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A Extension of Fixed Point Theorem in Banach Space

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Abstract

The object of the present paper is to extend the fixed point theorem of previous authors.

Key word: Common fixed point, Fixed point and non contracting point Metric Space, Banach Space.

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Introducation

Let C be a closed subset of a Banach Space X. The well known Banach contraction principal states that a contraction mapping of C into itself has a unique fixed point. The same conclusion hold if we assume that only some positive power of mapping are contraction (e.g. Bryant[1]). But it is not true for non-expansive mappings. Goebel and Zlotkiewicz[2] have proved this problem in applying some restriction. The purpose of this paper to generalize the result of Goebel and Zlotkiewicz[2] and others, for mappings satisfying more general conditions. Now we prove the following theorem four mappings.

Theorem Let K be a non empty closed convex subset of a Banach Space, let E, F, G, H, T be a mapping of K in to itself such that :

(i) $E^2 = I$, $F^2 = I$ $G^2 = I$, $H^2 = I$, $T^2 = I$

(ii) $EF = FF$, $EG = GF$, $EH = HE$, $ET = TE$, $FG = GH$, $FH = HF$, $FT = TF$, $GH = HG$, $GT = TG$, $HT = TH$.

Main Results

Theorem 1 : Let K be a closed convex subset of a Banach space X and let E, F, G and H be mappings of K into itself, such that

(i) $E^2 = I, F^2 = I\, G^2 = I, H^2 = I$ (1)

(ii) $EF = FE, EG = GE, EH = HE, FG = GH, FH = HF, GH = HG$ (2)

(iii) $\| Ex - Ey \| \leq a_1 \| FGHx - FGHy \| + a_2 \| FGHx - Ex \| + a_3 \| FGHy - Ey \|$

$$+ a_4 \| FGHx - Ey \| + a_5 \| FGHy - Ex \| + a_6 \frac{\| FGHx - Ex \| . \| FGHy - Ey \|}{\| FGHx - FGHy \|} + a_7$$

$$\frac{\| FGHx - Ey \| . \| FGHy - Ex \|}{\| FGHx - FGHy \|} \text{ for every } x, y \in X, x \neq y \text{ and } a_i > o, i = 1, 2 ..7 \quad (3)$$

such that

$a_1 + 2a_2 + 2a_3 + 2a_4 + 2a_5 + 4a_6 + 3a_7 < 2$, and $a_1 + a_4 + a_5 + a_7 < 1$.

Then there exist at least one fixed point $x_0 \in K$, such that $Ex_0 = FG\,Hx_0$, $Hx_0 = EFGx_0$ and x_0 is also a unique common fixed point of E, F, G and H.

We require that following lemma – I for the proof of the theorem-I

Lemma I :

Let E be a mapping of a Banach space X into it self such that,

(i) $E^2 = 1$ (4)

(ii) $\| Ex - Ey \| \leq a_1 \| x - y \| + a_2 \| x - Ex \| + a_3 \| y - Ey \| + a_4 \| x - Ey \| + a_5 \| y - Ex \|$

$$+ a_6 \frac{\| x - Ex \| . \| y - Ey \|}{\| x - y \|} + a_7 \frac{\| x - Ey \| . \| y - Fx \|}{\| x - y \|} \quad (5)$$

for every $x, y \in X$, and $x \neq y$. $O \leq a_1, a_2, a_3, a_4, a_5, a_6, a_7$ and $a_1 + 2a_2 + 2a_3 + 2a_4 + 2a_5 + 4a_6 + 3a_7 < 2$ and $a_1 + a_4 + a_5 + a_7 < 1$. Then E has at least one fixed point.

Proof :

Let x be an arbitrary point of X and taking $y = ½ (E + I)\, x$, $z = Ey$, $u = 2y - z$, then (i) and (ii) we have

$\| Ex - y \| = ½ \| x - Ex \|$

$\& \| x - y \| = ½ \| Ex - x \|$

$\therefore \| z - x \| = \| Ey - E^2x \| \| \leq a_1 \| \| y - Ex \| + a_2 \| y - Ey \| + a_3 \| Ex - E^2x \|$

$$+ a_4 \| y - E^2x \| + a_5 \| Ex - Ey \| + a_6 \frac{\| y - Ey \| . \| Ex - E^2x \|}{\| y - Ex \|}$$

$$+ a_7 \frac{\| y - E^2x \| . \| Ex - Ey \|}{\| y - Ex \|}$$

$= a_1 \| y - Ex \| + a_2 \| y - Ey \| + a_3 \| Ex - x \| + a_4 \| y -x \| + a_5 \| Ex - Ey \|$

$$+ a_6 \frac{\| y - Ey \| . \| Ex - x \|}{\| y - Ex \|} + a_7 \frac{\| y - x \| . \| Ex - Ey \|}{\| y - Ex \|}$$

