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Fixed Point Theorems and its Applications in Partial and Generalized Partial Cone Metric Spaces

VARSHA SHRIVASTAVA[1], RIZWANA JAMAL[2] and ANIL RAJPUT[3]

[1]H.No. 100, Gomti colony, Nehru Nagar, Bhopal (India)

[2]Professor, Department of Mathematics, Safia Science College, Bhopal (India)

[3]Professor and HOD, Department of Mathematics, Chandra Shekhar Azad Govt. P.G. College, Sehore (India)

Corresponding Author Email: dranilrajput@hotmail.com

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Abstract

The main purpose of this paper is to prove some fixed point theorems and its applications in partial and generalized partial cone metric spaces. Our results are satisfying various contractive conditions on cone spaces. We also prove the uniqueness of such fixed points theorems.

1. Introduction

During the sixties, the notion of 2-metric space introduced by G¨ahler (see [5,6] as a generalization of usual notion of metric space (X, d). But different authors proved that there is no relation between these two functions, for instance, Ha *et al*. in[7] show that 2-metric need not be continuous function, further there is no easy relationship between results obtained in the two settings.

In 1992, Bapure Dhage in his Ph.D. thesis introduce a new class of generalized metric space called *D*-metric spaces[1,2].

In 2005, Mustafa and Sims[11] introduced a new structure of generalized metric spaces which are called G-metric spaces as generalization of metric space (X, d) to develop and introduce a new ûxed point theory for various mappings in this new structure.

2. Partial Cone Metric Spaces :

Partial metric spaces have been originally developed by Matthews,[10] in 1992 to provide mechanism generalizing metric space theories. This new field has vast application in the study of computer domains and semantics.

In 1980, Rzepecki,[12] introduced a generalized metric d_E on a set X in a way that d_E: X × X → P, replacing the set of real numbers with a Banach space E in the metric function where P is a normal cone in E with a partial order ≤. Lin[9] considered the notion of cone metric spaces by replacing real numbers with a cone P in the metric function in which it is called a K - metric.

Twenty years after from Lin's work. Huang and Zhang,[8] announced the notion of a cone metric space by replacing real numbers with an ordering Banach space, which is the same as either the definition of Rzepecki or of Lin. Their discussion of some properties of convergence of sequences and proofs of the fixed point theorems of contractive mappings for cone metric spaces drew many authors to publish papers on topological properties of cone metric spaces, see[3, 4] and fixed point theorems of contractive mappings for cone metric spaces, see[8].

3. Preliminary Notes :

In this section we will have discussed theorems and existence results and examples are as under:

Definition 3.1 : A partial cone metric on a nonempty set X is a function p: X×X → E, where E be a real Banach space such that for all $x, y, z \in X$.

(p_1) $x = y \text{ } if \text{ } and \text{ } only \text{ } if \text{ } p(x,x) = p(x,y) = p(y,y)$,

(p_2) $0 \leq p(x,x) \leq p(x,y)$,

(p_3) $p(x,y) = p(y,x)$

(p_4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

A partial cone metric space is a pair (X,p) such that X is a nonempty set and p is a partial cone metric on X. It is clear that, if p(x,y) =0, then from (p_1) and (p_2) x = y. But if x = y, p(x,y) may not be zero. A cone metric space is a partial cone metric space. But there are partial cone metric spaces which are not cone metric spaces. The following two examples illustrate such two partial cone metric spaces.

Example 3.2 : E = R^2, P = {(x,y)∈ E|x,y ≥ 0}, X= R^+ and p: X × X → E defined by

$$p(x,y) = (\max\{x,y\}, \alpha \max\{x,y\})$$

where α ≥ 0 is a constant. Then (X,p) is a partial cone metric space which is not a cone metric space.

