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Geometry and Spectral Variation: the Operator Norm

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Abstract

In this paper, we will obtain if A is a q-**k**-normal matrix and B is any matrix close to A, then the optimal matching distance $d(\sigma(A),\sigma(B))$ is bounded by $\|A-B\|$.

Key words : q-**k**-Hermitian , q-**k**-Skew-Hermitian, q-**k**-normal path, q-**k**-unitary

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Introduction

We will use the notation $\sigma(A)$ for both the subset of the quaternion plane that consists of all the q-**k**-eigenvalues on $n\times n$ matrix A, and for the unordered n-tuple whose entries are the q-**k**-eigenvalues of A counted with multiplicity. Since we will be taking of the distances $s(\sigma(A),\sigma(B)), h(\sigma(A),\sigma(B))$ and $d(\sigma(A),\sigma(B))$, it will be clear which of the two objects is being represented by $\sigma(A)$.

We explore, how fare, these results can we carried over the q-**k**-normal matrices. The first difficulty we face is that, if the matrices re not q-**k**-Hermitian, there is no natural way to order their q-**k**-eigenvalues. So, the problem has to be formulated in terms of optimal matchings even after this has been done, analogues of the inequalities above turn out to be a little more complicated. Though several good results are known, many await discovery.

Definitions and Some Theorems

Theorem 2.1:

Let A be a q-**k**-normal and let B be any matrix such that $\|A-B\|$ is smaller half of the distance between any two-distinct q-**k**-eigenvalues of A. Then $d(\sigma(A),\sigma(B)) \leq \|A-B\|$.

Proof :

Let $\alpha_1, \alpha_2, ..., \alpha_k$ be all the distinct q-**k**-eigenvalues of A.

Let $\delta = \|A-B\|$, all the q-**k**-eigenvalues of B lie in union of the disks $\overline{D}(\alpha_j, \delta)$. By the hypothesis, these disks are mutually disjoint.

We will show that if q-**k**-eigenvalue $'\alpha_j'$ has multiplicity m_j, then the disk $\overline{D}(\alpha_j, \delta)$ contains exactly m_j q-**k**-eigenvalues of B, counted with their respective multiplicities. Once this is established, the statement of the theorem is seen to follow easily.

Let $A(t) = (1-t)A + B; \;\; 0 \leq t \leq 1.$

$\Rightarrow$ This is a continuous map from $[0,1]$ into the space of quaternion matrices.

$\Rightarrow A(0) = A$ and $A(1) = B$

$\Rightarrow \|A-B\| = \|A(0) - A(1)\|$

$\Rightarrow \|A - A(t)\| = t\delta$

So, all the q-**k**-eigenvalues of $A(t)$ also lie in the disks $\overline{D}(\alpha_j, \delta)$ for each $0 \leq t \leq 1$, as t moves from 0 to 1 the q-**k**-eigenvalues of $A(t)$ trace continuous curves can jump from one of the disks $\overline{D}(\alpha_j, \delta)$ to another. So, if we start off with m_j such curves in the disk $\overline{D}(\alpha_j, \delta)$. We must end up with exactly as many.

Hence proved.

Remark 2.2:

Let $H_{n\times n}$ denote the set of q-**k**-normal of a fixed size n. If A is an element of $H_{n\times n}$, then so is tA for all real $'t'$. Thus the set $H_{n\times n}$ is path connected. However, N is not an affine set.

Definition 2.3:

A continuous map $'\gamma'$ from any interval $[a,b]$ into $H_{n\times n}$ will be called a q-**k**-normal path or a q-**k**-normal curve. If $\gamma(a) = A$ and $\gamma(b) = B$, We say that γ is a path joining A and B, then A and B are end prints of γ. The length of γ is defined with respect to the norm $\|\cdot\|$ by $\mathbf{l}_{\|\cdot\|}(\gamma) = \sup \sum_{k=0}^{m-1} \|\gamma(t_{k+1}) - \gamma(t_k)\|$ (1)

Where the supremum is taken over all partitions of $[a,b]$ as $a = t_0 < t_1 < .... < t_m = b$.

