-open set subspace property.

Theorem 3.4. Let (X,T) be a space. Then the following are equivalent: (a) (X,T) is Urysohn, (b) for each proper T-open set O and $C= Cl_T(O)$, both (O,T_O) and $(C,(Ts)_C)$ are Urysohn, and for each z in X, there exist a proper Ts-open set containing z, and (c) for each proper T-open set O, (O,T_O) is Urysohn, for each T-open set U with $C= Cl_T(U)$ not X, $(C,(Ts)_C)$ is Urysohn, and for each z in X, there exists a proper regularly open set containing z.

Proof: (a) implies (b): By the results above, if O is a proper T-open or a proper Ts-open set, then (O,T_O) and $(O,(Ts)_O)$ are Urysohn and for each z in X, there exists a proper Ts-open set containing z. Let O be a proper T-open set. Then $U= Int_T(Cl_T(O))$ is regularly open [ ], which is in Ts, and $Cl_T(U)= Cl_T(O)$. Thus U is a proper Ts-open set and, by the results above, $(C,(Ts)_C)$ is Urysohn.

Clearly (b) implies (c).

(c) implies (a): Since Ts is a subset of T, then for each proper Ts-open set O, then (O,T_O) and thus $(O,(Ts)_O)$ are Urysohn. In the same manner, for each proper Ts-open set O with $C= Cl_T(O)$ not X, $(C,(Ts)_C)$ is Urysohn. Since the regularly open sets are a subset of Ts, then for each z in X, there exists a proper Ts-open set containing z. Hence, by Theorem 3.3, (X,T) is Urysohn.

Theorem 3.5. Let (X,T) be a space. Then the following are equivalent: (a) (X,T) is Urysohn, (b) for each regularly open set O, (O,T_O) is Urysohn, (c) for each proper regularly open set O and $C= Cl_T(O)$, both (O,T_O) and $(C,(Ts)c)$ are Urysohn and for each z in X, there exists a proper regularly open set containing z, and (d) for each proper regularly open set O, (O,T_O) is Urysohn, for each regularly open set U with $Cl_T(U)$ not X, $(C,(T_C)s)$ is Urysohn, and for each z in X, there exists a proper Ts-open set containing z, and (e) for each proper regularly open set O, $(O,(Ts)_O)$ is Urysohn, for each regularly open set U with $Cl_T(U)$ not X, $C,(Ts)_C)$ is Urysohn, and for each z in X, there exists a Ts-open set containing z.

Proof: Clearly, by the results above, (a) implies (b).

(b) implies (c): Since X is regularly open, then (X,T) is Urysohn and, by the results above statement (c) is true.

Clearly, by the results above (c) implies (d).

(d) implies (e): Since for each proper regularly open set O, (O,T_O) is Urysohn, then for each proper regularly open set O, (O,T_O) is T_2 and $(O,(T_O)s)$ is Urysohn. Since for each z in X, there exists a Ts-open set containing z, then (X,T) is T_2 [4]. Since for each regularly open set U with $C= Cl_T(U)$ not X, $(C,(T_C)s)$ is Urysohn, then for each regularly open set U with $C= Cl_T(U)$ not X, (C,T_C) is Urysohn.

Let x and y be distinct elements of X. Let z be in X distinct from x and y. Let U, V, and W be mutually disjoint T-open sets such that x is in U, y is in V, and z is in W. Then Y, the union of U and V, is T-open and $C= Cl_T(Y)$ is not X. Then $Z= Int_T(Cl_T(Y))$ is regularly open and $Cl_T(Z)= Cl_T(Y)$ is not X and $(C,(T_C)s)$ and thus (C,T_C) are Urysohn. Let A and B be T_C-open sets such that x is in A, y is in B and the (T_C)-closures of A and B are disjoint. Let D and E be T-open sets such that A is the intersection of D and C and B is the intersection of E and C. Then x is in F, the intersection of U and D, which is a T-open subset of A, y is in G, the intersection of V and E, which is a T-open subset of B, $Cl_T(F)$ is a subset of T_C-closure of A, and $Cl_T(G)$ is a subset of T_C-closure of B. Thus (X,T) is Urysohn and, by the results above, (e) is true.

(e) implies (a): Since the set of regularly open sets of (X,T) equals the set of regularly open sets of (X,Ts) and for each regularly open set O, $Cl_T(O)$ equals the Ts-closure of O, then for each proper Ts-regularly open set O, $(O,(Ts)_O)$ is Urysohn. For each Ts-regularly open set U with C, the Ts-closure of U not X, $(C,(Ts)_C$ is Urysohn, which implies $(C,((Ts)_C)s)$ is Urysohn. Since Ts= (Ts)s, then for each z in X, there exists a proper (Ts)s-open set containing z. Thus, by the argument above, (X,Ts) is Urysohn, which implies (X,T) is Urysohn.

Thus Urysohn is a regularly open set subspace property. If desired, above additional characterizations could have been given using for each space (X,T) and each T-open set O, $(T_O)s= (Ts)_O$.

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