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$(H,1)(E,1)$ Product Transform of Fourier Series and its Conjugate Series

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Abstract

In this paper, we introduce the concept of $(H,1)(E,1)$ product transform and obtained two quite new theorems on $(H,1)(E,1)$ product transform of Fourier series and its conjugate series. Our result extends several known result on single summability methods.

Key word: $(E,1)$ Summability, $(H,1)$ Summability, $(H,1)(E,1)$ Summability, Fourier series.

Mathematics Subject Classification: 42B05; 42B08.

1. Introduction

A good amount of work have been done in the field of summability of Fourier series and its conjugate series using single and product summability methods by several researchers like Sahney[5], Chandra[1], Sinha and Shrivastava[7], Chandra and Dikshit[2], Nigam and Sharma[3], Mursaleen and Alotaibi[8] and Singh[6] under different conditions. But nothing seems to have been done so far in the direction of $(H,1)(E,1)$ product summability of Fourier series and its conjugate series. Therefore, in the present paper, two new theorems have been established on $(H,1)(E,1)$ product summability of Fourier series and its conjugate Fourier series under general condition.

2. Definition and Notation :

Let $f(x)$ be a 2π- periodic function and Lebesgue integrable over $(-\pi,\pi)$. The Fourier series of $f(x)$

is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = \sum_{n=1}^{\infty} A_n(x) \tag{2.1}$$

The conjugate series of Fourier series is given by

$$\sum_{n=1}^{\infty}(b_n \cos nx - a_n \sin nx) = \sum_{n=1}^{\infty} B_n(x) \tag{2.2}$$

We shall use the following notations:

$$\Phi(\mathrm{t}) = f(x+t) + f(x-t) - 2\mathrm{s}$$

$$\Psi(t) = f(x+t) - f(x-t)$$

$$K_n(t) = \frac{1}{2\pi.\log n}\sum_{k=0}^{n}\left[\frac{1}{(k+1)2^k}\left\{\sum_{v=0}^{k}\binom{k}{v}\frac{\sin\left(v+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right]$$

$$\widetilde{K_n}(t) = \frac{1}{2\pi.\log n}\sum_{k=0}^{n}\left[\frac{1}{(k+1)2^k}\left\{\sum_{v=0}^{k}\binom{k}{v}\frac{\cos\left(v+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right]$$

And $\tau = \left[\frac{1}{t}\right]$, where τ denotes the greatest integer not greater than $\frac{1}{t}$.

Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series with sequence of its n^th^ partial sum of $\{s_n\}$. The $(H, 1)$ transform is defined as the n^th^ partial sum of $(H, 1)$ summability and is given by

$$H_n^1 = t_k(n) = \frac{1}{\log n}\sum_{k=0}^{n} \frac{s_{n-k}}{k+1} \; as\; n \to \infty \tag{2.3}$$

then infinite series $\sum_{n=0}^{\infty} u_n$ is summable to the definite number s by (E, q) method .

If ,

$$(E,1) = E_k^1 = \frac{1}{2^k}\sum_{v=0}^{k}\binom{k}{v} s_v \to s \; as\; n \to \infty \tag{2.4}$$

then the infinite series $\sum_{n=0}^{\infty} u_n$ is said to be summable $(E, 1)$ to the definite number s.

Now the $(H, 1)$ transform of the $(E, 1)$ summability defines $(H, 1)$ $(E, 1)$ summability and we denote it by $H_n^1 E_k^1$.

Thus if

$$H_n^1 E_k^1 = \frac{1}{\log n}\sum_{k=0}^{n}\frac{1}{k+1}E_k^1 \tag{2.5}$$

If $H_n^1 E_k^1 \to s$, as $\to \infty$, then the series $\sum_{n=0}^{\infty} u_n$ or the sequence $\{s_n\}$ is said to summable to the sum s by $H_n^1 E_k^1$.

