ISSN 2231-346X (Print)
JUSPS-A Vol. 31(2), 4-10 (2019). Periodicity-Monthly
ISSN 2319-8044 (Online)
Section A

JOURNAL OF ULTRA SCIENTIST OF PHYSICAL SCIENCES
An International Open Free Access Peer Reviewed Research Journal of Mathematics
website:- www.ultrascientist.org
Estd. 1989

Study of Cubic B Spline Interpolation

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http://dx.doi.org/10.22147/jusps-A/310201

Acceptance Date 30th January, 2019, Online Publication Date 14th February, 2019

Abstract

In this study, we discuss the numerical solution of the wave equation subject to non-local conservation condition, using cubic trigonometric B-spline collocation method (CuTBSM). Consider a vibrating elastic string of length L which is located on the x-axis of the interval [0, L].

It is also clear from the examples that the approximate solution is very close to the exact solution. The technique requires smaller computational time than several other methods and the numerical results are found to be in good agreement with known solutions and with existing schemes in the literature.

Key words: Cubic trigonometric B Spline Interpolation, collocation method, non-Newtonian fluid, non-classical diffusion equation.

Subject Classification: 65D,65L,65M

Introduction

There are quite a number of phenomena in science and engineering which can be modeled by the use of hyperbolic partial differential equations subject to non-local conservation condition instead of traditional boundary conditions[1] and these arise in the study of chemical heterogeneity[2,3], medical science, visco-elasticity, plasma physics[4] and thermo elasticity[5,6]. This type of problems also arises in non-local reactive transport in underground water flows in porous media, semi-conductor modeling, non-Newtonian fluid flows and radioactive nuclear decay in fluid flows[7]. The temperature distribution of air near the ground over time during calm clear nights is a good example of such models[8]. The analysis, development and implementation of numerical methods for the solution of such problems have received wide attention in the literature.

One of the most interesting equation in physical phenomena is reaction-diffusion equation. Since the equationis a model equation used in biology, chemistry, metallurgy and combustion, both analytical and numerical solutions are searched to investigate new physical phenomena[25].

This study deals with the numerical solution of a non-classical diffusion problem with two nonlocal boundary constraints using cubic trigonometric B-splines. This problem arises in several branches of science. In particular, electrochemistry[1], heat conduction process[2], thermo-elasticity[3], plasma physics[4], semiconductor modeling[5], biotechnology[6], control theory, and inverse problems[7]. The analysis, development, and implementation of numerical methods for the solution of such diffusion problems have received wide attention in the literature.

Consider an insulated rod of length L located on the x-axis of the interval $[0, L]$. Let the rod have a source of heat.

Let (x, t) denote the temperature in the insulated rod with ends held at constant temperature T_1 *and* T_2 and the initial temperature distribution along the rod is $g(x)$. The problem is to study the flow of heat in the rod and in this paper the partial differential equation governing the flow of heat in the rod is given by the diffusion equation with specification of energy

$$\frac{\partial u}{\partial t}(x,t) = \propto^2 \frac{\partial^2 u}{\partial x^2}(x,t) + q(x,t) \qquad 0 \le x \le L, 0 \le t \le T \tag{1}$$

With the initial constraints

$$u(x, t = 0) = g_1(x) \qquad 0 \le x \le L \tag{2}$$

And the nonlocal boundary constraints

$$\xi_1 u(x = 0, t) + \xi_2 u(x = 0, t) = \int_0^L g_2(x) u(x,t) dx + h_1(t) = T_1,$$

$$\xi_3 u(x = L, t) + \xi_4 u(x = L, t) = \int_0^L g_3(x) u(x,t) dx + h_2(t) = T_2 \qquad 0 < t \le T \tag{3}$$

where ξ_i, $i = 1, 2, 3, 4$ are known constants, g_i, $i=1,2,3,4$ are known continuous functions.

