ISSN 2231-346X (Print)

JUSPS-A Vol. 31(8), 69-78 (2019). Periodicity-Monthly

Section A

(Online) ISSN 2319-8044

Estd. 1989

JOURNAL OF ULTRA SCIENTIST OF PHYSICAL SCIENCES

An International Open Free Access Peer Reviewed Research Journal of Mathematics

website:- www.ultrascientist.org

An Orthogonal Stabilization of Quadratic and Generalized Quadratic Functional Equations

[1]KIRAN YADAV and [2]A. K. MALIK

[1]Department of Mathematics,
Singhania University, Pacheri Bari, Jhunjhunu (Rajasthan), (India)
[2]Department of Mathematics
B. K. Birla Institute of Engineering & Technology, Pilani (Rajasthan), (India)
Corespondinding author Email.ajendermalik@gmail.com, raoavi11@gmail.com

http://dx.doi.org/10.22147/jusps-A/310801

Acceptance Date 21st August, 2019, **Online Publication Date 26th August, 2019**

Abstract

This study is devoted to the stabilization of following quadratic and modified quadratic functional equations in orthogonal space

$$h(3x \pm y) = 16h(x) + h(x \pm y),$$

and $$h(x + ay) + h(x - ay) = 2a^2h(y) + 2h(x).$$

Keywords: Orthogonal spaces, Quadratic and Modified functional equations.

1. Introduction

In 1975, Gudder *et. al.*[12] first established the orthogonal stability of the Cauchy functional equations $h(r+s) = h(r) + h(s)$ with $r \perp s$. This result was further extended and studied to examine the orthogonal stability for the mapping h by Ger and Sikorska[11] on the steps of Ratz[6]. Further, the stability of the functional equation

$$h(r+s) + h(r-s) = 2h(r) + 2h(s), \text{ with } x \perp y$$

On Hilbert orthogonal space was studied by famous mathematician Vajzovic[3]. The results of Vajzovic[3] were generalized by Szabo[2], Drljevic[5], Fochi[8]. Furthermore, for more study on orthogonal spaces one may refer to[1, 7, 10, 13, 14].

This article deals with the orthogonal stabilization of the following functional equations defined as

$$h(3x \pm y) - 16h(x) - h(x \pm y) = 0$$

$$h(x + ay) + h(x-ay) - 2a^2h(y) - 2h(x) = 0$$

The paper is divided into four sections. Section 2 is introductory in nature. Sections 3 and 4 present the stability of quadratic and modified quadratic functional equations.

2. Preliminaries :

This section contains the following orthogonality result studied by many researchers such as Ratz[6], James[4], Birkhoff [9], etc.

Definition 1. Let X be a linear space with dimension greater then equal to two and perpendicular (⊥) is the operator defined on X which satisfies the following conditions:

(A1) $r \perp 0$, $0 \perp r$, $\forall r \in X$ (Totality)

(A2) $r, s \in X - \{0\} \Rightarrow r \perp s$, (Independence)

(A3) if $r \perp s$, then $\alpha r \perp \beta s$ for $\alpha, \beta \in$ □ and for all $r, s \in X$, (Homogeneity)

(A4) Let Y is a subspace of *X*, $r \in Y$ and λ be a positive scalar number, then for $y_0 \in Y$ and $r \perp y_0$ we have $r + y_0 \perp \lambda r - y_0$. (Thalesian property)

Then the combination $(X, \perp)$ is known asorthogonality space. It is also known as symmetric if $r \perp s$ and $s \perp r$, $\forall r, s \in X$.

Definition 2. Let $(X, \perp)$ be an orthogonal space and Z be a Banach space. Then, the relation $h : X \to Z$ is called orthogonal quadratic map if it satisfies the system (1).

3. Orthogonal stability of quadratic equations :

In this section we prove that orthogonal stability of following quadratic functional equations

$$P(h) = h(3r \pm s) - 16h(r) - h(r \pm s) \tag{1}$$

Theorem 1. Let us consider *h* be the quadratic function which satisfies

$$\|P(h))\|_Z \le \eta(\|r\|_X^p + \|s\|_X^p) \tag{2}$$

$\forall r, s \in X$ with $x \perp y$ and p<2. Then, the mapping $R : X \to Z$ satisfying

$$\|h(r) - R(r)\|_Z \le \frac{\eta}{2(3^2 - 3^p)}\|r\|_X^p \tag{3}$$

is uniqueorthogonality solution.

