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Reduction of Certain Type of Parabolic Partial Differential Equations to Heat Equation

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Abstract

As we all know that the solution of Heat Equation is found more easily than other Partial Differential Equations of Parabolic Type. So, if we enable us to convert Parabolic Partial Differential Equations to Heat Equation, then it becomes easier to find solutions. In this paper we are considering a method introduced by Harper and with the help of a method for reduction of some types of Partial Differential Equations to their Canonical Form it is shown that all the equations of this type are reduced to Heat Equation by following some definite steps. To illustrate the method, we have taken some PDEs of this type and converted them to their Canonical Form and then to Heat Equation.

Keywords: Parabolic Partial Differential Equation; Heat Equation; Solution of First Order Linear PDE.

AMS Subject Classification(2010): 35K05; 35K10; 35F05; 35A25

1 Introduction

Here we claim that every parabolic partial differential equation of the form,

$$R \frac{\partial^2 u}{\partial x^2} + S \frac{\partial^2 u}{\partial x \partial y} + T \frac{\partial^2 u}{\partial y^2} + f(x, y, u, u_x, u_y) = 0 \quad (1.1)$$

where,

$$f(x, y, u, u_x, u_y) = F \frac{\partial u}{\partial x} + G \frac{\partial u}{\partial y} + Hu$$

is reducible to Heat Equation.

Here, R, S, T, F, G & H are functions of x and y.

In this paper we are considering a method introduced in¹ and with the help of method for reduction of some types of Partial Differential Equations to their canonical form it is shown that all the equations of this type are reducible to heat equation.

In section-2 the criteria for a Partial Differential Equation (1.1) to be a Parabolic PDE is explained and if it is not in canonical form then the method of converting it to canonical form is also explained.

In section-3 the method of reduction of a Parabolic PDE in the canonical form into Heat equation is elaborated and also illustrated by giving some examples.

2 Reduction to Canonical Form :

For a Partial Differential Equation in the form,

$$R \frac{\partial^2 u}{\partial x^2} + S \frac{\partial^2 u}{\partial x \partial y} + T \frac{\partial^2 u}{\partial y^2} + f(x, y, u, u_x, u_y) = 0 \quad (2.1)$$

if $S^2 - 4RT = 0$, then it is categorized as Parabolic Partial Differential Equation.

We note that, the Canonical form of Parabolic Partial Differential Equations is,

$$\frac{\partial^2 u}{\partial x^2} + \phi(x, y, u, u_x, u_y) = 0$$

which contains only one term with second ordered derivative either with respect to x or with respect to y.

Hence, if $S^2 - 4RT = 0$, then,

• Case-1:

$S = 0$ & $T = 0$, $R \neq 0$.

In this case, (2.1) becomes,

$$R \frac{\partial^2 u}{\partial x^2} + f(x, y, u, u_x, u_y) = 0$$

Which contains only one term with second ordered derivative and hence, it is in Canonical form.

• Case-2:

$S = 0$ & $R = 0$, $T \neq 0$.

Here, (2.1) becomes,

$$T \frac{\partial^2 u}{\partial y^2} + f(x, y, u, u_x, u_y) = 0$$

Which also contains only one term with second ordered derivative therefore it is in canonical form.

• Case-3 :

We must note that, as $S^2 - 4RT = 0$ if $S \neq 0$ then $R \neq 0$ and $T \neq 0$.

Therefore we consider, $S \neq 0$, $R \neq 0$, & $T \neq 0$.

In this case we convert (2.1) in to canonical form using method of characteristics² which is shown below.

Let $u(x, y) = V(\xi, \eta)$

where, $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$.

For R, S , & T , if $S^2 - 4RT = 0$, then the quadratic equation

$$R\alpha^2 + S\alpha + T = 0$$

has two equal roots,

$$\alpha = -\frac{S}{2R}.$$

Here we will choose ξ as follows,

$$\xi_x = \alpha \xi_y.$$

this implies,

$$\frac{dy}{dx} + \alpha = 0.$$

Hence,

$$dy = \frac{S}{2R} dx.$$

On integrating both the sides we will get ξ and we choose η independent of ξ .

Now,

$$u_x = V_\xi \xi_x + V_\eta \eta_x$$

$$u_y = V_\xi \xi_y + V_\eta \eta_y$$

$$u_{xx} = V_{\xi\xi} (\xi_x)^2 + 2V_{\xi\eta} \xi_x \eta_x + V_{\eta\eta} (\eta_x)^2 + V_\xi \xi_{xx} + V_\eta \eta_{xx}$$

$$u_{yy} = V_{\xi\xi} (\xi_y)^2 + 2V_{\xi\eta} \xi_y \eta_y + V_{\eta\eta} (\eta_y)^2 + V_\xi \xi_{yy} + V_\eta \eta_{yy}$$

$$u_{xy} = V_{\xi\xi} \xi_x \xi_y + V_{\xi\eta} (\xi_x \eta_y + \eta_x \xi_y) + V_{\eta\eta} \eta_x \eta_y + V_\xi \xi_{xy} + V_\eta \eta_{xy}$$

