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Similarity Reductions and Intermediate Integrals of The Phi-Four Equation

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Abstract

This study bases attention on new similarity solution and new similarity reductions of Phi-four equation has been subjected to Lie's group theoretic method of infinitesimal transformation (Transformation from nonlinear PDE to nonlinear ODE) .Also produce another new similarity reduction of that equation obtained by direct method. To the best of our knowledge, we are first to obtain a new similarity solution is expressed in terms of trigonometric function and new similarity reductions are successfully reported. Therefore we reported Intermediate integrals of reduced ordinary differential equation.

Key words : Group theoretic method, Direct method, Similarity solution, Intermediate integrals.

2010 Mathematics Subject Classification: 35C06,35C09,35Q35,35Q40.

1. Introduction

A large variety of physical, chemical, and biological phenomena is governed by nonlinear evolution equations. The analytical study of nonlinear partial differential equation was of great interest during the last decades. These theories were an important field of research for researcher in the past decades. The Phi-four equation studies in various areas of physics includes plasma physics, Fluid Dynamics, Quantum Field Theory, solid state physics and others.

Thus, the Phi-four equation reads

$$u_{tt} - au_{xx} - u + u^3 = 0 \ , \quad a > 0 \ ,$$

Where a is real constant. This equation can be investigated as a special form of the kelin-Gorden equation that patterns the phenomenon in particle physics where kink and anti-kink solitary waves interact[1].

Many scientist have used exact and numerical solutions of Phi-four to research some methods such as the sine-cosine method[2], the auxiliary equation method[3], the modified simple equation method[4], homotophy perturbation method[5], homotophy analysis method[5], Adomian decomposition method[5] and MEFM method[6].

In this paper, the basic interest is to construct new classical similarity solution of Phi-four equation via Lie's Group theoretic method, a new similarity reduction via Direct method and intermediate integrals of reduced ordinary differential equation.

The classical method for finding similarity reductions of a given partial differential equation is to use the Lie group method of infinitesimal transformations initially developed by Lie[7]. Latter discussed by Bluman and Cole[8], Bluman and Kumai[9], and Olver[10] provide an excellent description of Lie's classical group theoretic method of obtaining Similarity Solutions. Though the method is fully algorithmic, it often involves a large amount of tedigious algebra and auxiliary calculations which are virtually unmanageable manually. Symbolic manipulation programs have been developed , particularly in MACSYMA [11,12] and REDUCE[13] in order to facilitate the determination of the associated similarity reductions.

Bluman and Cole[14] proposed a generalization of Lie's method and defined it as "nonclassical method of group invariant solutions", which itself has been generalized by Olver and Rosenau[15,16]. All these methods determine Lie point transformation of a given partial differential equation.

As reported by Nother[17], Lie's method could be generalized by allowing the transformation to depend upon the derivatives of the dependent variable as well as the independent and dependent variables. The associated symmetries, called Lie-B$\ddot{a}$cklund symmetries, can also be determined by an algorithmic method.

Bluman, Kumei and Reid[18] introduced an algorithmic method which yields new classes of symmetries of a given partial differential equation that are neither Lie point nor Lie-Backlund symmetries.

In 1989, Clarkson and Kruskal[19] developed the direct method to obtain new similarity reduction of the Boussinesq equation. Levi and winternitz[20] subsequently gave a group theoretical explanation of these results by showing that all the new reduction of Boussinesq equation can also be obtained using the non-classical method of Bluman and Cole. However, the direct method appears to be simpler to implement than the nonclassical method. That Direct method involves no group theoretical techniques.

The outline of this paper as follows: in Section 2 a classical similarity reduction of the Phi-four equation is obtained by Lie's classical group theoretic method[22], new similarity solution is given and also provide intermediate integrals of reduced ordinary differential equation; in Section 3 a new similarity reduction obtained by direct method; in Section 4 we conclusion our results.