3. Main Theorems :

We prove the following theorems:

3.1 Theorem :

Let $\{P_n\}$ be a positive, monotonic, non-increasing sequence of real constants such that

$$P_n = \sum_{v=0}^{\infty} P_v \to \infty \text{ , } as\ n \to \infty$$

If,

$$\emptyset(t) = \int_0^t |\emptyset(u)| du = o\left[\frac{t}{\alpha\left(\frac{1}{t}\right).P_t}\right], as\ t \to +0 \tag{3.1.1}$$

Where, $\alpha(t)$ is positive, monotonic and non-increasing function of t and

$$\log n = O[\{\alpha(n)\}.P_n], as\ as\ n \to \infty \tag{3.1.2}$$

then the Fourier series (2.1) is summable $(H, 1)\ (E, 1)$ to $f(x)$.

3.2 Theorem :

Let $\{P_n\}$ be a positive, monotonic, non-increasing sequence of real constants such that

$$P_n = \sum_{v=0}^{\infty} P_v \to \infty,\ as\ n \to \infty$$

$$\Psi(t) = \int_0^t |\Psi(u)| du = o\left[\frac{t}{\alpha\left(\frac{1}{t}\right),P_t}\right], as\ t \to +0 \tag{3.2.1}$$

Where, $\alpha(t)$ is positive, monotonic and non-increasing function of t then the conjugate Fourier series (2.2) is Summable $(H, 1)\ (E, 1)$ to

$$\tilde{f}(x) = \frac{-1}{2\pi} \int_0^{2\pi} \Psi(t) \cos\frac{t}{2} dt$$

at any point where this point exists.

4. Lemmas :

Lemma 1. $|k_n(t)| = O(n), \quad for\ 0 \le t \le \frac{1}{n}; \sin nt \le \sin nt; |\cos nt \le 1|.$

Proof:

$$|k_n(t)| \le \frac{1}{2\pi.\log n}\left|\sum_{k=0}^{n}\left[\frac{1}{(k+1)2^k}\sum_{v=0}^{k}\binom{k}{v}\ \frac{\sin\left(v+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right]\right|$$

$$\le \frac{1}{2\pi.\log n}\left|\sum_{k=0}^{n}\left[\frac{1}{(k+1)2^k}\sum_{v=0}^{k}\binom{k}{v}\ \frac{(2v+1)\sin\frac{t}{2}}{\sin\frac{t}{2}}\right]\right|$$

$$\le \frac{1}{2\pi.\log n}\left|\sum_{k=0}^{n}\left[\frac{1}{(k+1)2^k}(2k+1)\sum_{v=0}^{k}\binom{k}{v}\right]\right|$$

$$= \frac{1}{2\pi.\log n}\ \sum_{k=0}^{n}\ \frac{1}{(k+1)}(2k+1)$$

$$= \frac{1}{2\pi.\log n}(2n+1)\ \sum_{k=0}^{n}\frac{1}{(k+1)}$$

$$= \frac{(2n+1)}{2\pi.\log n}$$

$$= O(n)$$

Lemma 2. $|k_n(t)| = O\left(\frac{1}{t}\right),\quad for\ \frac{1}{n} \le t \le \pi\,; \sin\frac{t}{2} \ge \frac{t}{2}\ \ and \sin nt \le 1.$

Proof:

$$|k_n(t)| \le \frac{1}{2\pi.\log n}\left|\sum_{k=0}^{n}\left[\frac{1}{(k+1)2^k}\sum_{v=0}^{k}\binom{k}{v}\ \frac{\sin\left(v+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right]\right|$$

$$\le \frac{1}{2\pi.\log n}\left|\ \sum_{k=0}^{n}\ \left[\frac{1}{(k+1)2^k}\ \sum_{v=0}^{k}\ \binom{k}{v}\ \frac{1}{\frac{t}{2}}\right]\right|$$

$$\le \frac{1}{\pi t.\log n}\left[\ \sum_{k=0}^{n}\left(\frac{1}{(k+1)2^k}\right)\ \sum_{v=0}^{k}\ \binom{k}{v}\right]$$