Proof. Putting $s = 0$ in (2), we get

$$\|2h(3r) - 16h(r) - 2h(r)\|_Z \le \eta(\|r\|_X^p + \|0\|_X^p)$$

$$\|2h(3r) - 18h(r)\|_Z \le \eta(\|r\|_X^p)$$

$$\left\| h(r) - \frac{h(3r)}{3^2} \right\|_Z \leq \frac{\eta}{2.3^2} \|r\|_X^p \tag{4}$$

Changing $r = 3r$ and then dividing throughout by 3^2 in inequality (4) and also summing the obtained result with (4), we get

$$\left\| h(r) - \frac{h(3^2 r)}{3^4} \right\|_Z \leq \frac{\eta}{2.3^2} \left(1 + \frac{3^p}{3^2} \right) \|r\|_X^p \tag{5}$$

Proceeding in this way n-times, we get the following inequality

$$\left\| h(r) - \frac{h(3^n r)}{3^{2n}} \right\|_Z \leq \frac{\eta}{2.3^2} \sum_{k=0}^{n-1} \frac{3^{pk}}{3^{2k}} \|r\|_X^p \tag{6}$$

Now, to prove that the sequence $< h(3^n r)/3^{2n} >$ is a Cauchy sequence. Changing r with $3^m r$ and then dividing throughout by 3^{2m} in (6) we get for all $n, m > 0$.

$$\left\| \frac{h(3^m r)}{3^{2m}} - \frac{h(3^{n+m} r)}{3^{2n+2m}} \right\|_Y \leq \frac{\eta}{2.3^2} \sum_{k=0}^{n-1} \frac{3^{p(k+m)}}{3^{2k+2m}} \|r\|_X^p ,$$

$$\frac{1}{3^{2m}} \left\| h(3^m r) - \frac{h(3^{n+m} r)}{3^{2n}} \right\|_Z \leq \frac{\eta}{2.3^2.3^{2m-pm}} \sum_{k=0}^{n-1} \frac{3^{pk}}{3^{2k}} \|r\|_X^p \tag{7}$$

As $m \to \infty$ for all $r \in X$ and p<2 the sequence $< h(3^n r)/3^{2n} >$ is converges to a point in *Z*. Further, as the Banach space *Z* is a complete, thus $< h(3^n r)/3^{2n} >$ is a Cauchy sequence. Thus, we can say

$$R(r) = \lim_{n\to\infty} \{h(3^n r)/3^{2n}\}, \ \forall r \in X . \tag{8}$$

Putting $3^n r$ and $3^n s$ for *r* and *s* in (2) respectively and then dividing by the number 3^{2n}, we have

$$\left\| \frac{P(h)}{3^{2n}} \right\|_Z \leq \frac{\eta}{3^{2n}} (\|3^n r\|_X^p + \|3^n s\|_X^p) \tag{9}$$

Letting $n \to \infty$, we obtain

$$\|R(3r \pm s) - 16R(r) - R(r \pm s)\|_Z \leq 0$$

$$R(3r \pm s) = 16R(r) + R(r \pm s), \ \forall r, s \in X .$$

Hence *R* is orthogonally quadratic relation.

Taking $n \to \infty$ in (6) we get

$$\|R(r) - h(s)\|_Z \leq \frac{\eta}{2(3^2 - 3^p)} \|r\|_X^p , \ \forall r \in X .$$

For uniqueness of $R: X \to Z$, let us consider the relation $R': X \to Z$ which satisfies (2), then we get

$$\|R'(r)-R(r)\|_Z \leq \frac{1}{3^{2n}}\{\|h(3^n r)-R'(3^n r)\|_Z + \|R(3^n r)-h(3^n r)\|_Z\}$$

$$\leq \frac{\eta}{(3^2-3^p)3^{n(2-p)}}\|r\|_X^p \to 0 \text{ as } n \to \infty$$

Thus, $R' = R$, that means *R* is unique.

