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Existence of Smooth Epimorphism from a Fuchsian Group to the point Group of Sulfur-Hexafloride

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Abstract

The theory of Fuchsian group plays an important role in the study of compact Riemann surfaces Automorphism groups, which was initially studied by A.M. Macbeth. The biholomorphic self transformations of a compact Riemann surface S of genus $g(\geq 2)$ forms a finite group whose order cannot exceed 84 $(g-1)$. This maximum bound is attained for infinitely many values of g, the least being 3. The groups for which this bound is attained are called Hurwitz groups, and this groups belong to the class of perfect groups which are non soluble. In context to the class of soluble groups, the corresponding bound is 48 $(g-1)$ and this bound is also attained for infinitely many values of g. The problem of finding such bounds for various sub-classes of the finite soluble groups and the number of values of g for which these bounds are attained has been the theme of many research papers during the last few decades.

In this paper, a set of necessary and sufficient conditions for the existence of smooth epimorphism from a Fuchsian group to the point group O_h, which belongs to the sub-class of octahedral group considering the symmetries of Sulfur-Hexafloride (SF_6) molecule Having the abstract group representation $\langle \alpha^2=\beta^4=(\alpha\beta)^2=1\rangle$ is established to fulfill the objective.

Key words : Compact Riemann Surface, Fuchsian group, Molecular Symmetries, Smooth Epimorphism.

1991 Mathematics Subject Classification: 20H10, 30F10

1.Introduction

The Hurwitz bound is defined as the maximum bound 84 $(g-1)$, of the order of the finite group that formed by the Automorphisms of a compact Riemann surface **S** of genus $g\,(\geq 2)$, and it is attained for infinitely many values of g, the least being 3[A.M. Macbeath]. It is observed that every finite group is isometric to the Automorphism group of some Compact Riemann Surface of genus g (≥ 2). The study of the group of automorphisms of Compact Riemann surface was originally studied up by A.M. Macbeath and announced numerous problems on this topic at the Dundee Summer School Lecture, as well as declaring some partial solutions to his problems. Macbeath formulated his announcements as follows[3,4].

- All large groups of automorphisms, including Hurwitz group, are to be found.
- For any finite group **G,** the minimum value of genus g such that G acts as a group of automorphisms on a Compact Riemann Surface of genus g is to be found.

There are some significant results already obtained in this topic which have been solved for different classes of finite groups namely- Cyclic groups[7], Abelian groups (Maclachlan 1966), Solvable groups[4] etc.

The study of symmetries is one of the most appealing applications of group theory. The set of all symmetries found in a non-linear molecular structure always forms a finite group (point group) under the operation of symmetries, e.g. rotation, reflection, etc.[1,11]

The theory of Fuchsian group is intimately related to the theory of Riemann surface Automorphism groups[4,5,7,8,9,10]. A Fuchsian group Γ is an infinite group having presentation of the form:

$$\langle x_1,\ x_2, x_3,\dots\dots\dots, x_k; b_1, c_1, b_2c_2, \dots\dots, b_\gamma c_\gamma;\ x_1^{m_1} = x_2^{m_2} = \cdots \dots\dots\dots = x_k^{m_k} = \prod_{i=1}^{k} a_i = \prod_{j=1}^{\gamma} [b_jc_j] = 1\rangle. \quad (1)$$

Where $[b_jc_j]=b_j^{-1}\, c_j^{-1}\, b_jc_j$ and $\delta(\Gamma)=2\gamma-2+\Sigma\,(1-\frac{1}{m_i})>0$ (2)

It is known that if Γ_1 is a subgroup of Γ of finite index then Γ_1 is a Fuchsian group and

$$[\Gamma:\Gamma_1] = \frac{\delta(\Gamma_1)}{\delta(\Gamma)} \quad (3)$$

A homomorphism φ from a Fuchsian group Γ to a finite group G is called smooth if the kernel is a surface subgroup of Γ. Another remarkable result is that a finite group G is represented as an Automorphism group of compact Riemann surface of genus g if and only if there is a smooth epimorphism φ from a Fuchsian group Γ to G such that $ker\varphi$ has genus g[3].