$$\le \frac{1}{\pi t.\log n}\left[\ \sum_{k=0}^{n}\ \frac{1}{(k+1)}\right]$$

$$= \frac{1}{\pi t.\log n}$$

$$= O\left(\frac{1}{t}\right)$$

Lemma 3. $\left|\widetilde{k_n}(t)\right| = O\left(\frac{1}{t}\right),$ for $0 \le t \le \frac{1}{n}\,; \sin\frac{t}{2} \ge \frac{t}{2}$ and $|\cos nt| \le 1$

$$\left|\widetilde{k_n}(t)\right| \le \frac{1}{2\pi.\log n}\left|\ \sum_{k=0}^{n}\left[\frac{1}{(k+1)2^k}\ \sum_{v=0}^{k}\binom{k}{v}\ \frac{\cos\left(v+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right]\right|$$

$$\le \frac{1}{2\pi.\log n}\sum_{k=0}^{n}\left[\frac{1}{(k+1)2^k}\ \sum_{v=0}^{k}\binom{k}{v}\ \left|\frac{\cos\left(v+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right|\right]$$

$$\le \frac{1}{\pi t.\log n}\left[\sum_{k=0}^{n}\frac{1}{(k+1)2^k}\sum_{v=0}^{k}\binom{k}{v}\right]$$

$$\le \frac{1}{\pi t.\log n}\left[\ \sum_{k=0}^{n}\frac{1}{(k+1)}\right]$$

$$= \frac{1}{\pi t.\log n}$$

$$= O\left(\frac{1}{t}\right)$$

Lemma 4. $\left|\widetilde{k_n}(t)\right| = O\left(\frac{1}{t}\right)$, for $\frac{1}{n} \leq t \leq \pi\,; \sin\frac{t}{2} \geq \frac{t}{2}$.

Proof:

$$\left|\widetilde{k_n}(t)\right| \leq \frac{1}{2\pi.\log n}\left|\sum_{k=0}^{n}\left[\frac{1}{(k+1)2^k}\sum_{v=0}^{k}\binom{k}{v}\frac{\cos\left(v+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right]\right|$$

$$\leq \frac{1}{\pi t.\log n}\left|\sum_{k=0}^{n}\left[\frac{1}{(k+1)2^k} Re\left\{\sum_{v=0}^{k}\binom{k}{v} e^{i\left(v+\frac{1}{2}\right)t}\right\}\right]\right|$$

$$\leq \frac{1}{\pi t.\log n}\left|\sum_{k=0}^{n}\left[\frac{1}{(k+1)2^k} Re\left\{\sum_{v=0}^{k}\binom{k}{v} e^{ivt}\right\}\right]\right|\left|e^{\frac{it}{2}}\right|$$

$$\leq \frac{1}{\pi t.\log n}\left|\sum_{k=0}^{n}\left[\frac{1}{(k+1)2^k} Re\left\{\sum_{v=0}^{k}\binom{k}{v} e^{ivt}\right\}\right]\right|$$

$$\leq \frac{1}{\pi t.\log n}\left|\sum_{k=0}^{r-1}\left[\frac{1}{(k+1)2^k} Re\left\{\sum_{v=0}^{k}\binom{k}{v} e^{ivt}\right\}\right]\right|$$

$$+\frac{1}{\pi t.\log n}\left|\sum_{k=r}^{n}\left[\frac{1}{(k+1)2^k} Re\left\{\sum_{v=0}^{k}\binom{k}{v} e^{ivt}\right\}\right]\right| \tag{4.4.1}$$