Theorem 2. Let h be the quadratic function which satisfies the inequality (2) for all $r, s \in X$ with $r \perp s$ and $p > 2$. Then, the mapping $R: X \to Z$ satisfying

$$\|h(r)-R(s)\|_Z \leq \frac{\eta}{2(3^p-3^2)}\|r\|_X^p \tag{10}$$

is a unique quadratic orthogonal mapping.

Proof. Putting r/3 at the place of r and then multiplying by 3^2 in inequality (4), we get

$$\left\|3^2 h(\frac{r}{3})-h(r)\right\|_Z \leq \frac{\eta}{2}.\left\|\frac{r}{3}\right\|_X^p,$$

$$\left\|3^2 h(\frac{r}{3})-h(r)\right\|_Z \leq \frac{\eta}{2.3^p}\|r\|_X^p \tag{11}$$

$r \perp 0$ for all $r \in X$. Proceeding in this way n-times we get the following inequality

$$\left\|3^4 h(\frac{r}{3^2})-h(r)\right\|_Z \leq \frac{\eta}{2.3^p}\left(1+\frac{3^2}{3^p}\right)\|r\|_X^p \tag{12}$$

$$\left\|3^{2n} h(\frac{r}{3^n})-h(r)\right\|_Z \leq \frac{\eta}{2.3^p}\sum_{k=0}^{n-1}\frac{3^{2k}}{3^{kp}}\|r\|_X^p \tag{13}$$

$$\leq \frac{\eta}{2.3^p}\sum_{k=0}^{\infty}\frac{3^{2k}}{3^{kp}}\|r\|_X^p$$

Now, to prove the sequence $< h(3^n r)/3^{2n} >$ is convergent. Replacing *r* with $r/3^m$ and then multiplying by 3^{2m} in the inequality (13), we get

$$\left\|3^{2n+2m} h(\frac{r}{3^{n+m}})-3^{2m}h(\frac{r}{3^m})\right\|_Z \leq \frac{\eta}{2.3^{m(p-2)}}\sum_{k=0}^{\infty}\frac{3^{2k}}{3^{p(k-1)}}\|r\|_X^p \tag{14}$$

Which tends to 0 as $m \to \infty$ for all in the right hand side of (14). Therefore, we prove that the sequence $< 3^{2n} h(r/3^n) >$ converges in the Banach space *Y*, hence the $< 3^{2n} h(r/3^n) >$ is a Cauchy sequence. Thus, we get the orthogonal quadratic system $R : X \to Z$ such that

$$\lim_{n\to\infty}\{3^{2n} h(r/3^n)\} = R(r) \quad \text{for all } r \in X. \tag{15}$$

Taking $n \to \infty$ in (14) and using (15), we get the required result.

4. Orthogonal stability for generalized quadratic equation :

This section deals with the orthogonal stability of the following modified quadratic equation

$$h(x+ay) + h(x-ay) - 2a^2h(y) - 2h(x) = 0 \tag{16}$$

Theorem 3. Let us consider X be a normed linear space, Y be a Banach space and $\zeta : X \times X \to [0, \infty)$ be a mapping such that

$$\lim_{n\to\infty} \frac{\zeta(a^n x, a^n y)}{a^{2n}} = 0 \tag{17}$$

for all $x, y \in X$. If the function $h : X \to Y$ with $h(0) = 0$, satisfies

$$\|h(x+ay) + h(x-ay) - a^2h(y) - 2h(x)\| \le \zeta(x, y) \tag{18}$$

for all $x, y \in X$. Then, the map $R : X \to Y$ is a unique quadratic function satisfying the relation

$$\| R(y) - h(y)\| \le \frac{1}{2a^2} \sum_{i=0}^{\infty} \frac{1}{a^{2i}} \zeta(0, a^i y) \tag{19}$$