Following these remarkable results in this paper we find a set of necessary and sufficient conditions on periods and genus of Fuchsian group Γ for which we have a smooth epimorphism from Γ to the point group formed by all the symmetries of the Sulfur-hexa-flouride molecule (SF_6) which possess octahedral point group O_h[Fig.1]

$O_h = \{E, 6C_4, 8C_3, 9C_2, , i, \sigma_h, 2\sigma_v, 6\sigma_d, 8S_6, 6S_4\}$

Where E is the identity element, i the centre of inversion and C_4 is the principal axis.

Observing the group multiplication table of the point Group of sulfur-hexa-flouride molecule we obtained the group presentation with two generators a and b such that the group is nothing but O_h, Group of order 48.[2]

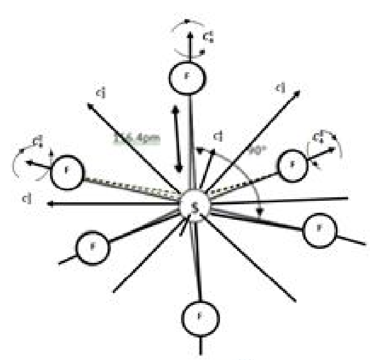

Fig:1 Symmetry elements of SF_6

2. Group Representation of the octahedral point group of Sulfur hexafluoride[2]

Table 1. List of generators

GENERATORS		
(1)$C = C_4^3 = \alpha$	(2) $W = S_6^4 = \beta$	(3)$CW = S = \sigma_d^6 = \alpha\beta$
Presentation:- $\alpha^4 = \beta^6 = (\alpha\beta)^2 = 1$		

Table 2. symmetry elements

Sl. no	Symmetry elements	presentation	Alphabetical symbol	Sl. no	Symmetry elements	permutation	Alphabetical symbol
1	E	1	E	25	i	β^3	i
2	C_4^1	$\alpha\beta^2$	A	26	σ_h	$\beta^5\alpha^2\beta^4$	O
3	C_4^{-1}	$\beta^5(\alpha\beta)$	a	27	σ_v^1	$\alpha^3\beta^2\alpha^2$	o
4	C_4^2	$\beta^3(\alpha\beta)\alpha$	B	28	σ_v^2	$\alpha^2\beta^3$	P
5	C_4^{-2}	$\alpha^3\beta^4$	b	29	σ_d^1	$\alpha\beta\alpha^2$	p
6	C_4^3	α	C	30	σ_d^2	$\beta^5\alpha\beta^2$	Q
7	C_4^{-3}	α^3	c	31	σ_d^3	$\beta\alpha$	q
8	C_3^1	$\alpha^2\beta^2$	D	32	σ_d^4	$\alpha^2\beta\alpha^3$	R
9	C_3^{-1}	$\alpha^3\beta^2\alpha$	d	33	σ_d^5	$\alpha^3\beta\alpha^2$	r
10	C_3^2	$\alpha^2\beta^4$	F	34	σ_d^6	$\alpha\beta$	S
11	C_3^{-2}	$\beta^3(\alpha\beta)\alpha^3$	f	35	S_6^1	$\alpha\beta\alpha^3$	T
12	C_3^3	β^2	G	36	S_6^{-1}	$\alpha^2\beta$	t
13	C_3^{-3}	β^4	g	37	S_6^2	$\alpha^2\beta\alpha^2$	U
14	C_3^4	$\alpha\beta^2\alpha$	H	38	S_6^{-2}	$\alpha^3\beta\alpha^3$	u
15	C_3^{-4}	$\beta^4\alpha^2\beta^4$	h	39	S_6^3	$\alpha^2\beta^5$	V
16	C_2^1	$\beta^2\alpha^2\beta^4$	J	40	S_6^{-3}	$\beta\alpha^2$	v
17	C_2^2	α^2	j	41	S_6^4	β	W
18	C_2^3	$\beta^4\alpha^2\beta^2$	K	42	S_6^{-4}	β^5	w
19	C_2^4	$\alpha^3\beta\alpha^2\beta^5$	k	43	S_4^1	$\beta\alpha^3$	X
20	C_2^5	$\beta^4\alpha^3\beta^4$	L	44	S_4^{-1}	$\alpha\beta^5$	x
21	C_2^6	$\alpha\beta^4$	l	45	S_4^2	$\alpha^3\beta^3$	Y
22	C_2^7	$\alpha^2\beta^4\alpha$	M	46	S_4^{-2}	$\alpha\beta^3$	y
23	C_2^8	$\alpha^2\beta^2\alpha$	m	47	S_4^3	$\alpha\beta\alpha^2$	Z
24	C_2^9	$\alpha^3\beta^2$	N	48	S_4^{-3}	$\alpha^3\beta$	z