Substituting all these expressions in (2.1), we get,

$$A(\xi_x, \xi_y) V_{\xi\xi} + 2B(\xi_x, \xi_y, \eta_x, \eta_y) V_{\xi\eta} + A(\eta_x, \eta_y) V_{\eta\eta} + \phi(\xi, \eta, V, V_\xi, V_\eta) = 0 \quad (2.2)$$

Where,

$$A(\xi_x, \xi_y) = R(\xi_x)^2 + S(\xi_x)(\xi_y) + T(\xi_y)^2$$

$$A(\eta_x, \eta_y) = R(\eta_x)^2 + S(\eta_x)(\eta_y) + T(\eta_y)^2.$$

Now as $\xi_x = \alpha \xi_y$,

$$A(\xi_x, \xi_y) = R\alpha^2 (\xi_y)^2 + S\alpha (\xi_y)^2 + T(\xi_y)^2$$

$$= (\xi_y)^2 (R\alpha^2 + S\alpha + T)$$

$$= (\xi_y)^2 (0)$$

$$= 0$$

∴ $A(\xi_x, \xi_y)$ vanishes.

Also, if $A(\eta_x, \eta_y) = 0$, then,

$$R(\eta_x)^2 + S(\eta_x)(\eta_y) + T(\eta_y)^2 = 0.$$

Hence,

$$R \frac{(\eta_x)^2}{(\eta_y)^2} + S \frac{(\eta_x)}{(\eta_y)} + T = 0.$$

Therefore,

$$\frac{(\eta_x)}{(\eta_y)} = \alpha.$$

This implies,

$$\frac{(\eta_x)}{(\eta_y)} = \alpha = \frac{(\xi_x)}{(\xi_y)}$$

as the equation $R\alpha^2 + S\alpha + T = 0$ has only one root α .

This is not possible as we have chosen η independent of ξ .

Hence,

$$A(\eta_x, \eta_y) \neq 0$$

Now, it is easily seen that ²,

$$A(\xi_x, \xi_y) A(\eta_x, \eta_y) - B^2 = (4RT - S^2)(\xi_x \eta_y - \xi_y \eta_x)^2$$

Hence,

$$B(\xi_x, \xi_y, \eta_x, \eta_y) = 0$$

Therefore, (2.2) is rewritten as,

$$A(\eta_x, \eta_y) V_{\eta\eta} + \phi(\xi, \eta, V, V_\xi, V_\eta) = 0 \tag{2.3}$$

Which is the canonical form of (2.1).

Here, $\phi(\xi, \eta, V, V_\xi, V_\eta) = \psi(\xi)V_\xi + \psi(\eta)V_\eta + HV$.

Where,

$$\begin{aligned} \psi(\xi) &= R\xi_{xx} + S\xi_{xy} + T\xi_{yy} + F\xi_x + G\xi_y \\ &\quad \& \\ \psi(\eta) &= R\eta_{xx} + S\eta_{xy} + T\eta_{yy} + F\eta_x + G\eta_y \end{aligned}$$

Hence, in all three cases the equation of the form (2.1) is transformed to a canonical form which contains only one term of second ordered derivative.

According to Harper¹, equations in the above said canonical form are reduced to heat equation using some definite transformations.

The method of conversion to Heat Equation is described in next section.

3 Reduction to Heat Equation from Canonical Form :

We consider the equation,

$$\frac{\partial^2 u}{\partial y^2} + a \frac{\partial u}{\partial x} + b \frac{\partial u}{\partial y} + cu = 0 \quad (3.1)$$

- First we avoid the second order derivative and solve the first order linear partial differential equation,

$$a \frac{\partial u}{\partial x} + b \frac{\partial u}{\partial y} + cu = 0$$

using known methods.

Hence we get the solution of the form

$$u = \theta \omega(\mu);$$

where θ and μ are functions of x & y .

- We consider $u = \theta \omega(\mu, \tau)$ as solution of (3.1), where τ is chosen independent of the variable y (as the term containing second ordered derivative is u_{yy}) so that (3.1) gets reduced to Heat Equation.

Examples :

Here, we convert some Parabolic Partial Differential Equations to Canonical Form and then into Heat Equation, which will illustrate the discussions in the paper.

1. We consider the BSM (i.e. Black-Schole Merton) Partial Differential Equation. Here we will compare the transformation we get to convert BSM Equation to Heat Equation with the transformation used in³ and⁴. The BSM Equation is,

$$\frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} - rV = 0 \quad (3.2)$$

Where, $V(S, t)$ is the value of European Call option.

S = Spot price of asset (i.e. the price of asset at time $t = 0$)
 r = Risk free interest rate
 σ = Volatility
 X = Exercise price or Strike price
 T = Total period of time.

As there is only one term which contains second ordered derivative it is in the canonical form.

Hence, we directly convert it into Heat equation using the process explained in section-2.