2. New Similarity Solution :

The Phi-four equation is,

$$u_{tt} - au_{xx} - u + u^3 = 0 \ , \tag{1}$$

Where $a > 0$ is a constant. We seek to obtain Lie group of infinitesimal transformations which takes the (x, t, u) space into itself and under which (1) is invariant, viz.,

$$\begin{aligned} x^* &= x + \epsilon X(x,t,u) + o(\epsilon^2) \ , \\ t^* &= t + \epsilon T(x,t,u) + o(\epsilon^2) \ , \\ u^* &= u + \epsilon U(x,t,u) + o(\epsilon^2) \ , \end{aligned} \tag{2}$$

Invariance of equation (1) under (2) gives

$$\theta(U_u) + \theta_x(-X_{tt} + a\,X_{xx} - 2a\,U_{xu}) + \theta_x^2(-a\,U_{uu} + 2a\,X_{xu}) + \theta_x^3(a\,X_{uu}) + \theta_{xx}(2a\,X_x) + \theta_t(2\,U_{tu} - T_{tt} + a\,T_{xx}) + \theta_t^2(U_{uu} - 2\,T_{tu}) + \theta_t^3(-T_{uu}) + \theta_{tt}(-2\,T_t) + \theta_{xt}(-2\,X_t + 2a\,T_x) + \theta\theta_x(-X_u) + \theta_x\theta_t(-2\,X_{tu} + 2a\,T_{xu}) + \theta_x^2\theta_t(a\,T_{uu}) + \theta_x\theta_t^2(-X_{uu}) + \theta_x\theta_{xx}(2a\,X_u) + \theta_x\theta_{xt}(2a\,T_u) + \theta_t\theta_{xt}(-2\,X_u) + \theta^3(-U_u) + \theta^3\theta_x(X_u) + \theta^3\theta_t(T_u) + \theta_t\theta_{tt}(-2\,T_u) + \theta\theta_t(-T_u) + [U_{tt} - a\,U_{xx} - U + U^3] = 0. \quad (3)$$

Successively equating to zero the coefficients of $\theta\theta_t$, $\theta^3\theta_x$, θ^3, $\theta_t\theta_{xt}$, $\theta_x\theta_t^2$, $\theta_x^2\theta_t$, θ_{tt}, θ_{xx}, in (3), we find that

$$T_u = X_u = U_u = X_{uu} = T_{uu} = T_t = X_x = 0. \quad (4)$$

Equating the coefficients of θ_t^2 in (3) to zero and using (4), we get

$$U_{uu} = 0. \quad (5)$$

Equating the coefficients of θ_x, θ_t, θ_{xt} and θ^0 in (3) to zero and using (4) and (5), we have

$$-X_{tt} - 2a\,U_{xu} = 0, \quad (6)$$

$$2\,U_{tu} + a\,T_{xx} = 0, \quad (7)$$

$$-2\,X_t + 2a\,T_x = 0, \quad (8)$$

$$U_{tt} - a\,U_{xx} - U + U^3 = 0. \quad (9)$$

Equation (4) lead to

$$X = X(t), \quad T = T(x). \quad (10)$$

Using (4) in Equation (5) requires that

$$U = f(x, t). \quad (11)$$

In view of (11), equation (6)-(9) take the form

$$X_{tt} = 0, \quad (12)$$

$$T_{xx} = 0, \quad (13)$$

$$-2\,X_t + 2a\,T_x = 0, \quad (14)$$

$$f_{tt} - a\,f_{xx} - f + f^3 = 0. \quad (15)$$

Equation (15) is meaningful if

$$f = 0 \quad \text{or} \quad f = \pm 1. \quad (16)$$

Suppose

$$f = 0. \quad (17)$$

On insertion of (17), equation (11) becomes

$$U = 0. \quad (18)$$

Equation (12) and (13) requires that

$$X = a_1 t + b_1, \quad (19)$$

$$T = a_0 x + b_0. \quad (20)$$

Where a_0, b_0, a_1 and b_1 are arbitrary constant.