Hence the presentation of the point group of Sulphur-hexa-flouride molecule is-

$$G = \langle \alpha^4 = \beta^6 = (\alpha\beta)^2 = 1 \rangle \tag{4}$$

3. Smooth Homeomorphism from a Fuchsian group Γ to G

In this section we establish a set of necessary and sufficient conditions on the genus g and periods of a Fuchsian group Γ, for the existence of a smooth epimorphism from Γ to G.

The presentation of Γ and G as given by (1) and (4) respectively, then we have the following theorem:

Theorem:[5,6] There is a smooth epimorphism $\phi: \Gamma \to G$ if and only if

(1) When $k = 0$ i.e. $\Gamma = \Delta(\gamma: -)$ a surface group then $\gamma \geq 2$.

(2) When $k \neq 0$, $\phi(x_i) = m_i$, m_i divides 48.

Moreover

(i) If all $\phi(x_i) \in \langle \alpha \rangle$ then m_i divides 4 and $\gamma \geq 1$.

(ii) If all $(x_i) \in \langle \beta \rangle$ then m_i divides 6 and $\gamma \geq 1$.

(iii) If all $\phi(x_i) \in \langle \alpha\beta \rangle$ then $m_i = 2$, k is even and $\gamma \geq 1$.

(iv) If some $\phi(x_i) \in \langle \alpha \rangle$ for $i = 1,2, \ldots, s; s < k$ and remaining $(x_{s+j}) \in \langle \beta \rangle, j = 1,2, \ldots, t$.
Then $s + t = k$ and $sl \equiv 0 (mod\ 4)$ and $1 < l < 3$ and $tm \equiv 0 (mod\ 6)$ and $1 < m < 5$.

(v) If some $\phi(x_i) \in \langle \alpha \rangle$ for $i = 1,2, \ldots, s; s < k$ and remaining $\phi(x_{s+j}) \in \langle \alpha\beta \rangle, j = 1,2, \ldots, t$.
Then $s + t = k$, t is even and $sl \equiv 0 (mod\ 4)$ and $1 < l < 3$.

(vi) If some $\phi(x_i) \in \langle \beta \rangle$ for $i = 1,2, \ldots, s; s < k$ and remaining $(x_{s+j}) \in \langle \alpha\beta \rangle, j = 1,2, \ldots, t$.
Then $s + t = k$, t is even and $sl \equiv 0 (mod\ 6)$ and $1 < l < 5$.

(vii) If some $\phi(x_i) \in \langle \alpha \rangle$ for $i = 1,2, \ldots, s; s < k$, some $\phi(x_{s+j}) \in \langle \beta \rangle, j = 1,2, \ldots, t$, and remaining $(x_{s+t+j}) \in \langle \alpha\beta \rangle, j = 1,2, \ldots, p$ such that $s + t + p = k, p,$ is even, $t + p$ is even if t is even and $t + p$ is odd if t is odd.