The quadratic map R is defined as

$$R(y) = \lim_{n\to\infty} \frac{h(a^n y)}{a^{2n}} \quad . \tag{20}$$

Proof: Letting $x = 0$ in the relation (18), we obtain

$$\| 2a^2h(y) - 2h(ay) \| \le \zeta(0, y) \tag{21}$$

So that

$$\left\| h(y) - \frac{h(ay)}{a^2} \right\| \le \zeta(0, y)\frac{1}{2a^2} \tag{22}$$

Now, putting $y = ay$ in (22) and dividing throughout with a^2 and then adding the final equation with (22), we have

$$\left\| h(y) - \frac{h(a^2 y)}{a^4} \right\| \le \frac{1}{2a^2}\zeta(0, y) + \frac{1}{2a^4}\zeta(0, a y) \tag{23}$$

$$\le \frac{1}{2a^2}\left[\frac{1}{a^2}\zeta(0, ay) + \zeta(0, y)\right]$$

Proceeding in this way n-times for a positive integer n, we get

$$\left\| h(y) - \frac{h(a^n y)}{a^{2n}} \right\| \leq \frac{1}{2a^2} \sum_{i=0}^{n-1} \frac{1}{a^{2i}}\ \zeta(0, a^i y) \tag{24}$$

$$\leq \frac{1}{2a^2} \sum_{i=0}^{\infty} \frac{1}{a^{2i}} \zeta(0,\ a^i y)$$

Now, we will prove the convergence of the sequence $< h(a^n y)/a^{2n} >$, changing y with $a^k y$ and then dividing relation (24) by a^{2k}, we obtain for $n, k > 0$,

$$\left\| \frac{h(a^k y)}{a^{2k}} - \frac{h(a^n.a^k y)}{a^{2n+2k}} \right\| = \left\| \frac{h(a^k y)}{a^{2k}} - \frac{h(a^{n+k} y)}{a^{2(n+k)}} \right\|,$$

$$\leq \frac{1}{a^{2k}} \left\| h(a^k y) - \frac{h(a^{n+k} y)}{a^{2n}} \right\|$$

$$\leq \frac{1}{2a^2} \frac{1}{a^{2k}} \sum_{i=0}^{\infty} \frac{\zeta(0, a^{i+k} y)}{a^{2i}}$$

$$\leq \frac{1}{2a^2} \sum_{i=0}^{\infty} \frac{\zeta(0, a^{i+k} y)}{a^{2(i+k)}} \tag{25}$$

As $k \to \infty$, the sequence $< h(a^n y)/a^{2n} >$ is a Cauchy sequence. Further, as Y is a Banach space, the sequence $< h(a^n y)/a^{2n} >$ approaches to a point $R(y) \in$ Y and thus R can be defined as

$$R(y) = \lim_{n \to \infty} \frac{h(a^n y)}{a^{2n}}.$$

Now, we replace x and y with $a^n x$, $a^n y$ in (16) and then dividing throughout with a^{2n}, to show that R is a solution of (16)

$$\left\| \frac{h(a^n (x+ay))}{a^{2n}} - \frac{2h(a^n x) + 2a^2 h(a^n y)}{a^{2n}} + \frac{h(a^n (x-ay))}{a^{2n}} \right\| \leq \frac{\zeta(a^n x,\ a^n y)}{a^{2n}}.$$

As $n \to \infty$, then R satisfies (16).

Now, Let us consider $R' : X \to Y$ be the second quadratic mapping which is the solution of (16) and (19). Thus, we get

$$\| R'(y) - R(y)\| = \frac{1}{a^{2n}} \|R'(a^n y) - R(a^n y)\|$$

$$\leq \frac{1}{a^{2n}} (\|R'(a^n y) - h(a^n y)\| + \|R(a^n y) - h(a^n y)\|)$$

$$\leq \frac{1}{a^2} \sum_{i=0}^{\infty} \frac{\zeta(0, a^{i+n} y)}{a^{2(i+n)}} \tag{26}$$

As $n\to\infty$, we get $R(y) = R'(y)$ for all $y\in X$. This completes the result.