Now considering the first part of (4.4.1),

$$\frac{1}{\pi t.\log n}\left|\sum_{k=0}^{r-1}\left[\frac{1}{(k+1)2^k} Re\left\{\sum_{v=0}^{k}\binom{k}{v} e^{ivt}\right\}\right]\right|$$

$$\leq \frac{1}{\pi t.\log n}\left|\sum_{k=0}^{r-1}\left[\frac{1}{(k+1)2^k}\left\{\sum_{v=0}^{k}\binom{k}{v}\right\}\right]\right|\left|e^{ivt}\right|$$

$$\leq \frac{1}{\pi t.\log n}\left|\sum_{k=0}^{r-1}\frac{1}{(k+1)2^k}\sum_{v=0}^{k}\binom{k}{v}\right|$$

$$\leq \frac{1}{\pi t.\log n}\sum_{k=0}^{r-1}\left(\frac{1}{(k+1)}\right)$$

$$= \frac{1}{\pi t.\log n}$$

$$= O\left(\frac{1}{t}\right)$$

Considering the second part of (4.4.1),

$$\frac{1}{\pi t\,.\log\, n}\left|\sum_{k=r}^{n}\left[\frac{1}{(k+1)2^k}Re\left\{\sum_{v=0}^{k}\binom{k}{v}e^{ivt}\right\}\right]\right|$$

$$\leq \frac{1}{\pi t\,.\log\, n}\sum_{k=r}^{n}\frac{1}{(k+1)2^k}\quad \underset{0\leq m\leq k}{max}\left|\sum_{v=0}^{k}\binom{k}{v}\right.$$

$$\leq \frac{1}{\pi t\,.\log\, n}\sum_{k=r}^{n}\frac{1}{(k+1)}$$

$$= \frac{1}{\pi t.\,\log n}\sum_{k=r}^{n}\frac{1}{(k+1)}$$

$$= O\left(\frac{1}{t}\right)$$

5. Proof of Main Theorems :

5.1 Proof of theorem 3.1

Following Titchmarsh[4] and using Riemann-Lebesgue theorem, $s_n(f\,;x)$ of the series (2.1) is given by

$$s_n(f;x)-f(x)=\frac{1}{2\pi}\int_0^{\pi}\emptyset(t)\frac{\sin\left(v+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt$$

Therefore using (1), the $(E,1)$, transform E_n^q of $s_n(f\,;x)$ is given by

$$E_n^1-f(x)=\frac{1}{2\pi 2^k}\int_0^{\pi}\emptyset(t)\left\{\sum_{v=0}^{k}\binom{k}{v}\;\frac{\sin\left(v+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}dt$$

Now denoting $(H,1)\,(E,1)$ transform of $s_n(f\,;x)$ by $H_n^1E_n^1$, we write

$$H_n^1E_n^1-f(x)=\frac{1}{2\pi.\,\log n}\sum_{k=0}^{n}\left[\frac{1}{(k+1)2^k}\int_0^{\pi}\emptyset(t)\left\{\sum_{v=0}^{k}\binom{k}{v}\;\frac{\sin\left(v+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}dt\right]$$

$$=\int_0^{\pi}\emptyset(t)k_n(t)dt \qquad (5.1.1)$$

We have to show that, under the hypothesis of theorem

$$\int_0^{\pi}\emptyset(t)k_n(t)dt=o(1),\, as\ n\to\infty$$

For $0<\delta<\pi$, we have

$$\int_0^{\pi}\emptyset(t)k_n(t)dt=\left[\int_0^{\frac{1}{n}}\emptyset(t)+\int_{\frac{1}{n}}^{\delta}\emptyset(t)+\int_{\delta}^{\pi}\emptyset(t)\right]k_n(t)dt$$

$$= I_1 + I_2 + I_3 \text{ (say)} \tag{5.1.2}$$

We consider,

$$|I_1| \le \int_0^{\frac{1}{n}} |\emptyset(t)||k_n(t)|dt$$

$$= O(n)\left[\int_0^{\frac{1}{n}} \emptyset(t)k_n(t)dt\right] \text{ by Lemma 1}$$

$$= O(n)\left[\int_0^{\frac{1}{n}} o\left\{\frac{1}{n\alpha(n).p_n}\right\}\right] \text{ by (3.1.1)}$$

$$= o\left\{\frac{1}{n\alpha(n).p_n}\right\}$$

$$= o\left\{\frac{1}{\log n}\right\} \quad \text{using (3.1.2)}$$

$$= o(1), as \ n \to \infty \tag{5.1.3}$$

Now,

$$|I_2| \le \int_{\frac{1}{n}}^{\delta} |\emptyset(t)||k_n(t)|dt$$

$$= O(n)\left[\int_{\frac{1}{n}}^{\delta} |\emptyset(t)|\left(\frac{1}{t}\right)dt\right] \text{ by Lemma 1}$$