   (a) If $t + p$ is even then $sl \equiv 0 (mod\ 4)$ and $1 < l < 3$.

   (b) If $t + p$ is odd then $sl \equiv 0 (mod\ 4)$ and $1 < l < 3$.

Proof: Let $\phi: \Gamma \to G$ be a smooth epimorphism, then we see that –

(1) If $k = 0$ then from [2], the measure of Fuchsian group it is clear that $2\gamma - 2 > 0$

i.e. if $k = 0$ then $\gamma \geq 2$.

Therefore the condition is necessary.

Again the condition is sufficient because, if $= 0, \gamma \geq 2$, then we can define $\phi: \Gamma \to G$ by

$$\phi(a_1) = \alpha^2 = \phi(b_1)$$
$$\phi(a_2) = \beta^3 = \phi(b_2)$$
$$\phi(a_j) = 1 = \phi(b_j); 3 \leq j \leq \gamma$$

Then $\prod_{j=1}^{\gamma} [\phi(a_j), \phi(b_j)] = [\alpha^2, \beta^3]$

$= \alpha^{-2}.\beta^{-3}.\alpha^{2}.\beta^{3}$
$= \alpha^{-2}.\alpha^{-2}.\beta^{3}.\beta^{3}$
$= \alpha^{-4}.\beta^{6}$
$= 1$, (from [3])

Then clearly ϕ is a homeomorphism as $\prod_{j=1}^{\gamma}[\phi(a_j),\phi(b_j)] = 1$

Also $\alpha,\beta \in \phi(\Gamma)$ gives ϕ is a smooth epimorphism.

(2) When $k \neq 0$, $\phi(x_i) = m_i$, m_i divides 48.

(i) If all $\phi(x_i) \in \langle\alpha\rangle$ then clearly m_i divides 4. If $\gamma = 0$ then $\phi(\Gamma) = \langle\alpha\rangle$ which gives ϕ is not onto which contradicts our assumption that ϕ is onto, hence if all $\phi(x_i) \in \langle\alpha\rangle$ then $\gamma \geq 1$.

(ii) If all $\phi(x_i) \in \langle\beta\rangle$ then clearly m_i divides 6. If $\gamma = 0$ then $\phi(\Gamma) = \langle\alpha\rangle$ which gives ϕ is not onto which contradicts our assumption that ϕ is onto, hence if all $\phi(x_i) \in \langle\beta\rangle$ then for $m_i = 2,3,6;\ \gamma \geq 1$.

(iii) If all $\phi(x_i) \in \langle\alpha\beta\rangle$ then $m_i = 2$ clearly $\gamma \geq 1$.

(iv) If some $\phi(x_i) \in \langle\alpha\rangle$ for $i = 1,2,\dots,s; s < k$ and remaining $(x_{s+j}) \in \langle\beta\rangle, j = 1,2,\dots,t$. Then $s + t = k$, as s and t both are even so k must be greater than equal 4 i.e. $k \geq 4$, furthermore

(a) If $k = 4$, then $s = 2, t = 2$, so that using [2] we get $\gamma \geq 0$.

(b) If $k > 4$, say $k = 6$ then either $s = 4, t = 2$ or $s = 2, t = 4$; in both cases using [1.3.1] we get $\gamma \geq 0$. i.e. if $k > 4$ then $\gamma \geq 0$.

(v) If some $\phi(x_i) \in \langle\alpha\rangle$ for $i = 1,2,\dots,s; s < k$ and remaining $(x_{s+j}) \in \langle\alpha\beta\rangle, j = 1,2,\dots,t$. Then $s + t = k$, as s and t both are even so k must be greater than equal 4 i.e. $k \geq 4$, furthermore

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(vi) If some $\phi(x_i) \in \langle\beta\rangle$ for $i = 1,2,\dots,s; s < k$ and remaining $(x_{s+j}) \in \langle\alpha\beta\rangle, j = 1,2,\dots,t$.