Remark 2.4:

If this length is finite, the path γ is said to be rectifiable. If the function γ is piecewise H' function then $\mathbf{l}_{\|.\|}(\gamma) = \int_a^b \|\gamma'(t)\| dt$ (2)

Theorem 2.5:

Let A and B be q-**k**-normal matrices, and let γ be rectifiable q-**k**-normal path joining them T , then $d(\sigma(A), \sigma(B)) \leq \mathbf{l}_{\|.\|}(\gamma)$ (3)

Proof:

For our convenience, let us choose the parameter $'t'$ to vary in $[0,1]$.

For $0 \leq r \leq 1$, let γ_r be that part of the curve which is parameterised by $[0, r]$.

Let $G = \{r \in [0,1];\ d(\sigma(A), \sigma(\gamma(r)) \leq \mathbf{l}_{\|.\|}(\gamma_r)\}$. The theorem will be proved if we show that the point 1 is in G.

Since the function γ_1 the arc length, and the distance $'d'$ are all continuous in their arguments, the set G is closed. So it contains the point $g = \sup G$.

We have to show that $g = 1$. Suppose $g < 1$, let $S = \gamma(g)$ lying theorem (2.1). We can find a point t in (g,1]. Such that, if $T = \gamma(t)$, then $d(\sigma(B), \sigma((T)) \leq \|S - T\|$.

But then $d(\sigma(A), \sigma(\gamma(t)) \leq d(\sigma(A), \sigma(S)) + d(\sigma(S), \sigma(T))$

$$\leq \mathbf{l}_{\|.\|}(\gamma_g) + \|S - T\|$$

$$\leq \mathbf{l}_{\|.\|}(\gamma_t)$$

By the definition of g, this is not possible. So $g = 1$.

Hence proved.

Remark 2.6:

An effective estimate of $d(\sigma(A), \sigma(B))$ can thus be obtained if one could find that the length of the shortest normal path joining A and B. This is a difficult problem since the geometry of the set $H_{n\times n}$ is poorly understood. However, the theorems above have several interesting consequences.

Definition 2.7:

Let S be any subset of $H_{n\times n}$. We will say that S is metrically flat in the metric induced by the norm $\|.\|$. If any two points A and B in S can be joined by a path that lies entirely with in S and has length $\|A - B\|$.

Remark 2.8:

Every affine set in metrically flat . A non-trivial exchange of a $\|.\|$ flat set is given by the theorem below. Let U be the set of $n \times n$. q-**k**-unitary matrices and H.U the set of all constant multiple of q-**k**-unitary matrices.

Theorem 2.9:

The set H.U is || . || flat.

Proof:

First note that H.U consists of just non-negative real multiplies of q-**k**-unitary matrices.

Let $A_0 = r_0\ U_0$ and $A_1 = r_1\ U_1$ be any two elements of this set, where $r_0, r_1 \geq 0$.

Choose an orthonormal basis in which the q-**k**-unitary matrix is $U_1U_0^{-1}$ diagonal.

$U_1U_0^{-1} = dia(e^{i\theta_1}, ..., e^{i\theta_n})$ with $|\theta_n| \leq |\theta_{n-1}| \leq .... \leq |\theta_1| \leq \pi$.

We, Reduce to such a form can be achieved by a q-**k**-unitary conjugation. Such a process changes neither q-**k**-eigenvalues nor norms. So, we may assume that all q-**k**-matrices are written with respect to the above orthonormal basis.

Let $K = dia\left(i\theta_1, i\theta_2, ..., i\theta_n\right)$, then K is q-**k**-Skew-Hermitian matrix whose q-**k**-eigenvalues are in the interval $(-i\pi, i\pi]$.