Example 3.3 : E = l_1, P= {{x_n} ∈ l_1 |x_n≥0}, Let X = {(x_n)|(x_n) ∈ $(R^+)^w$, $\sum x_n<\infty$}, where $(R^+)^w$ be the set of all infinite sequences over R^+, and p: X× X → E defined by

$$p(x,y)= (x_1\vee y_1, x_2 \vee y_2,....,x_n \vee y_n, ...)$$

where the symbol ∨ denotes the maximum, i.e., x ∨ y = max {x,y}, Then (X,p) is a partial cone metric space which is not a cone metric space.

Let (X, d) be a partial cone metric space, x ∈ X and A be a non-empty subset of X. We modify the concepts of distance between the set A and the singleton {x} and the distance between two subsets A and B of X in the following.

p (x, A) = inf {p(x, a)|a ∈A}, p(A, B) = inf {p(a, b)|a ∈ A,B ∈B}

Theorem 3.4 : Let (X, p) be a partial cone metric space and P be a normal cone with normal constant K. then (X, p) is topological space To.

Definition 3.5 : Let (X, p) be a partial cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c\in$ intP there is positive integer N such that for all n > N, $p(x_n, x) << c+p(x,x)$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x, and x is the limit of $\{x_n\}$ We denote this by

$$\lim x_n = x \text{ or } x_n \to x, \text{ as } n \to\infty$$

Theorem 3.6 : Let (X, p) be a partial cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $p(x_n x) \to p(x,x)$, as $n \to\infty$.

Lemma 3.7 : Let $\{x_n\}$ be a sequence in partial cone metric space (X,p). If a point x is the limit of (x_n) and $p(y,y)= p(y,x)$, then y is the limit point of (x_n).

Definition 3.8 : Let (X,p) be a partial cone metric space, $\{x_n\}$ be a sequence in X, $\{x_n\}$ is Cauchy sequence if there is $a \in P$ such that for every $\in>0$ there is positive integer N such that for all n, m >N,

$$\|p(x_n, x_m) - a\| < \in.$$

Lemma 3.9 : If (X,p)be a partial cone metric space, then the function $d_p: X \times X \to P$ defined by

$$d_p(x, y) = p(x, y) - p(x, x)$$

is a quasi-cone metric space on X. If we denote the quasi-cone metric topology Td_p and partial cone metric topology T_p, then $T_p= Td_p$

Theorem 3.10 : Let (X, p) be a partial cone metric space. If $\{x_n\}$ is a Cauchy sequence in (X,p), then it is a Cauchy sequence in the cone metric space (X,d).

Theorem 3.11 : Let (X, G) be a G-metric space and let $T: X \to X$ be a mapping such that T satisfies that

(A_1) $G(Tx, Ty, Tz) \le aG(x, Tx, Tx) + bG(y, Ty, Ty) + cG(z, Tz, Tz)$ for all $x,y,z \in X$ where $0 < a + b + c < 1$,

(A_2) T is G-continuous at a point $u \in X$

(A_3) there is $x \in X$; $\{T^n(x)\}$ has a subsequence $\{T^{ni}(x)\}$ G-converges to u. Then u is a unique fixed point (i. e. Tu = u)

Theorem 3.12 : Let (X, G) be a complete G-metric space and let $T: X \to X$ be a mapping satisfies the following condition for all $x,y,z \in X$

$G(Tx, Ty, Tz) \le aG(x, Tx, Tx) + bG(y, Ty, Ty) + cG(z, Tz, Tz) + dG(x, y, z)$,

where $0 \le a + b + c + d < 1$,

then T has a unique fixed point, say u and T is G-continuous at u.

Theorem 3.13 : Let (X, G) be a complete G-metric space and let $T: X \to X$ be a mapping satisfies the following condition for all $x,y,z \in X$

$$G(Tx, Ty, Tz) \le aG(x, Tx, Tx) + bG(y, Ty, Ty) + cG(z, Tz, Tz)$$

$where\ 0 \le a + b + c < 1, then\ T\ has\ a\ unique\ fixed\ point$,

say u and T is G-continuous at u.