Corollary 1. Let us consider X and Y are normed linear and Banach spaces, respectively. Let $h: X\to Y$ with the condition $h(0) = 0$ satisfies

$\|h(x + ay) + h(x-ay) - 2a^2 h(y) - 2h(x)\| \leq \varepsilon$, $\varepsilon \geq 0$ be a real number

Then, $\exists$ a unique quadratic mapping $R: X\to Y$ defined by

$$\lim_{n\to\infty} \frac{h(a^n y)}{a^{2n}} = R(y)$$

Satisfying the inequality (20) and the relation

$$\|R(y) - h(y)\| \leq \frac{\varepsilon}{2(a^2 - 1)}, \text{ for all } y\in X.$$

Moreover, for each $y\in X$ the function $m\to h(my)$ from R to Y is continuous function, then we get $a^2 R(y) = R(ay)$.

Corollary 2. Let us consider X and Y are normed linear and Banach spaces, respectively. Let $h: X\to Y$ with the condition $h(0) = 0$ satisfies

$\|h(x + ay) + h(x-ay) - 2a^2 h(y) - 2h(x)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$, where $\varepsilon \geq 0$, $0 < p < 2$.

Then, $\exists$ a unique quadratic mapping $R: X\to Y$ satisfying the inequality (16) and the relation

$$\| R(y) - h(y)\| \leq \frac{\varepsilon}{2(n^2 - a^p)} \|y\|^p$$

Where the function R is defined as

$$\lim_{n\to\infty} \frac{h(a^n y)}{a^{2n}} = R(y).$$

Moreover, for each $y\in X$ the function $m\to h(my)$ from R to Y is continuous function, then we get $a^2 R(y) = R(ay)$.

Theorem 2.3.4. Let us consider X and Y are normed and Banach spaces, respectively and $\zeta : X\times Y\to [0, \infty)$ is a mapping such that

$$\lim_{n\to\infty} a^n \, \zeta\left(\frac{x}{a^n}, \frac{y}{a^n}\right) = 0 \tag{27}$$

If the function $h : X \to Y$ with $h(0) = 0$, satisfies

$$\|h(x + ay) + h(x - ay) - 2a^2h(y) - 2h(x)\| \le \zeta(x, y) \tag{28}$$

Then, the map $R : X \to Y$ is a unique quadratic function satisfying the relation

$$\|R(y) - h(y)\| \le \frac{1}{2}\sum_{i=0}^{\infty} a^{2i}\ \zeta\left(0,\ \frac{y}{a^{i+1}}\right) \tag{29}$$

where the quadratic map R is defined as

$$\lim_{n\to\infty} a^{2n}\, h\left(\frac{y}{a^n}\right) = R(y), \quad \text{for all } y \in X. \tag{30}$$

Proof: Putting $y = y/a$ in (16) and multiplying throughout by a^2, then, we have

$$\left\| a^2h\left(\frac{y}{a}\right) - h(y) \right\| \le \frac{1}{2}\,\zeta\left(0,\ \frac{y}{a}\right) \tag{31}$$

Again changing $y = y/a$ and then multiplying throughout by a^2 in (31).

$$\left\| a^4h\left(\frac{y}{a^2}\right) - h(y) \right\| \le \frac{a^2}{2}\,\zeta\left(0,\ \frac{y}{a^2}\right) + \frac{1}{2}\zeta\left(0, \frac{y}{a}\right)$$

Thus, we obtain

$$\|h(y) - R(y)\| \le \frac{1}{2}\sum_{i=0}^{\infty} a^{2i}\ \zeta\left(0,\ \frac{y}{a^{i+1}}\right) \tag{32}$$

For the convergence of $\left\{a^{2n}\ h\left(\frac{y}{a^n}\right)\right\}$, putting $y = \frac{y}{a^k}$ and then multiplying throughout by a^{2k} in (32), we get

$$\left\| a^{2k}h\left(\frac{y}{a^k}\right) - a^{2n+2k}h\left(\frac{y}{a^{n+k}}\right) \right\| \le \frac{1}{2}\sum_{i=0}^{\infty} a^{2(i+k)}\ \zeta\left(0, \frac{y}{a^{i+k}}\right)$$

Then, from (32) the sequence $\left\{a^{2n}\ h\left(\frac{y}{a^n}\right)\right\}$, is a Cauchy sequence. But Y is a Banach space thus the sequence $\left\{a^{2n}\ h\left(\frac{y}{a^n}\right)\right\}$ converges in *Y*. So, let us define a mapping $h: X \to Y$ by

$$\lim_{n\to\infty} a^{2n}\ h\left(\frac{y}{a^n}\right) = R(y)$$

Then, using Theorem 3, the map $R: X\to Y$ is quadratic. Further, the remaining part is similar to the Theorem 3.