$$= O\left(\frac{1}{n}\right)\left[\left\{\frac{1}{t}\emptyset(t)\right\}_{\frac{1}{n}}^{\delta} + \int_{\frac{1}{n}}^{\delta} \frac{1}{t^2}\emptyset(t)dt\right]$$

$$= O\left(\frac{1}{n}\right)\left[o\left\{\frac{1}{\alpha\left(\frac{1}{t}\right).p_t}\right\}_{\frac{1}{n}}^{\delta} + \int_{\frac{1}{n}}^{\delta} o\left(\frac{1}{t\alpha\left(\frac{1}{t}\right).p_t}\right)dt\right] \quad \text{by (3.1.1)}$$

Putting $\frac{1}{t} = u$ in second term,

$$= O\left(\frac{1}{n}\right)\left[o\left\{\frac{1}{\alpha(n).p_n}\right\} + \int_{\frac{1}{\delta}}^{n} o\left(\frac{1}{u\alpha(u).p_u}\right)du\right]$$

$$= o\left\{\frac{1}{\alpha(n).p_n}\right\} + o\left\{\frac{1}{n\alpha(n).p_n}\right\}\int_{\frac{1}{\delta}}^{n} 1.du$$

$$= o\left\{\frac{1}{\log n}\right\} + o\left\{\frac{1}{\log n}\right\} \qquad \text{by (3.1.2)}$$

Using second mean value theorem for the integral in the second term as $\alpha(n)$ is monotonic

$$= o(1) + o(1)\,, as \ n \to \infty$$

$$= o(1) \ \ as, n \to \infty \tag{5.1.4}$$

By Rieman-Lebesgue theorem and by regularity condition of the method of Summability,

$$|I_3| \le \int_\delta^\pi |\emptyset(t)||k_n(t)|dt$$

$$= o(1) \ \ as, n \to \infty \tag{5.1.5}$$

Combining (5.1.3), (5.1.4) and (5.1.5), we have

$$H_n^1 E_n^1 - f(x) = o(1) \text{ , } as\ n \to \infty$$

This completes the proof of theorem 3.1.

5.2 Proof of Theorem 3.2.

Let $\widetilde{s_n}(f,x)$ denotes the partial sum of series (2.2) .

Using Riemann-Lebesgue Theorem, $\widetilde{s_n}(f,x)$ of series (2.2) and the $(E, 1)$ transform E_n^1 of $\widetilde{s_n}(f,x)$ is given by

$$\tilde{E}_n^1 - \tilde{f}(x) = \frac{1}{2\pi 2^k}\int_0^\pi \Psi(t)\left\{ \sum_{v=0}^k \binom{k}{v}\frac{\cos\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}dt$$

Now denoting $(H, 1)$ $(E, 1)$ transform of $\widetilde{s_n}(f,x)$ by $\overline{H_n^1 E_n^1}$, we write

$$\overline{H_n^1 E_n^1} - \tilde{f}(x) = \frac{1}{2\pi.\log n}\sum_{k=0}^n\left[\frac{1}{(k+1)2^k}\int_0^\pi \Psi(t)\left\{\sum_{v=0}^k\binom{k}{v}\frac{\cos\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}dt\right]$$

$$= \int_0^\pi \Psi(t)\widetilde{k_n}(t)dt \tag{5.2.1}$$

In order to prove the Theorem, we have to show that, under the hypothesis of theorem

$$\int_0^\pi \Psi(t)\widetilde{k_n}(t)dt = o(1) \text{ ,} \qquad as\ n \to \infty$$