Then $s + t = k$, as s and t both are even so k must be greater than equal 4 i.e. $\gamma \geq 4$, furthermore

(a) If $k = 4$, then $s = 2, t = 2$, so that using [1.3.1] we get $\gamma \geq 0$.

(b) If $k > 4$, say $k = 4$ then either $s = 4, t = 2$ or $s = 2, t = 4$; in both cases using [1.3.1] we get $\gamma \geq 0$ i.e. if $k = 4$ then $\gamma \geq 0$.

(vii) If some $\phi(x_i) \in \langle\alpha\rangle$ for $i = 1,2,\dots,s; s < k$ and some $\phi(x_{s+j}) \in \langle\beta\rangle, j = 1,2,\dots,t$ and remaining $(x_{s+t+j}) \in \langle\alpha\beta\rangle, j = 1,2,\dots,p$. Then $s + t + p = k$. Then clearly k must be even and ≥ 6.

(i) If $k = 6; s = 2, t = 2, p = 2$ then from [1.3.1] $\gamma \geq 0$.

(ii) If $k \geq 6$ then also $\gamma \geq 0$.

Hence the condition is necessary for $k \neq 0$.

Next we see that the conditions are sufficient for $k \neq 0$.

2. (i) All $\phi(x_i) \in \langle\alpha\rangle, m_i = 4$ and $\gamma \geq 1$. Then let us construct ϕ as follows-

$\phi(x_i) = \alpha^2$, $\phi(a_1) = \beta^2$, $\phi(b_1) = \alpha^{-k}\beta$

And $\phi(a_j) = 1 = \phi(b_j)$ $\forall j$ if any

Then $\prod_{i=1}^{k}\phi(x_i)\prod_{j=1}^{\gamma}[\phi(a_j),\phi(b_j)] = \alpha^{2k}.[\beta^2,\alpha^{-k}\beta]$

$=\alpha^{2k}.\beta^{-2}.(\alpha^{-k}\beta)^{-1}.\beta^2\ .\alpha^{-k}\beta$

$=\alpha^{2k}.\beta^{-2}.\beta^{-1}.\alpha^{k}.\beta^2\ .\alpha^{-k}.\beta$

$=\alpha^{2k}.\beta^{-3}.\beta^{-2}.\alpha^{-k}.\ \alpha^{-k}.\beta$

$=\alpha^{2k}.\beta^{-5}.\alpha^{-2k}.\beta$

$=\alpha^{2k}.\alpha^{2k}.\beta^{5}.\beta$

$=\alpha^{4k}.\beta^{6}.$

$=1$

Also, $\langle\beta\rangle,\langle\alpha\beta\rangle \in \phi(\Gamma)$ gives ϕ is a smooth epimorphism.

2 (ii) If all $\phi(x_i) \in \langle\beta\rangle$ then clearly m_i divides 6. $\gamma \geq 1$. Then let us construct ϕ as

$\phi(x_i) = \beta^2$, $\phi(a_1) = \alpha^2$, $\phi(b_1) = \alpha\beta^{-k}$

And $\phi(a_j) = 1 = \phi(b_j)$ $\forall j$ i if any

Then $\prod_{i=1}^{k}\phi(x_i)\prod_{j=1}^{\gamma}[\phi(a_j),\phi(b_j)] = \beta^{2k}.[\alpha^2,\alpha\beta^{-k}]$

$=\beta^{2k}.\alpha^{-2}.(\alpha\beta^{-k})^{-1}.\alpha^2.\alpha\beta^{-k}$

$=\beta^{2k}.\alpha^{-2}.\beta^{k}.\alpha^{-1}.\alpha^2.\alpha\beta^{-k}$

$=\beta^{2k}.\beta^{-k}.\alpha^{2}.\alpha^{-1}.\alpha^2.\alpha\beta^{-k}$

$=\beta^{2k}.\beta^{-k}.\alpha^{4}.\beta^{-k}$

$=\beta^{2k}.\beta^{-2k}$, $\because \alpha^4 = 1$

$=1$

Also, $\langle\alpha\rangle,\langle\alpha\beta\rangle \in \phi(\Gamma)$ gives ϕ is a smooth epimorphism.