$= a_1 \| ½ (E + I) x - Ex \| + a_2 \| y - Ey \| + a_3 \| Ex - x \| + a_4 \| ½ (E + I) x - x \|$

$+ a_5 [\| Ex - y \| + \| y - Ey \|] + a_6 \dfrac{\| y - Ey \| . \| Ex - x \|}{\| ½ (E + I) x - Ex \|}$

$+ a_7 \dfrac{\| ½ (E + I) x - x \| . [\| Ex - y \| + \| y - Ey \|]}{\| ½ (E + I) x - Ex \|}$

$= a_1/2 \| x - Ex \| + a_2 \| y - Ey \| + a_3 \| Ex - x \| + a_4 /2 \| Ex - x \| + a_5 [½ \| Ex - x \|$

$+ \| y - Ey \|] + 2 a_6 \| y - Ey \| + a_7 [½ \| x - Ex \| + \| y - Ey \|]$

$= (a_1/2 + a_3 + a_4/2 + a_5/2 + a_7/2) \| x - Ex \| + \| (a_2 + a_5 + 2a_6 + a_7) \| y - Ey \|$ (6)

Now $\| u - x \| = \| 2y - z - x \| = \| (E + I) x - Ey - x \| = \| Ex - EY \| < a_1 \| x - y \|$

$+ a_2 \| x - Ex \| + a_3 \| y - Ey \| + a_4 \| x - Ey \| + a_5 \| y - Ex \|$

$+ a_6 \dfrac{\| x - Ex \| . \| y - Ey \|}{\| x - ½ (E + I) x \|} + a_7 \dfrac{[\| x - y \| + \| y - Ey \|] . \| ½ (E + I) x - Ex \|}{\| x - ½ (E + I) x \|}$

$= a_1/2 \| Ex - x \| + a_2 \| x - Ex \| + a_3 \| y - Ey \| + a_4 [\| ½ \| x - Ex \| + \| y - Ey \|]$

$+ a_5/2 \| x - Ex \| + 2a_6 \| y - Ey \| + a_7 \| [½] x - Ex \| + \| y - Ey \|$

$= (a_1/2 + a_2 + a_4/2 + a_5/2 + a_7/2) \| x - Ex \|$

$+ (a_3 + a_4 + 2a_5 + a_7) . \| y - x \|$ (7)

$\therefore \| z - u \| \ \| z - x \| + \| u - x \|$

$\leq (a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7) \| x - Ex \| + (a_2 + a_3 + a_4 + a_5 + 4a_6 + 2a_7) \| y - Ey \|$ (8)

Also $\| z - u \| = \| Ey - 2y + z \| = 2 \| y - Ey \|$ (9)

Comparing (8) and (9), we get.

$2 \| y - Ey \| \leq (a_1 + a_2 + a_3 + a_4 + a_5 + a_7) \| x - Ex \| + (a_2 + a_3 + a_4 + a_5 + 4a_6 + 2a_7) \| y - Ey \|$

Or $\| y - Ey \| \leq q \| x - Ex \|$, where $q = \dfrac{a_1 + a_2 + a_3 + a_4 + a_5 + a_7}{2 - a_2 - a_3 - a_4 - a_5 - a_6 - 2a} < 1$

Let G $= 1/2 (E + I)$, them for any $x \in X$, $\| G^2x - Gx \| = \| Gy - y \|$

$= \| 1/2 (E + I) y - y \| = 1/2 \| y - Ey \| \leq q/2 \| x - Ex \|$

$= q/2 \| x - (2Gx - x) \| = q \| Gx - x \|$

By definition of q, we claim that $\{G^n{}_x\}$ is a cauchy sequence in X. By completeness, $\{G^n{}_x\}$ converges to some element x_0 in X.i.e.

Lim $G^n{}_x = x_0$. This implies $G x_0 = x_0$. Hence $E x_0 = x_0$ i.e. x_0 is a $n \to \infty$ fixed point of E.

For the uniqueness, if possible y_0 ($\neq x_0$) be another fixed point of E.