Solution of Non-classical Diffusion Problem :

Consider a uniform mesh Ω with grid points (x_i, t_n) to discretize the grid region $\Delta = [a, b] \times [0,]$ with $x_i = a + ih$ $i = 0, 1, 2, . . . , N$ and $t_n = nk$, $n = 0, 1, 2, 3, . . .$ M, $Mk = T$.

Here the quantities h and k are mesh space size and time step size, respectively. The time derivative can be approximated by using the standard finite difference formula:

$$\frac{\partial u^n}{\partial t} = \frac{u^{n+1} - u^n}{k} \tag{4}$$

Using the approximation of (4), (1) becomes

$$\frac{u^{n+1} - u^n}{k} = \propto^2 \frac{\partial^2 u^n}{\partial x^2} + q(x_i, t_{n+1}) \tag{5}$$

Using $\emptyset$-weighted technique, the space derivatives of (5) can be written as

$$\frac{u^{n+1} - u^n}{k} = \emptyset\left(\propto^2 \frac{\partial^2 u^{n+1}}{\partial x^2}\right) + (1-\emptyset)\left(\propto^2 \frac{\partial^2 u^n}{\partial x^2}\right) + q(x_i, t_{n+1}) \tag{6}$$

where $0 \le \emptyset \le 1$ and the subscripts n and $n + 1$ are successive time levels. It is noted that the system becomes an explicit scheme when $\emptyset = 0$, a fully implicit scheme when $\emptyset = 1$, and a

Crank-Nicolson scheme when $\emptyset = 1/2$. In this paper, we use the Crank-Nicolson approach. Hence, (6) becomes

$$\frac{u^{n+1}-u^n}{k} = \frac{1}{2}(\propto^2 \frac{\partial^2 u^{n+1}}{\partial x^2}) + \frac{1}{2}(\propto^2 \frac{\partial^2 u^n}{\partial x^2}) + q(x_i, t_{n+1}) \tag{7}$$

On simplification:

$$2u^{n+1} - k\propto^2 u^{n+1} = 2u^n + k\propto^2 u^n + 2kq(x_i, t_{n+1}) \tag{8}$$

The space derivatives are approximated by using cubic trigonometric B-spline and are discussed in the next section.

3. Cubic Trigonometric B-Spline Technique :

In this section, we discuss the cubic trigonometric B-spline collocation method (CuTBSM) for the numerical solution of the non-classical diffusion equation (1). Consider a mesh $a \le x \le b$ which is equally divided by knots x_i into N subintervals

$[x_i, x_{i+1}]$, $i = 0, 1, 2, \ldots, N-1$ where $a = x_0 < x_1 < \cdots < x_N = b$.

Our approach for the non-classical diffusion equation using collocation method with cubic trigonometric B-spline is to seek an approximate solution as

$$U(x,t) = \sum_{i=-3}^{N-1} C_i(t) TB_i(x) \tag{9}$$

where $C_i(t)$ are to be determined for the approximated solutions $U(x, t)$ to the exact solutions $u(x, t)$, at the point $(x_i\ t_n)$. $TB_i(x)$ are twice continuously differentiable piecewise cubic trigonometric B-spline basis functions over the mesh defined by

$$\mathrm{T}B_i(x) = \frac{1}{w}\begin{cases} p^3(x_i) & x \in [x_i, x_{i+1}] \\ p(x_i)\big(p(x_i)q(x_{i+2}) + q(x_{i+3})p(x_{i+1}) & \\ +q(x_{i+4})p^2(x_{i+1}), & x \in [x_{i+1}, x_{i+2}] \\ q(x_{i+4})\big(p(x_{i+1})q(x_{i+3}) + q(x_{i+4})p(x_{i+2})\big) + & \\ p(x_i)q^2(x_{i+3}), & x \in [x_{i+2}, x_{i+3}] \\ q^3(x_{i+4}), & x \in [x_{i+3}, x_{i+4}] \end{cases} \tag{10}$$