Therefore, we have,

$$\|A_0 - A_1\| = \|r_0U_0 - r_1U_1\|$$

$$= \|r_0I - r_1U_1U_0^{-1}\|$$

$$= \max_j \left|r_0 - r_1e^{i\theta_j}\right|$$

$$= \left|r_0 - r_1e^{i\theta_1}\right|.$$

This last quantity is the length of the straight line joining the points r_0 and $r_1\ e^{i\theta_1}$ in the quaternion space. Parameterise this line segment as $r(t)e^{it\theta_1}$, $0 \leq t \leq 1$. This can be done except when $|\theta_1| = \pi$, an exceptional case to which we will return later. The equation above can then be written as

$$\|A_0 - A_1\| = \int_0^1 \left|r(t)e^{it\theta_1}\right|' dt$$

$$= \int_0^1 |r'(t) + r(t)i\theta_1| dt$$

Now, let $A(t) = r(t)e^{(tk)U_0}$, $0 \leq t \leq 1$.

This is a smooth curve in H.U with end points A_0 and A_1. The length of this curve is

$$\int_0^1 \|A'(t)\| dt = \int_0^1 \left\|r'(t)e^{(tk)U_0} + r(t)ke^{(tk)U_0}\right\| dt$$

$$= \int_0^1 \|r'(t)I + r(t)k\| dt,$$

Since, $e^{(tk)U_0}$ is a q-**k**-unitary matrix.

$$\|r'(t)I + r(t)K\| = \max_j |r'(t) + ir(t)\theta_j|$$

$$= |r'(t) + ir(t)\theta_1|$$

We put the last equation together, we see that the path $A(t)$ joining A_0 and A_1 has length $\|A_0\text{-}A_1\|$.

The exceptional case $\|\theta_1\| = \pi$ is much simpler. The piecewise linear path that joins A_0 to 0 and then to A_1 has length $r_0 + r_1$.

This is equal to $\left|r_0 + r_1 e^{i\theta_1}\right|$ and hence to $\|A_0 - A_1\|$.

Thus H.U is flat

Hence proved.

Theorem 2.10:

The set $H_{n\times n}$ q-**k**-normal matrices is || . || flat if and only if $n \le 2$.

Proof :

Let A and B be 2×2 q-**k**-normal matrices. If the q- **k**-eigenvalues of A and these of B lie on two parallel lines, We assume that these two lines are parallel to real axis.

Then the q-**k**-Skew-Hermitian part of $A-B$ is scalar and hence $A-B$ is q-**k**-normal.

The straight line joining A and B then they are lying in $H_{n\times n}$.

If the q-**k**-eigenvalues of A and B do not lie on parallel lines, then they lie on two concentric circles. If α is common centre of these circles then A and B are in the set α + H.U.

This set is || . || flat. Thus, in either case, A and B can be joined by q-**k**-normal path of length $\|A-B\|$.

Hence proved.

Remark 2.11:

If $n \ge 3$ then $H_{n\times n}$ cannot be || . || flat because of theorem (2.5).

Example 2.12:

Here is an example of a q-**k**-Hermitian A and a q-**k**-Skew-Hermitian matrix B that cannot be joined by a q-**k**-normal path of length $\|A-B\|$.

Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$; $B = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$

Then $\|A-B\| = 2.$

If there were a q-**k**-normal path of length 2 joining A,B then the midpoint of this path would be a normal matrix C such that $\|A-C\| = \|B-C\| = 1.$

Since each entry of a matrix is dominated by its norm, this implies that $|C_{21}-1| \le 1$ and $|C_{21}+1| \le 1$

Hence $C_{21} = 0$.

By the same argument, $C_{32} = 0$.

So $A-C = \begin{pmatrix} * & * & * \\ 1 & * & * \\ * & 1 & * \end{pmatrix}$

Where * represents an entry whose value is not yet known. But if $\|A-C\| = 1.$

We must have $A-C = \begin{pmatrix} 0 & 0 & * \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

Hence, $C = \begin{pmatrix} 0 & 1 & * \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

But then C could not have been normal.

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