Example 3.14 : Let $X = [0,1), Tx = \frac{x}{4}$ $and\ G(x,y,z) = max\{|x-y|, |y-z|, |x-z|\}.$

Then (X, G) is G-metric space but not complete, since the sequence $x_n = 1 - \frac{1}{n}$ is G-Cauchy which is not

G-convergent in (X, G) However condition (20 and (3) in theorem 2.3.1 are satisfied.

Theorem 3.15 : Let (X, G) be a G-metric space and let $T : X \to X$ be a G-continuous mapping satisûes the following conditions:

(B1) $G(Tx, Ty, Tz) \leq k\{G(x, Tx, Tx) + G(y, Ty, Ty) + G(z, Tz, Tz)\} for\ all\ x, y, z \in M$, where M is an every where dense subset of X (with respect the topology of G–metric convergence) and $0 < k < 1/6$,

(B2) there is $x \in X$ such that $\{T^n(x)\} \to x_0$. Then x_0 is unique ûxed point.

4 Main Result

Now we extend Banach fixed point theorem to partial cone metric spaces in the following

Theorem 4.1 : Let (X,p) be a complete partial cone metric space, P be a normal cone with normal constant K. Suppose that the mapping T : X→X satisfies the contractive condition.

$$p(Tx,Ty) \leq cp(x,y) \text{ for all } x,y \in X:$$

where $c \in (0,1)$ is a constant. Then T has a unique fixed point in X and for any $x \in X$ iterative sequence $\{T^n(x)\}$ converge to the fixed point.

Now we give a generalization of following theorem which is also new for partial metric spaces as a special case.

Theorem 4.2 : Let (X,p) be a complete cone partial metric space. P a normal cone with normal constant K. Suppose the mapping T : X→X satisfies the contractive condition.

$$p(Tx, Ty) \leq \alpha p(Tx, x) + \beta p(Ty, y), \text{ for all } x, y \in X,$$

where $\alpha, \beta > 0$ are constant and $\alpha + \beta < 1$. Then T has a unique fixed point X, And for any $x \in X$, iterative sequence $\{T^n(x)\}$ converges the fixed point.

Proof. Choose $x_o \in X$. Define the sequence $x_1 = Tx_0$, $x_2 = Tx_1 = T^2 x_o \ldots x_{n+1} = Tx_n = T^{n+1} x_o \ldots$ Then we have

$$p(x_{n+1}, x_n) = p(Tx_n, Tx_{n-1})$$
$$\leq \alpha p(Tx_n, x_n) + \beta p(Tx_{n-1}, x_{n-1})$$
$$\leq \alpha p(x_{n+1}, x_n) + \beta b(x_n, x_{n-1})$$

So, we get $p(x_{n+1}, x_n) \leq \dfrac{\beta}{1-\alpha} p(x_n, x_{n-1})$

$$= hp(x_n, x_{n-1})$$
$$\leq h^2 p(x_{n-1}, x_{n-2})$$
$$\leq h^n p(x_1, x_0)$$

where $h = \dfrac{\beta}{(1-\alpha)}$ and $0 < h < 1$. For $m > n$,

$$p(x_m, x_n) \leq p(x_m, x_{m-1}) + p(x_{m-1}, x_{m-2}) + \ldots + p(x_{n+2}, x_{n+1}) + p(x_{n+1}, x_n) - \sum_{k=1}^{m-n-1} p(x_{m-k}, x_{m-k})$$
$$\leq p(x_m, x_{m-1}) + p(x_{m-1}, x_{m-2}) + \ldots + p(x_{n+2}, x_{n+1}) + p(x_{n+1}, x_n),$$
$$\leq [h^{m-1} + h^{m-2} + \ldots + h^n) p(x_1, x_o)$$
$$\leq \frac{h^n}{(1-h)} p(x_1, x_o)$$

We get $\| p(x_n, x_m) \| \leq \frac{h^n}{(1-h)} K \| p(x_1, x_0) \|$.