2 (iii) If all $\phi(x_i) \in \langle\alpha\beta\rangle$ then clearly $m_i = 2$ and $\gamma \geq 1$. Then let us construct ϕ as

$\phi(x_i) = \alpha^{-2k}\beta^4$, $\phi(a_1) = \alpha^3$, $\phi(b_1) = \beta^k$

And $\phi(a_j) = 1 = \phi(b_j)$ $\forall j$ if any

Then $\prod_{i=1}^{k}\phi(x_i)\prod_{j=1}^{\gamma}[\phi(a_j),\phi(b_j)] = (\alpha^{-2k}\beta^4)^k.[\alpha^3,\beta^k]$

$=\alpha^{-2}.\beta^{4k}.\alpha^{-3}.\beta^{-k}.\alpha^3\beta^k$

$=\alpha^{-2}.\alpha^{3}.\beta^{-4k}.\beta^{-2k}.\alpha^{-3}.$

$=\alpha.\beta^{-6k}.\alpha^{-3}.$

$=\alpha.\alpha^{3}.\beta^{6k}$

$=\alpha^{4}.\beta^{6k}$

$=1$

Hence $\langle\alpha\rangle,\langle\beta\rangle \in \phi(\Gamma)$ gives ϕ is a smooth epimorphism.

2 (iv) If some $\phi(x_i) \in \langle\alpha\rangle$ for $i = 1,2,\ldots,s; s < k$ remaining $(x_{s+j}) \in \langle\beta\rangle, j = 1,2,\ldots,t,$ such that $s + t = k; k \geq 4$ then $\gamma \geq 0$.

(a) If $k = 4$ then $s = 2$, $t = 2, \gamma \geq 0$, then let us construct ϕ as

$$\phi(x_1) = \alpha, \phi(x_2) = \alpha^3, \phi(x_3) = \beta^2, \phi(x_4) = \beta^4$$

And $\phi(a_j) = 1 = \phi(b_j)\ \ \forall j$ if any

Then $\prod_{i=1}^{4} \phi(x_i).1 = \alpha.\alpha^3.\beta^2.\beta^4 = \alpha^4.\beta^6 = 1$

(b) If $k \geq 4$, say $k = 6$, then $s = 4$, $t = 2, \gamma \geq 0$, then let us construct ϕ as

$$\phi(x_1) = \alpha, \phi(x_2) = \alpha^2\ \phi(x_3) = \alpha^3, \phi(x_4) = \alpha^{-2}, \phi(x_5) = \beta^2, \phi(x_6) = \beta^4$$

And $\phi(a_j) = 1 = \phi(b_j)\ \ \forall j$ if any

Then $\prod_{i=1}^{6} \phi(x_i).1 = \alpha.\alpha^2.\alpha^3.\alpha^{-2}.\beta^2.\beta^4 = \alpha^4.\beta^6 = 1$

Hence the condition is sufficient to yield ϕ is a smooth epimorphism.

2(v) If some $\phi(x_i) \in \langle\alpha\rangle$ for $i = 1,2, \ldots, s; s < k$ and remaining $(x_{s+j}) \in \langle\alpha\beta\rangle, j = 1,2, \ldots, t.$ such that $s + t = k; k \geq 4$ then $\gamma \geq 0$.