Table 1: Values TB(x) and its derivatives

x	x_i	x_{i+1}	x_{i+2}	x_{i+3}	x_{i+4}
$\mathrm{T}B_i$	0	a_1	a_2	a_1	0
$\mathrm{T}B_i'$	0	a_3	0	a_4	0
$\mathrm{T}B_i''$	0	a_5	a_6	a_5	0

Where $p(x_i) = \sin(\frac{x - x_i}{2})$

$$q(x_i) = \sin(\frac{x_i - x}{2})$$

$$w = \sin\left(\frac{h}{2}\right)\sin(h)\sin(\frac{3h}{2}) \tag{11}$$

and where h = $(b - a)/n$. The approximations U_i^n at the point $(x_i,\ t_n)$ over subinterval $[x_i,\ x_{i+1}]$ can be defined as

$$U_i^n = \sum_{j=i-3}^{i-1} C_k^n TB_j(x) \tag{12}$$

In order to obtain the approximations to the solutions, the values of $TB_i(x)$ and its derivatives at nodal points are required and these derivatives are tabulated in Table 1, where

$$a_1 = \frac{sin^2(\frac{h}{2})}{\sin(h)\sin(\frac{3h}{2})}$$

$$a_2 = \frac{2}{1+2\cos(h)}$$

$$a_3 = -\frac{3}{4\sin(\frac{3h}{2})}$$

$$a_4 = \frac{3}{4\sin(\frac{3h}{2})}$$

$$a_5 = \frac{3(1+3\cos(h))}{16sin^2\left(\frac{h}{2}\right)(2\cos\left(\frac{h}{2}\right)+\cos(\frac{3h}{2})}$$

$$a_6 = -\frac{3cos^2(\frac{h}{2})}{sin^2\left(\frac{h}{2}\right)(2+4\cos(h))}$$

Using approximate functions (10) and (12) the values at the knots of $U_i^n(x)$ and their derivatives up to second order are determined in terms of time parameters C_j^n as

$$\begin{aligned} U_i^n &= a_1 C_{i-3}^n + a_2 C_{i-2}^n + a_1 C_{i-1}^n \\ (U_x)_i^n &= a_3 C_{i-3}^n + a_4 C_{i-1}^n \\ (U_{xx})_i^n &= a_5 C_{i-3}^n + a_6 C_{i-2}^n + a_5 C_{i-1}^n \end{aligned} \tag{14}$$

Substituting (12) into (8) gives the following equation:

$$2\sum_{j=i-3}^{i-1} C_j^{n+1} TB_j(x_i) - k \propto^2 \sum_{j=i-3}^{i-1} C_j^{n\_1} TB_j''(x_i)$$
$$=2\sum_{j=i-3}^{i-1} C_j^n TB_j(x_i) + k \propto^2 \sum_{j=i-3}^{i-1} C_j^n TB_j''(x_i) + 2kq(x_i, t_{n+1}) \tag{15}$$

The system thus obtained on simplifying (15) consists of $N+1$ linear equations in $N+3$ unknowns

$C^{n+1} = (C_{-3}^{n+1}, C_{-2}^{n+1}, C_{-1}^{n+1}, \ldots, C_{N-1}^{n+1})$ at the time level $t = t_{n+1}$. Equation (9) is applied to the boundary constraints (2) and (3) for two additional linear equations to obtain a unique solution of the resulting system:

$$\xi_1 U(0, t_{n+1}) + \xi_2 U_x(0, t_{n+1}) = \int_0^L g_2(x) U(x, t_{n+1}) dx + h_1(t_{n+1}) \tag{16}$$

$$\xi_3 U(L, t_{n+1}) + \xi_4 U_x(L, t_{n+1}) = \int_0^L g_3(x) U(x, t_{n+1}) dx + h_2(t_{n+1}) \tag{17}$$

