ISSN 2231-346X (Print)
JUSPS-A Vol. 30(10), 384-388 (2018). Periodicity-Monthly
(Online) ISSN 2319-8044
Section A

JOURNAL OF ULTRA SCIENTIST OF PHYSICAL SCIENCES
An International Open Free Access Peer Reviewed Research Journal of Mathematics
website:- www.ultrascientist.org
Estd. 1989

On trilateral generating functions of biorthogonal polynomials suggested by Laguerre polynomials

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http://dx.doi.org/10.22147/jusps-A/301002

Acceptance Date 04th February, 2018, Online Publication Date 2nd October, 2018

Abstract

In this note, we have obtained some novel results on trilateral generating functions involving $Y_n^{\alpha+nk}(x;k)$, a modified form of Konhauser biorthogonal polynomials, $\{Y_n^{\alpha}(x;k)\}$ with Tchebycheff polynomials by group theoretic method. As special cases, we have obtained the corresponding results on Lagurrepolynomials.

Key words : Biorthogonal polynomials, Lagurre polynomials, trilateral generating functions.
AMS-2010 Subject Classification Code: 33C45, 33C47, 33C65.

Introduction

In 1967[1], Konhauser introduced two sets of polynomials $\{Y_n^{\alpha}(x;k)\}$ and $\{Z_n^{\alpha}(x;k)\}$, which are biorthogonal with respect to the weight function $x^{\alpha}e^{-x}$ over the interval $(0,\infty)$, $\alpha > -1$, k is a positive integer. These polynomials satisfy the following condition:

$$\int_0^{\infty} x^{\alpha}\exp(-x)\,Y_i^{\alpha}(x;k)Z_j^{\alpha}(x;k)\,dx \begin{cases} = 0, & i \neq j, \\ \neq 0, & i = j; \end{cases} \qquad i,j = 0,1,2,\dots .$$

For $k = 1$, these polynomials reduce to the generalized Laguerre polynomials, $L_n^{\alpha}(x)$. In the present paper we are interested only on $Y_n^{\alpha}(x;k)$. In[2], Carlitz gave an explicit representation for the polynomials $Y_n^{\alpha}(x;k)$ in the following form:

$$Y_n^{\alpha}(x;k) = \frac{1}{n!}\sum_{i=0}^{n}\frac{x^i}{i!}\sum_{j=0}^{i}(-1)^j\binom{i}{j}\left(\frac{j+\alpha+1}{k}\right)_n,$$

where $(a)_n$ is the pochhammer symbol defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & if\ n = 0, \quad a \neq 0 \\ a(a+1)\ldots(a+n-1), & \forall\ n \in \{1,2,3\ldots\}. \end{cases}$$

In[3], Chongdar and Chatterjea gave a general method based on the theory of one parameter group of continuous transformations, with the help of which any unilateral generating relation involving one special function can be transformed into a trilateral generating relation with Tchebycheff polynomials.

In fact in[3], a unilateral generating relation is converted to bilateral generating relation with the help of one parameter group of continuous transformations and then this bilateral generating relation is converted into a trilateral generating relation with the Tchebycheff polynomial by means of the relation

$$T_n(x) = \frac{1}{2}\left[\left(x+\sqrt{x^2-1}\right)^n + \left(x-\sqrt{x^2-1}\right)^n\right].$$

The aim at presenting this paper is to obtain some novel result on trilateral generating relations for the polynomials, $Y_n^{\alpha+nk}(x;k)$, a modified form of Konhauser biorthogonal polynomials, $Y_n^{\alpha}(x;k)$ with Tchebycheff polynomial by utilizing the above mentioned method of Chongdar[4-8] and Chatterjea.For previous works on Konhauser biorthogonal polynomials one can see the works. As special cases, we obtain the corresponding results on Laguerre polynomials, $L_n^{(\alpha)}(x)$. The main result of our investigation is stated in the form of the following theorem:

Theorem 1: If there exista unilateral generating relation of the form

$$G(x,w) = \sum_{n=0}^{\infty} a_n\, Y_n^{\alpha+nk}(x;k)\, w^n \tag{1.1}$$

then

$$\sum_{n=0}^{\infty}\sigma_n(x,v)T_n(u)w^n$$

$$= \frac{1}{2}\Bigg[(1+k\rho_1)^{\frac{(1+\alpha-k)}{k}}\exp\left(x\left[1-(1+k\rho_1)^{\frac{1}{k}}\right]\right)G\left(x(1+k\rho_1)^{\frac{1}{k}},\ v\rho_1(1+\rho_1 k)\right)$$

$$+ (1+k\rho_2)^{\frac{(1+\alpha-k)}{k}}\exp\left(x\left[1-(1+k\rho_2)^{\frac{1}{k}}\right]\right)G\left(x(1+k\rho_2)^{\frac{1}{k}},\ v\rho_2(1+\rho_2 k)\right)\Bigg], \tag{1.2}$$

where

$$\sigma_n(x,v) = \sum_{p=0}^{n} a_p\, k^{n-p} \binom{n}{p} Y_n^{(\alpha-nk+2pk)}(x;k) v^p,$$

$$\rho_1 = w\left(u + \sqrt{u^2-1}\right) \quad \text{and} \quad \rho_2 = w\left(u - \sqrt{u^2-1}\right).$$

2. Proof of theorem 1

For the biorthogonal polynomials, $Y_n^{\alpha+nk}(x;k)$, we consider the following linear partial differential operator:

$$R = xyz^{-2k}\frac{\partial}{\partial x} + k\,y^2 z^{-2k}\frac{\partial}{\partial y} + yz^{1-2k}\frac{\partial}{\partial z} - (x+k-1)yz^{-2k}$$

such that

$$R\left(Y_n^{\alpha+nk}(x;k)y^n z^\alpha\right) = k(n+1)\,Y_{n+1}^{\alpha+nk-k}(x;k)y^{n+1}z^{\alpha-2k}. \tag{2.1}$$

The extended form of the group generated by R is given by

$$e^{wR}f(x,y,z) = (1+kwyz^{-2k})^{\frac{1-k}{k}}\exp\left\{x - x(1+kwyz^{-2k})^{\frac{1}{k}}\right\}$$
$$\times f\left(x(1+kwyz^{-2k})^{\frac{1}{k}},\ y(1+kwyz^{-2k}),\ z(1+kwyz^{-2k})^{\frac{1}{k}}\right), \tag{2.2}$$

where $f(x,y,z)$ is an arbitrary function and w is an arbitrary constant.
At first, we consider the generating relation of the form:

$$G(x,w) = \sum_{n=0}^{\infty} a_n Y_n^{\alpha+nk}(x;k)\, w^n. \tag{2.3}$$

Replacing w by wyv and multiplying both sides of (2.3) by z^α, we get

$$z^\alpha\, G(x,wvy) = \sum_{n=0}^{\infty} a_n \left(Y_n^{\alpha+nk}(x;k)y^n z^\alpha\right)(wv)^n. \tag{2.4}$$

Operating e^{wR} on both sides of (2.4), we get

$$e^{wR}\left(z^\alpha\, G(x,wvy)\right) = e^{wR}\left(\sum_{n=0}^{\infty} a_n \left(Y_n^{\alpha+nk}(x;k)y^n z^\alpha\right)(wv)^n\right). \tag{2.5}$$

Now the left member of (2.5), with the help of (2.2), reduces to

$$(1+wkyz^{-2k})^{\frac{(1+\alpha-k)}{k}}\exp\left(x - x(1+wkyz^{-2k})^{\frac{1}{k}}\right)z^\alpha$$

$$\times\, G\left(x(1+wykz^{-2k}z)^{\frac{1}{k}}, \quad wvy(1+wkyz^{-2k})\right). \tag{2.6}$$