This $p(x_n, x_m) \to 0$ as $n,m \to \infty$

Hence $\{x_n\}$ is a Cauchy sequence. By the completeness of X, there is $x \in X$ such $x_n \to x$, as $n \to \infty$

$$\lim p(x_n, x) = p(x,x) = \lim p(x_n, x) = 0$$

Since $p(Tx,x) \leq p(Tx, Tx_n) + p(Tx_n, x) - p(Tx_n, Tx_n)$

$$\leq \alpha p(Tx, x) + \beta p(Tx_n, Tx_n)(x_{n+1}, x)$$

$$p(Tx,x) \leq \frac{1}{(1-\alpha)} [\beta p(Tx_n, Tx_n)(x_{n+1}, x)]$$

$$\| p(Tx, x) \| \leq K[\frac{1}{(1-\alpha)} [\beta \| p(Tx_n, x_n) \| + \| p(x_{n+1}, x) \|]$$

$$\| p(Tx, x) \| \leq K[\frac{1}{(1-a)} [\beta p(Tx_n - x_n) \| + \| p(xn+1, x) \|]]$$

Hence $p(Tx,x)=0$, But since

$$p(Tx, Tx) \leq \alpha p(Tx, x) + \beta p(Tx, x)$$

$$= (\alpha + \beta) p(Tx, x)$$

$$= 0$$

We have $p(Tx, Tx) = p(Tx, x) = p(x, x) = 0$ which implies that $Tx = x$. So x is fixed point of T.

Uniqueness : Now if y is another fixed point of T, then

$p(x,y) = p(Tx,Ty) \leq \alpha p(Tx,x) + \beta p(Ty,y) = \beta p(y,y)$

Now $0 < \beta < 1$ and $0 \leq p(y,y) < p(x,y)$

given $p(y,y) = 0$ etc. Hence $p(x,y) = p(x,x) = p(y,y) = 0$. We get $x=y$, thus the fixed point of T is unique.

Theorem 4.3 : Let (X, p) be a complete cone partial metric space. P a normal cone with normal constant K. Suppose the mapping $T : X \to X$ satisfies the contractive condition.

$$p(Tx,Ty) \leq \alpha p(Tx,x) + \beta p(Ty,y), + yp(x,y) + \eta p(\tau x,y) + \nu p(x,Ty)$$

where $\alpha, \beta, \gamma, \eta, \nu > 0$ are constant and $\alpha + \beta + \gamma + \eta + \nu < 1$. Then T has a unique fixed point X, And for any $x \in X$, iterative sequence $(T(x)\}$ converges the fixed point.

Proof. Choose $x_o \in X$. Define the sequence $x_1 = Tx_0$, $x_2 = Tx_1 = T^2 x_o \ldots x_{n+1}$

$x_1 = T_x o$, $x_2 = Tx_1 = Tx^2 x_o, \ldots, x_{n+1} = Tx_n$

$p(x_{n+1, xn}) = p(Tx_n, Tx_{n-1})$

$\leq \alpha p(T_{xn}.x_n) + \beta P(Tx_{n-1}, x_{n-1}) + y p(x_n, x_{n-1}) + \eta p(Tx_n, x_{n-1}) + \nu (x_n, y_{n-1})$

$\leq \alpha p(x_{n+1}, x_n) + \gamma p(x_n, x_{n-1}) + \gamma p(x_n, x_{n-1}) + \eta p(x_{n+1}, x_{n-1}) + \nu (x_n, x_n)$

$\Rightarrow p(x_{n+1}, x_n) \leq (\alpha p(x_{n+1}, x_n) + \beta p(x_n, x_{n-1}) + \gamma p(x_n, x_{n-1}) + \eta [p(x_{n+1}, x_n] + p(x_n, x_{n-1}) - p(x_n, x_n)](1-\alpha-\eta) p(x_{n+1}, x_n)$