Substituting (19) and (20) into (14), we get

$$a_1 = aa_0 . \tag{21}$$

The invariant surface condition is

$$\frac{dx}{X} = \frac{dt}{T} = \frac{du}{U}. \tag{22}$$

Substituting (19)-(21), with $b_1 = 0$, $b_0 = 0$, equation (22) becomes

$$\frac{dx}{at} = \frac{dt}{x} = \frac{du}{0}. \tag{23}$$

Integration of the first two ratios of (23) gives rise to the similarity variable

$$z(x,t) = \frac{x^2}{2} - a\frac{t^2}{2}. \tag{24}$$

In a similar manner, the second and third ratios give the similarity form of u as

$$u(x,t) = f(z). \tag{25}$$

Thus, the similarity transform of (1) is

$$u(x,t) = f(z), \tag{26}$$

$$z(x,t) = \frac{x^2}{2} - a\frac{t^2}{2} \tag{27}$$

Putting (26)-(27) in (1), we get the following non linear ordinary differential equation for the similarity function $f(z)$:

$$f''(-2a\,z) - f'(2a) - f + f^3 = 0. \tag{28}$$

We solve the equation (28), we provide an intermediate integral as well as a new similarity solution is expressed in terms of trigonometric function.

A first integral of (28) is (see [21])

$$f' = \frac{f}{2}\sqrt{\frac{f^2-2}{az}}. \tag{29}$$

Where we've the constant of integration equal to zero. And an exact solution of (28) is found to be

$$f = \sqrt{2}\,\sec\left(\sqrt{\frac{2z}{a}} + c\right). \tag{30}$$

Where c is an arbitrary constant.

Substituting (27) and (30) into (26), we obtain the new similarity solution of (1):

$$u(x,t) = \sqrt{2}\,\sec\left(\sqrt{\frac{x^2-at^2}{a}} + c\right). \tag{31}$$

3. New Similarity Reductions :

In this section, To obtain new similarity reductions of the Phi-four equation

$$u_{tt} - au_{xx} - u + u^3 = 0\,, a > 0\,, \tag{32}$$

From the section 2, Eq.,(16), suppose

$$f = \pm 1 \tag{33}$$

To the same process of Eq., (18)-(25), we obtain new similarity reduction form is,

$$u(x,t) = \frac{\pm t}{x} + f(z), \tag{34}$$

$$z(x,t) = \frac{x^2}{2} - a\frac{t^2}{2} \tag{35}$$

To obtain another new similarity reductions of the Phi-four equation (32), According to the direct method due to Clarkson and Kruskal[19], it is sufficient to seek a solution of Eq.(32) in the form

$$u(x,t) = \alpha(x,t) + \beta(x,t)w(z), \tag{36}$$

Where $\alpha(x, t)$, $\beta(x, t)$ and $z = z(x, t)$, are functions to be determined and the substitution of Eq.(36) into Eq.(32) yields

$$\Gamma_1 + \Gamma_2 w + \Gamma_3 w' + \Gamma_4 w'' + \Gamma_5 w^2 + \Gamma_6 w^3 - w'' = 0 \tag{37}$$

For this to be an ordinary differential equation for $w(z)$, then coefficients must be of the form $\beta z_t^2 \Gamma(z)$ (using the coefficient βz_t^2 of w'' as the normalizing coefficient).

The function $\Gamma_n(z)$, n=1,2,….6 are introduced according to

$$\beta z_t^2 \Gamma_1(z) = \alpha_{tt} - a\,\alpha_{xx} - \alpha + \alpha^3 \ , \tag{38}$$

$$\beta z_t^2 \Gamma_2(z) = \beta_{tt} - a\beta_{xx} - \beta + 3\alpha^2\beta, \tag{39}$$

$$\beta z_t^2 \Gamma_3(z) = 2\beta_t z_t + \beta z_{tt} - 2a\beta_x z_x - a\beta z_{xx} \ , \tag{40}$$

$$\beta z_t^2 \Gamma_4(z) = -a\beta z_x^2 \ , \tag{41}$$

$$\beta z_t^2 \Gamma_5(z) = 3\alpha\beta^2 \ , \tag{42}$$

$$\beta z_t^2 \Gamma_6(z) = \beta^3, \tag{43}$$

Where $\Gamma_n(z)$, n=1,2,….6 are functions to be determined and we make use of the following simplifying assumptions to determine α, β and z :

Assumption 1: If $\alpha(x, t)$ has the form $\alpha(x,t) = \tilde{\alpha}(x,t) + \beta(x,t)\Gamma(z)$ then we may choose $\Gamma(z) = 0$.