Corollary 3. Let $h: X\to Y$ be a mapping and $h(0) = 0$ which satisfies the inequality

$$\|h(x + ay) + h(x-ay) - 2a^2h(y) - 2h(x)\| \le \varepsilon$$

for all $x, y \in X$, then, $\exists$ a mapping $R : X \to Y$ which satisfies the relation

$$\| R(y) - h(y)\| \le \frac{q}{2(1-a^2)}$$

where the mapping R is defined as

$$\lim_{n\to\infty} a^{2n}\, h\left(\frac{y}{a^n}\right) = R(y)\ , \text{ for all } y \in X.$$

Corollary 4. Let $h: X\to Y$ be a mapping and $h(0) = 0$ which satisfies the inequality

$$\|h(x + ay) + h(x-ay) - 2a^2h(y) - 2h(x)\| \le \varepsilon(\|x\|^p + \|y\|^p)$$

for some $p > 2$, then, $\exists$ a mapping $R : X \to Y$ which satisfies the relation

$$\| R(y) - h(y)\| \le \frac{1}{2}\frac{\varepsilon}{(a^p - a^2)}\|y\|^p$$

where the mapping R is defined as

$$R(y) = \lim_{n\to\infty} a^{2n}h\left(\frac{y}{a^n}\right), \text{ for all } y \in E_1.$$

References

1. D. H. Hyers, On the Stability of the Linear Functional Equation, Proc. Nat. Acad. Sci. U.S.A. (27), 222–224 (1941).
2. F. Drljevic, On a functional which is quadratic on A-orthogonal vectors, publ. Inst. Math. (Beograd), (54), 63-71 (1986).
3. F. Vajzovic, Uber das functional H mit der Eigenschaft: $(x, y) = 0$ implies $H(x+y)+H(x-y) = 2H(x)+2H(y)$, Glasnik Mat. Ser. III, (22), 73-81 (1967).
4. G. Birkhoff, Orthogonality in linear metric spaces, Duke Math. J., (1), 169-172 (1995).
5. Gy. Szabo, Sesqulinear-orthogonally quadratic mappings, Aequationes Math., (40), 190-200 (1990).
6. J. Ratz, Onorthogonality additive mappings, Aeq. Math. (28), 35-49 (1985).
7. M. S. Moslehian, On the stability of the orthogonal Pexiderized Cauchy equation, J.Math.Anl. Appl.,(318)(1),

211-223 (2006).

8. M. Fochi, Functional equations in A-orthogonal vectors, Aequationes Math., (38), 28-40 (1989).
9. R. C. James, Orthogonality in normed linear spaces, Duke Math. J., (12), 291-302 (1945).
10. R. Chugh and Ashish, On the stability of generalized Cauchy linear functional equations, Int. J. of Math. Anal., (6)(29), 1403 – 1413 (2012).
11. R. Ger and J. Sikorska, Stability of the orthogonal additivity , Bull. Polish Acad. Sci. Math.,(43),143-151 (1995).
12. S. Gudder and D. Strawther, Orthogonality additive and Orthogonality increasing functions on vector spaces, Pacific J. Math.,(58), 427-436 (1975).
13. Th. M. Rassias, On the stability of the Linear mapping in Banach spaces, Procc. of the Amer. Math. Soc., (72)(2), 297-300 (1978).
14. W. Towanlong, p. Nakmahachalasiant, A quadratic functional equation and its generalized Hyers-Ulam-Rassias stability, Thai j. of Math., Special Issue (Annual meeting in Mathematics), 85-91 (2008).