The right member of (2.5), with the help of (2.1), becomes

$$= \sum_{n=0}^{\infty} (wy)^n \sum_{p=0}^{n} a_{n-p} k^p \binom{n}{p} Y_n^{\alpha+nk-2pk}(x;k) z^{\alpha-2pk} v^{n-p}. \tag{2.7}$$

Now equating (2.6) and (2.7) and then substituting $y = z = 1$, we get

$$(1+kw)^{\frac{(1+\alpha-k)}{k}} \exp\left(x\left[1-(1+kw)^{\frac{1}{k}}\right]\right) G\left(x(1+kw)^{\frac{1}{k}}, \ wv(1+wk)\right)$$

$$= \sum_{n=0}^{\infty} w^n \sigma_n(x,v), \tag{2.8}$$

where

$$\sigma_n(x,v) = \sum_{p=0}^{n} a_p\, k^{n-p} \binom{n}{p} Y_n^{\alpha-nk+2pk}(\mathrm{x};\mathrm{k})\, v^p.$$

We now proceed to convert the above bilateral generating relation into a trilateral generating relation with Tchebycheff polynomial.

Now L.H.S. of (1.2),

$$\sum_{n=0}^{\infty} \sigma_n(x,v) T_n(u) w^n$$

$$= \frac{1}{2}\Bigg[(1+k\rho_1)^{\frac{(1+\alpha-k)}{k}} \exp\left(x\left[1-(1+k\rho_1)^{\frac{1}{k}}\right]\right) G\left(x(1+k\rho_1)^{\frac{1}{k}}, \ v\rho_1(1+\rho_1 k)\right)$$

$$+ (1+k\rho_2)^{\frac{(1+\alpha-k)}{k}} \exp\left(x\left[1-(1+k\rho_2)^{\frac{1}{k}}\right]\right) G\left(x(1+k\rho_2)^{\frac{1}{k}}, \ v\rho_2(1+\rho_2 k)\right)\Bigg],$$

= R.H.S. of (1.2),

where

$$\sigma_n(x,v) = \sum_{p=0}^{n} a_p\, k^{n-p} \binom{n}{p} Y_n^{(\alpha-nk+2pk)}(x;k) v^p,$$

$$\rho_1 = w\left(u+\sqrt{u^2-1}\right) \quad \text{and} \quad \rho_2 = w\left(u-\sqrt{u^2-1}\right).$$

This completes the proof of the theorem and does not seem to have appeared in the earlier works.

Special case 1 Now putting $k=1$ in our Theorem 1 we get the following result on generalized Laguerre polynomials:

Theorem 2 If there exists a generating relation of the form

$$G(x,w) = \sum_{n=0}^{\infty} a_n \, L_n^{(\alpha+n)}(\mathrm{x}) \, w^n \tag{2.9}$$

then

$$\sum_{n=0}^{\infty} \sigma_n(x,v) T_n(u) w^n$$

$$= \frac{1}{2}\Big[(1+\rho_1)^{\alpha} \exp(-x\rho_1) G\big(x(1+\rho_1), \quad v\rho_1(1+\rho_1)\big)$$

$$+(1+\rho_2)^{\alpha} \exp(-x\rho_2) G\big(x(1+\rho_2), \quad v\rho_2(1+\rho_2)\big)\Big], \tag{2.10}$$

where

$$\sigma_n(x,v) = \sum_{p=0}^{n} a_p \binom{n}{p} L_n^{(\alpha-n+2p)}(\mathrm{x}) \, v^p,$$

$$\rho_1 = w\big(u + \sqrt{u^2-1}\big) \quad \text{and} \quad \rho_2 = w\big(u - \sqrt{u^2-1}\big).$$

Conclusion

From the above discussion, it is clear that whenever one knows a unilateral generating relation of the form (1.1, 2.9) then the corresponding trilateral generating relation can at once be written down from (1.2, 2.10). So one can get a large number of trilateral generating relations by attributing different suitable values to a_n in (1.1, 2.9).

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