From (15), (16), and (17), the system can be written in the matrix vector form as follows:

$$MC^{n+1} = NC^n + b \tag{18}$$

Where

$$C^{n+1} = [C_{-3}^{n+1}, C_{-2}^{n+1}, C_{-1}^{n+1}, \ldots, C_{N-1}^{n+1}]^T,$$
$$C^n = [C_{-3}^n, C_{-2}^n, C_{-1}^n, \ldots, C_{N-1}^n]^T.$$
$$n = 0, 1, 2, \ldots, M \tag{19}$$

and M and N are $N + 3$-dimensional matrix given by

$$\propto_1 (t_{n+1}) = \int_0^L g_2(x) u(x, t_{n+1}) dx + h_1(t_{n+1}),$$
$$\beta_1(t_{n+1}) = \int_0^L g_3(x) u(x, t_{n+1}) dx + h_2(t_{n+1}). \tag{20}$$

Thus, the system (18) becomes a matrix system of dimension $(N + 3) \times (N + 3)$ which is a tridiagonal system that can be solved by the Thomas Algorithm[16].

3.1. Initial State Vector C^0. After the initial vectors C^0 have been computed from the initial constraints, the approximate solutions U_i^{n+1} at a particular time level can be calculated repeatedly by solving the recurrence relation (15) . The initial vectors C^0 can be obtained from the initial condition and boundary values of the derivatives of the initial condition as follows :

$$(U_i^0)_x = g_1'(x_i), \quad i = 0, \tag{21}$$
$$U_i^0 = g_1(x_i), \quad i = 0,1,2, \ldots \ldots N \tag{22}$$
$$(U_i^0)_x = g_1'(x_i), \quad i = \mathrm{N}$$

Thus (22) yields a $(N + 3)\times(N + 3)$ matrix system, of the form

$$\mathrm{A}C^n = d_1 \tag{23}$$

Where

$$C^n = [C_{-3}^n, C_{-2}^n, C_{-1}^n, \ldots \ldots C_{N-1}^n]^T.$$
$$\mathrm{d} = [g_1'(x_0), g_1(x_0), \ldots \ldots \ldots g_1(x_N), g_1'(x_N)]^T \tag{24}$$

Table 2 comparison between TMOL[10] and present method (CuTBSM)

t	K=0.01 TMOL[10]	Present method (CuTBSM)	K= 0.005 TMOL[10]	Present method (CuTBSM)	K= 0.001 TMOL[10]	Present method (CuTBSM)
0.1	4.5E-04	4.31E-04	4.0E-05	1.96E-04	1.5E-05	8.25E-06
0.3	1.4E-03	5.88E-04	1.4E-04	2.69E-04	2.5E-05	1.33E-05
0.5	2.5E-03	5.20E-04	2.5E-04	2.38E-04	2.4E-05	1.20E-05
0.7	4.0E-03	4.23E-04	4.0E-04	1.92E-04	2.7E-05	8.36E-06
0.9	5.5E-03	3.32E-04	5.5E-04	1.50E-04	6.0E-05	4.03E-06
1.0	6.0E-04	2.91E-04	6.0E-04	1.31E-04	6.8E-05	1.90E-06

4. Results and Discussions

In this section, the cubic trigonometric B-spline collocation method is employed to obtain the numerical solutions for one-dimensional non classical diffusion problem with nonlocal boundary constraints given in (1)–(3). Two numerical examples are discussed in this section to exhibit the capability and efficiency of the proposed trigonometric spline method. Numerical results are compared with existing methods in the literature and with the exact solution at the different nodal points x_i for some time levels t_n using some particular space step size h and time step k.

5. Conclusion

In this paper, a new two-time level implicit scheme based on cubic trigonometric B-spline has been used to solve thenonclassical diffusion problem with known initial and with nonlocal boundary constraints instead of the usual boundary constraints. A usual finite difference discretization is used for time derivatives and cubic trigonometric B-spline is applied for space derivatives. It is noted that the accuracy of solution may reduce as time increases due to the time truncation errors of time derivative term[24]. The technique requires smaller computational time than several other methods and the numerical results are found to be in good agreement with known solutions and with existing schemes in this field.

Acknowledgement

Manuscript communication number (MCN) : IU/R&D/2018-MCN000513 office of research and development integral university, Lucknow

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