For $0 < \delta < \pi$, we have

$$\int_0^\pi \Psi(t)\widetilde{k_n}(t)dt = \left[\int_0^{\frac{1}{n}} \Psi(t) + \int_{\frac{1}{n}}^\delta \Psi(t) + \int_\delta^\pi \Psi(t)\right]\widetilde{k_n}(t)dt$$

$$= I_1 + I_2 + I_3 (\text{say}) \tag{5.2.2}$$

We consider,

$$|I_1| \le \int_0^{\frac{1}{n}} |\Psi(t)|\left|\widetilde{k_n}(t)\right|dt$$

$$= O\left[\int_0^{\frac{1}{n}} \frac{1}{t}|\Psi(t)|dt\right] \text{ by Lemma 3}$$

$$= O\left(\frac{1}{n}\right)\left[\int_0^{\frac{1}{n}}\frac{1}{t}|\Psi(t)|dt\right]$$

$$= O(n)\left(\frac{1}{n}\right)\left[o\left\{\frac{1}{n\alpha(n).p_n}\right\}\right] \text{ by } (3.2.1)$$

$$= o\left\{\frac{1}{\alpha(n).p_n}\right\}$$

$$= o\left\{\frac{1}{\log n}\right\} \quad \text{using } (3.2.2)$$

$$= o(1), as \ n \to \infty \tag{5.2.3}$$

Now,

$$|I_2| \le \int_{\frac{1}{n}}^{\delta}|\Psi(t)|\left|\widetilde{k_n}(t)\right|dt$$

$$= O\left[\int_{\frac{1}{n}}^{\delta}\frac{1}{t_n}|\Psi(t)|dt\right] \text{ by Lemma 4}$$

$$= O\left(\frac{1}{n}\right)\left[\int_{\frac{1}{n}}^{\delta}\frac{1}{t}|\Psi(t)|dt\right]$$

$$= O\left(\frac{1}{n}\right)\left[\left\{\frac{1}{t}\Psi(t)\right\}_{\frac{1}{n}}^{\delta} + \int_{\frac{1}{n}}^{\delta}\frac{1}{t^2}\Psi(t)dt\right]$$

$$= O\left(\frac{1}{n}\right)\left[o\left\{\frac{1}{\alpha\left(\frac{1}{t}\right).p_t}\right\}_{\frac{1}{n}}^{\delta} + \int_{\frac{1}{n}}^{\delta} o\left(\frac{1}{t\alpha\left(\frac{1}{t}\right).p_t}\right)dt\right] \quad \text{by} \tag{3.2.1}$$

Putting $\frac{1}{t} = u$ in second term,

$$= O\left(\frac{1}{n}\right)\left[o\left\{\frac{1}{\alpha(n).p_n}\right\} + \int_{\frac{1}{\delta}}^{n} o\left(\frac{1}{u\alpha(u).p_u}\right)du\right]$$

$$= o\left\{\frac{1}{\alpha(n).p_n}\right\} + o\left\{\frac{1}{n\alpha(n).p_n}\right\}\int_{\frac{1}{\delta}}^{n}1.du$$

$$= o\left\{\frac{1}{\log n}\right\} + o\left\{\frac{1}{\log n}\right\} \ by \ (3.1.2)$$

Using second mean value theorem for the integral in the second term as $\alpha(n)$ is monotonic

$$= o(1) + o(1), as \ n \to \infty$$

$$= o(1) \ as, n \to \infty \tag{5.2.4}$$

By Riemann-Lebesgue theorem and by regularity condition of the method of summability,

$$|I_3| \leq \int_{\delta}^{\pi} |\emptyset(t)||\widetilde{k_n}(t)|dt$$

$$= o(1), \ as \ n \to \infty \tag{5.2.5}$$

Combining (5.2.3), (5.2.4) and (5.2.5), we have

$$\overline{H_n^1 E_n^1} - \tilde{f}(x) = o(1), \qquad as \ n \to \infty$$

This completes the proof of theorem 3.2.

Conclusion

In the field of Summability theory, various results pertaining $(H, 1)$, (E, q) and $(E, q)X$ and $X(H, 1)$ Summability of Fourier series as well as its allied series have been reviewed. In future, the present work can be generalised under certain conditions.

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