Assumption 2: If $\beta(x, t)$ is found to have the form $\beta(x,t) = \tilde{\beta}(x,t)\Gamma(z)$ then we may put $\Gamma(z) = 1$.

Assumption 3: If $z(x, t)$ is to be determined from an equation of the form $w(z) = \tilde{z}(x,t)$, where $w(z)$ is any invertible function of z, then we take $w(z) = z$.

Taking $\Gamma_4(z) = \dfrac{-a}{z^2}$ and solving the equation (41), we have

$$z = \frac{x}{t} \ . \tag{44}$$

In view of Assumption 1 from (43), we have

$$\beta = z_t \quad , \quad \Gamma_6(z) = 1 \tag{45}$$

In view of Assumption 2 from (42), we have

$$\Gamma_5(z) = \alpha = 0. \tag{46}$$

On insertion (44) into (45), we have

$$\beta = -xt^{-2} \tag{47}$$

Now inserting (44),(46) and (47) in (38) and (40), we have

$$\Gamma_1(z) = 0\,, \quad \Gamma_2(z) = \frac{4}{z} - \frac{-2a}{z^3} - \frac{2}{z}\,. \tag{48}$$

Writing $\beta_{tt} = \beta\, f(t)$ into (39), we have

$$\Gamma_3(z) = 0\,, \quad f(t) = \frac{6}{t^2}\,. \tag{49}$$

Substituting (44), (46) and (47) into (36), we obtain the new similarity reduction of Phi-four equation,

$$u(x,t) = -xt^{-2}\, w(z) \tag{50}$$

$$z(x,t) = \frac{x}{t} \tag{51}$$

Where $w\,(z)$ is governed by the ODE

$$(z^2 + a)w'' - \left(4z - \frac{-2a}{z} - 2z\right) w - z^2\, w^3 = 0 \tag{52}$$

The infinitesimals for the Phi-four equation obtained by Lie classical method are

$$X = a_1 t + b_1, T = a_0 x + b_0\,, U = 0 \;\; or \;\; \pm 1 \tag{53}$$

Where a_0, b_0, a_1 and b_1 are arbitrary constant. we can easily shown that the new similarity reduction obtained by direct method (50)-(51) for Phi-four equation (1) can also be obtained similarity reductions (26)-(27) and (34)-(35) from these infinitesimals (53).

4. Conclusions

(1). Classical Lie group theoretic method of infinitesimal transformation has been successfully applied to the Phi-four equation (1) to derive a new similarity solution is given by

$$u(x,t) = \sqrt{2}\,\sec\left(\sqrt{\frac{x^2 - at^2}{a}} + c\right) \tag{54}$$

And a new similarity reduction is given by

$$u(x,t) = \frac{\pm t}{x} + f(z), \tag{55}$$

$$z(x,t) = \frac{x^2}{2} - a\frac{t^2}{2} \tag{56}$$

Thus the similarity solution and similarity reduction has not been reported previously. And also be provide intermediate integrals.

(2). Direct method has been successfully applied to the Phi-four equation (1) to derive a new similarity reduction is given by

$$u(x,t) = \frac{-x}{t^2}\, w(z) \tag{57}$$

$$z(x,t) = \frac{x}{t} \tag{58}$$

Also, Thus the similarity reduction has not been reported previously.

We record our impression that the new results presented in this work for the Phi-four equation (1).

Scope of Future work :

Thus the results are very helpful to various areas of physics includes plasma physics, Fluid Dynamics, Quantum Field Theory, solid state physics and others.

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