First we convert (3.2) into constant coefficient form as below.

let

$$V(S, t) = u(x, t)$$

such that,

$$x = \text{Log}(S)$$

Hence,

$$V_t = u_t$$

$$V_S = \frac{1}{S} u_x$$

$$V_{ss} = \frac{1}{S^2} (u_{xx} - u_x)$$

Therefore, (3.2) becomes,

$$\frac{\sigma^2}{2} u_{xx} + \left(r - \frac{\sigma^2}{2} \right) u_x + u_t = ru \quad (3.3)$$

Step-1:

First we obtain the solution of the first order PDE

$$\left(r - \frac{\sigma^2}{2} \right) u_x + u_t = ru.$$

Here,

$$\frac{dx}{r - \frac{\sigma^2}{2}} = \frac{dt}{1} = \frac{du}{ru}$$

Hence, we get the solution,

$$u(x, t) = e^{rt} f \left[x - \left(r - \frac{\sigma^2}{2} \right) t \right]$$

Step-2:

Now we consider the solution of (3.3) of the form,

$$u(x, t) = e^{rt} f[X, T]$$

Where

$$X = x - \left(r - \frac{\sigma^2}{2} \right) t$$

and

$$T = T(t)$$

i.e. T is independent of x.

Therefore,

$$\begin{aligned} u_x &= e^{rt} f_X \\ u_{xx} &= e^{rt} f_{XX} \\ u_t &= re^{rt} f - e^{rt} \left(r - \frac{\sigma^2}{2} \right) f_X + e^{rt} f_T T_t. \end{aligned}$$

Hence, substituting these expressions in (3.3), we get,

$$f_{XX} = - \frac{2}{\sigma^2} f_T T_t$$

So, if we take $T = - \frac{\sigma^2 t}{2}$, then,

$$T_t = -\frac{\sigma^2}{2}$$

$$\Rightarrow f_{XX} = f_T$$

Which is the required Heat Equation.

Now,

$$\begin{aligned} X &= x - \left(r - \frac{\sigma^2}{2} \right) t \\ &= \text{Log}[s] - \left(r - \frac{\sigma^2}{2} \right) t \end{aligned}$$

Hence, $V(S, t)$ is transformed to $f(X, T)$, where,

$$S = e^{X + \left(r - \frac{\sigma^2}{2} \right) t}$$

and

$$T = -\frac{\sigma^2 t}{2}$$

which is approximately same as the transformation used in³ and⁴.

2. We take another example⁵,

$$u_x - u_{yy} + \frac{y}{3x} u_x = 0$$

Here, $R=0$, $S=0$, $T=1$, $F=\frac{y}{3x}$ and $G=H=0$.

Also, as $S^2 - 4RT = 0$, this is a Parabolic Partial Differential Equation.

● We must note that this equation is already in canonical form.

● **Reduction to Heat Equation:**

Step-1:

First we solve the first order PDE

$$u_x + \frac{y}{3x} u_y = 0.$$

Hence we get solution

$$u(x, y) = f\left(\frac{y}{x^{\frac{1}{3}}}\right)$$

Step-2:

Now, we try to get solution $u = f(X, T)$ of the given PDE.

Where, $X = \left(\frac{y}{x^{\frac{1}{3}}} \right)$ and T is independent of y.

Hence,

$$u_x = -\frac{y}{3x^{\frac{4}{3}}} f_X + f_T T_x$$

$$u_y = \frac{f_X}{x^{\frac{1}{3}}}$$

$$u_{yy} = \frac{f_{XX}}{x^{\frac{2}{3}}}$$

Substituting all these values in given PDE, we get,

$$-\frac{y}{3x^{\frac{4}{3}}} f_X + f_T T_x - \frac{f_{XX}}{x^{\frac{2}{3}}} + \frac{y}{3x} \frac{f_X}{x^{\frac{1}{3}}} = 0$$

Hence

$$f_T T_x - \frac{f_{XX}}{x^{\frac{2}{3}}} = 0$$

Therefore if we take $T = 3 x^{\frac{1}{3}}$, then $T_x = x^{-\frac{2}{3}}$.

This implies,

$$f_{XX} = f_T$$

Which is the required Heat Equation.

Here, $u(x, y)$ is transformed to $f(X, T)$, where,

$$X = \left(\frac{y}{x^{\frac{1}{3}}} \right)$$

and

$$T = 3 x^{\frac{1}{3}}.$$

We must note that this transformation is different from the transformation used in⁵ to reduce this PDE into Heat Equation.

Conclusion

We have given the method of conversion (in section-2) of certain type of Parabolic Partial Differential Equations into their Canonical Form and (in section-3) a method is explained to convert equations in their Canonical Form to Heat Equation. Also, we have taken some Parabolic Partial Differential Equations and illustrated the method. We conclude that Parabolic Partial Differential Equations of certain type, which is described in this paper can be converted to Heat Equation.

Scope for Future Work :

Reduction of specific type of Parabolic Type Partial Differential Equations is carried out where the term $f(x, y, u, u_x, u_y)$ in (1.1) is

$$f(x, y, u, u_x, u_y) = F \frac{\partial u}{\partial x} + G \frac{\partial u}{\partial y} + Hu.$$

Further, we will work out on other Parabolic Type Partial Differential Equations in which the term $f(x, y, u, u_x, u_y)$ has different structure.

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