$\leq (\beta + n + y) p(x_n, x_{n-1})$

$$\therefore p(x_{n+1}, x_n) \leq \frac{B + n + y}{(1 - \alpha - \eta)} p(x_{n,} x_{n-1})$$

$$= np(x_n, x_{n-1})$$

$\leq h^2\, p(x_{n-1}, x_{n-2})$
$\leq h^n\, p(x_1, x_0)$

where $h = \dfrac{B + y + n}{1 - \alpha - \eta}$ and $0 < h < 1$

for m>n , $p(x_{n-1}, x_n) \leq p(x_{n1}, x_{-m1}) + p(x_{m-1}, x_{m-2}) + \ldots.. + p(x_{n+2}, x_{n+1}) + p(x_{n+1}, x_n) - \dfrac{m-n-1}{2}\, p(x_{m-k}, k_{m-k})$

$\leq p(x_m, X_{m-1}) + p(x_{m-1}, x_{m-2}) + .. + p(x_{n+2}, x_{n+1}) + p(x_{n+1}, x_n)$
$\leq [h^{m-1+} h^{m-2} + \ldots + h^n]\, p(x_1, x_0)$

$$\Sigma \frac{h^n}{(1-h)} = p(x_1, x_0)$$

We get $\| p(xn,xm)\| \leq \dfrac{h^n}{(1-h)}\, k \,\| p(x_1, x_0)\|$. This $p\,(xn, x_{n1}) \to 0$, $n,m \to \infty$. Hence $\{x_n\}$ is a cauchy sequence. By the completeness of x there is $x \leftarrow X$ such that $x_n \to x$ as $n \to \infty$

$$\lim_{n\to\infty} p(xn,x) = p(x,x) = \lim_{n\to\infty} p(xn,x) = 0$$

Since $p(Tx,x) \leq p(Tx,Txn) + p(Txn,k) - p(Txn,Txn)$
$\leq \alpha p(Tx,x) + \beta P\,(Txn,xn) + \gamma p(x,xn) + \eta\, p(T_x, x_n) + \nu p(x, x_n) + p(x_{n+1}, x)$

$p(Tx,x) \leq \dfrac{1}{(1-\alpha)}\, [p(Txn,xn) + \gamma p(xn] + \eta p(Tx,x_n) + \nu p(x,x_n) + p(xn+1,x)]$

$\| p(Tx,x)\| \leq [\dfrac{1}{(1-\alpha)}[\beta \,\| p(Tx_n, x_n)]\gamma \,\| p(x_1, x_n)\,\| + \eta\, \| p(Tx,xn)\| + \nu\, \| p(x,x_n)\| + \| p(x_{n+1}, x)\|]$

Hence p (Tx, x)=0, But since
$p(Tx,Tx) \leq \alpha\, p(Tx,x) + p(Tx,x) + \gamma p(x,x) + \eta p(Tx,x) + \nu\, p\,(x,Tx)$
$= (\alpha+\beta+\eta)\, p\,(Tx,x) + \gamma\, p(x,x) + \nu\, p(x,Tx)$
$= 0$
We have p(Tx, Tx) = p(Tx,x) = p(x,x) =0 which implies that Tx=x, So x is a fixed point of T.

Uniqueness : Now if y is another fixed point of T, then
$p(x,y) = p(Tx,Ty)$
$\leq \alpha\, p(Tx,x) + \beta p(\,(Ty,y) + \gamma\, p(x,y) + \eta p(Tx,y) + \nu p(x,Ty)$
$[1-\gamma-\eta]\, p(x,y) \leq \beta\, p\,(y,y)$
Now $0<\beta<1$ and $0 \leq p(y,y) \leq p(x,y)$ given (y,y) =0 etc.
Hence p(x,y) = p(x,x)- p(y,y), we get x=y, then the fixed point of T is unique.

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