Then $\| x_0 - y_0 \| = \| Ex_0 - Ey_0 \|$

$\leq a_1 \| x_0 - y_0 \| + a_2 \| x_0 - Ex_0 \| + a_3 \| y_0 - Ey_0 \| + a_4 \| x_0 - Ey_0 \| + a_5 \| y_0 - Ex_0 \|$

$+ a_6 \dfrac{\| x_0 - Ex_0 \| . \| y_0 - Ey_0 \|}{\| x_0 - y_0 \|} + a_7 \dfrac{\| x_0 - Ey_0 \| . \| y_0 - Ex_0 \|}{\| x_0 - y_0 \|}$

$\therefore (1 - a_1 - a_4 - a_5 - a_7) \| x_0 - y_0 \| \leq 0$

$\therefore\ x_0 = y_0$

Hence the result

Proof of theorem – I :

From (1) and (2), it follows that $(EFGH)^2 = I$ and (2) & (3), we have

$$\|EFGH.Fx - EFGH.Fy\| \le a_1 \|(FGH)^2.Fx - (FGH)^2.Fy\|$$
$$+ a_2 \|(FGH)^2.Fx - EFGH.Fx\| + a_3 \|(FGH)^2.Fy - EFGH.Fy\|$$
$$+ a_4 \|(FGH)^2.Fx - EFGH.Fy\| + a_5 \|(FGH)^2.Fy - EFGH.Fx\|$$
$$+ a_6 \frac{\|(FGH)^2 . Fx - EFGH . Fx\| . \|(FGH)^2 . Fy - EFGH . Fy\|}{\|(FGH)^2\ Fx - (FGH)^2 . y\|}$$
$$+ a_6 \frac{\|(FGH)^2 . Fx - EFGH . Fy\| . \|(FGH)^2 . Fy - EFGH . Fx\|}{\|(FGH)^2\ Fx - (FGH)^2 . Fy\|}$$

Now if we put $Fx = z$ and $Fy = w$, we get

$$\|EFGH.z - EFGH.w\| \le a_1 \|z - w\| + a_2 \|z - EFGH.z\|$$
$$+ a_3 \|w - EFGH.w\| + a_4 \|z - EFGH.w\| + a_5 \|w - EFGH.z\|$$
$$+ a_6 \frac{\|z - EFGH.z\| . \|w - EFGH.w\|}{\|z - w\|} + a_7 \frac{\|z - EFGH.w\| . \|w - EFGH.w\|}{\|z - w\|}$$

Where $(FGH)^2 = I$, then by lemma-1, we get EFGH have at least one fixed point say x_0 in X.i.e. $EFGH\ x_0 = x_0$ (10)

$FGH\ (EFGH)\ x_0 = FGH\ x_0$ or $Ex_0 = FGH\ x_0$ (11)

Also $H\ (EFGH)\ x_0 = H\ x_0$ or $EFG\ x_0 = H\ x_0$ (12)

Now using (1), (2), (3) and (10), (11), (12) we have

$$\|Hx_0 - x_0\| = \|EFG\ x_0 - E^2x_0\| = \|EFG\ x_0 - E.E\ x_0\|$$
$$\le a_1 \|FGH.FGx_0 - FGH.Ex_0\| + a_2 \|FGH.FGx_0 - EFGx_0\|$$
$$+ a_3 \|FGH.Ex_0 - E.Ex_0\| + a_4 \|FGH.FGx_0 - EFx_0\|$$
$$+ a_5 \|FGH.Ex_0 - EFGx_0\|$$
$$+ a_6 \frac{\|FGH.FGx_0 - EFGx_0\| . \|FGH.Ex_0 - E.Fx_0\|}{\|FGH.\ FGx_0 - FGH.Ex_0\|}$$
$$+ a_7 \frac{\|FGH.FGx_0 - FFx_0\| . \|FGH.Ex_0 - E.FGx_0\|}{\|FGH.\ FGx_0 - FGH.Ex_0\|}$$
$$= a_1 \|Hx_0 - x_0\| + a_2 \|Hx_0 - Hx_0\| + a_3 \|x_0 - x_0\| + a_4 \|Hx_0 - x_0\| + a_5 \|x_0 - Hx_0\|$$
$$+ a_6 \frac{\|Hx_0 - H^2\ x_0\| . \|x_0 - x_0\|}{\|Hx_0 - x_0\|} + a_7 \frac{\|Hx_0 - x_0\| . \|x_0 - Hx_0\|}{\|Hx_0 - x_0\|}$$
$$= (a_1 + a_4 + a_5 + a_7) \|Hx_0 - x_0\|$$

since $a_1 + a_4 + a_5 + a_7 < 1$, it follows that $Hx_0 = x_0$, i.e. x_0 is the fixed point of H. Then we have from (11) & (12) $Ex_0 = FGx_0$ (13)

and $EFGx_0 = x_0$ (14)

$FG(EFGx_0) = FGx_0$ or $Ex_0 = FGx_0$ (15)

$G(EFGx_0) = Gx_0$ or $EFx_0 = Gx_0$ (16)

Now using (1), (2), (3), (13), (16), we have

$$\| Gx_0 - x_0 \| = \| EFx_0 - E^2x_0 \| = \| EFx_0 - EEx_0 \|$$

$$\leq a_1 \| FGH.Fx_0 - FGH.Ex_0 \| + a_2 \| FGH.Fx_0 - EFx_0 \|$$

$$+ a_3 \| FGHFx_0 - EEx_0 \| + a_4 \| FGH.Fx_0 - EEx_0 \|$$

$$+ a_5 \| FGH.Ex_0 - EFx_0 \|$$

$$+ a_6 \frac{\| FGH.Fx_0 - E.Fx_0 \| . \| FGH.Ex_0 - E.Ex_0 \|}{\| FGH.Fx_0 - FGH.Ex_0 \|}$$

$$+ a_7 \frac{\| FGH.Fx_0 - E.Ex_0 \| . \| FGH.Ex_0 - E.Fx_0 \|}{\| FGH.Fx_0 - FGH.Ex_0 \|}$$

$$= a_1 \| Gx_0 - x_0 \| + a_2 \| Gx_0 - Gx_0 \| + a_3 \| x_0 - x_0 \| + a_4 \| Gx_0 - x_0 \| + a_5 \| x_0 - Gx_0 \|$$

$$+ a_6 \frac{\| Gx_0 - Gx_0 \| . \| x_0 - x_0 \|}{\| Gx_0 - x_0 \|}$$

$$+ a_7 \frac{\| Gx_0 - x_0 \| . \| x_0 - Gx_0 \|}{\| Gx_0 - x_0 \|}$$

$$= (a_1 + a_4 + a_5 + a_7) \| Gx_0 - x_0 \|$$

since $a_1 + a_4 + a_5 + a_7 < 1$, it follows that $Gx_0 = x_0$, i.e. x_0 is the fixed point of G, so we have from (15). $Ex_0 = Fx_0$

To Prove uniqueness, let us consider y_0 be another Common fixed point of E, F, G and H. Now using $\| x_0 - y_0 \| = \| E^2x_0 - E^2y_0 \| = \| EEx_0 - E.Ey_0 \|$

$$\leq a_1 \| FGH.Ex_0 - FGH.Ey_0 \| + a_2 \| FGH.Ex_0 - EEx_0 \|$$

$$+ a_3 \| FGH.Ey_0 - E.Ey_0 \| + a_4 \| FG.H.Ex_0 - F.Ex_0 \|$$

$$+ a_5 \| FGH.Ey_0 - E.Ex_0 \| + a_6 \frac{\| FGH.Ex_0 - EEx_0 \| . \| FGH.Ey_0 - EEy_0 \|}{\| FGH.Ex_0 - FGH.Ey_0 \|}$$

$$+ a_7 \frac{\| FGH.Ex_0 - EEx_0 \| . \| FGH.Ey_0 - EEx_0 \|}{\| FGH.Ex_0 - FGH.Ey_0 \|}$$

$$= a_1 \| x_0 - y_0 \| + a_2 \| x_0 - x_0 \| + a_3 \| y_0 - y_0 \| + a_4 \| x_0 - y_0 \|$$

$$+ a_5 \| y_0 - x_0 \| + a_6 \frac{\| x_0 - x_0 \| . \| y_0 - y_0 \|}{\| x_0 - y_0 \|} + a_7 \frac{\| x_0 - y_0 \| . \| y_0 - x_0 \|}{\| x_0 - y_0 \|}$$

$= (a_1 + a_4 + a_5 + a_7) \| x_0 - y_0 \|$, a contradiction since $a_1 + a_4 + a_5 + a_7 < 1$. It fallows that $x_0 = y_0$ and so proving the uniqueness of x_0. This complete the proof of the theorem.

Remarks

(i) If we take H = I, $a_2 = a_3$, $a_4 = a_5$, $a_6 = a_7 = 0$, we get the resulting Pathak and Maity (6).

(ii) If we put H = I, $a_2 = a_3$, $a_4 = a_5$, $a_6 = a_7 = 0$, we get the result of Jain and Jain (4).

(iii) If we take G = H, $a_2 = a_3$, $a_4 = a_5$, $a_6 = a_7 = 0$, we get the result Sharma and Bajaj (7).

(iv) On making F = G = H = I, $a_2 = a_3$, $a_4 = a_5$, $a_6 = a_7 = 0$, we obtain the result (Corollary 2.2) of Khan and Imdad (5).

(v) On taking F = G = H = I, $a_2 = a_3$, $a_4 = a_5$, $a_6 = a_7 = 0$, we get the result of Iseki (3).

(vi) On taking F = G = H = I, $a_2 = a_3$, $a_4 = a_5$, $a_6 = a_7 = 0$, we get the result due to Goebel and Zlotkiewicz (2).

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