(a) If $k = 4$ then $s = 2$, $t = 2, \gamma \geq 0$, then let us construct ϕ as

$$\phi(x_1) = \alpha^2, \phi(x_2) = \alpha^3, \phi(x_3) = \alpha\beta, \phi(x_4) = \alpha^2\beta$$

And $\phi(a_j) = 1 = \phi(b_j)\ \ \forall j$ if any

$$\begin{aligned} \text{Then } \prod_{i=1}^{4} \phi(x_i).1 &= \alpha^2.\alpha^3.\alpha\beta.\alpha^2\beta \\ &= \alpha^6.\alpha^{-2}.\beta^{-1}.\beta \\ &= \alpha^4.1 \\ &= 1 \end{aligned}$$

(b) If $k \geq 4$, say $k = 6$, then $s = 2$, $t = 4, \gamma \geq 0$, then let us construct ϕ as

$$\phi(x_1) = \alpha^2, \phi(x_2) = \alpha^3, \phi(x_3) = \alpha\beta^2, \phi(x_4) = \alpha^2\beta,$$
$$\phi(x_5) = \alpha^2\beta^3, \phi(x_6) = \alpha^2\beta^4$$

And $\phi(a_j) = 1 = \phi(b_j)\ \ \forall j$ if any

$$\begin{aligned} \text{Then } \prod_{i=1}^{6} \phi(x_i).1 &= \alpha^2.\alpha^3.\alpha\beta^2.\alpha^2\beta.\alpha^2\beta^3.\alpha^2\beta^4 \\ &= \alpha^6.\alpha^{-2}.\beta^{-2}.\beta.\beta^{-3}.\alpha^{-2}.\alpha^2\beta^4 \\ &= \alpha^4.\ \beta^{-4}.1.\beta^4 \\ &= 1.\ \beta^{-4}.\beta^4 \\ &= 1 \end{aligned}$$

Hence the condition that gives ϕ is a smooth epimorphism is sufficient .

2(vi) If some $\phi(x_i) \in \langle\beta\rangle$ for $i = 1,2, \ldots, s; s < k$ and remaining $\phi(x_{s+j}) \in \langle\alpha\beta\rangle, j = 1,2, \ldots, t,$ such that $s + t = k; k \geq 4$; then $\gamma \geq 0$.

(a) If $k = 4$ then $s = 2$, $t = 2, \gamma \geq 0$, then let us construct ϕ as

$$\phi(x_1) = \beta^2, \phi(x_2) = \beta^5, \phi(x_3) = \alpha\beta^3, \phi(x_4) = \alpha\beta^2$$

And $\phi(a_j) = 1 = \phi(b_j)\ \ \forall j$ if any

$$\begin{aligned} \text{Then } \prod_{i=1}^{4} \phi(x_i).1 &= \beta^2.\beta^5.\alpha\beta^3.\alpha\beta^2 \\ &= \beta^7.\beta^{-3}.\alpha^{-1}.\alpha\beta^2 \end{aligned}$$

$= \beta^4.1.\beta^2 = \beta^6 = 1$

(b) If $k \geq 4$, say $k = 6$, then $s = 2$, $t = 4$, $\gamma \geq 0$, then let us construct ϕ as

$\phi(x_1) = \beta, \phi(x_2) = \beta^4, \phi(x_3) = \alpha\beta^2, \phi(x_4) = \alpha^2\beta,$

$\phi(x_5) = \alpha^2\beta^3, \phi(x_6) = \alpha^3\beta^5$

And $\phi(a_j) = 1 = \phi(b_j)$ $\forall j$ if any

Then $\prod_{k=1}^{6} \phi(x_i).1 = \beta.\beta^4.\alpha\beta^2.\alpha^2\beta.\alpha^2\beta^3.\alpha^3\beta^5$

$= \beta^5.\beta^{-2}.\alpha^{-1}.\beta^{-1}.\beta^3.\alpha^3\beta^5 \qquad \because \alpha^2\beta.\alpha^2 = \beta^{-1}$

$= \beta^3.\alpha^{-1}.\beta^2.\alpha^3\beta^5$

$= \beta^3.\beta^{-2}.\alpha^1\alpha^3\beta^5$

$= \beta^1.\alpha.\alpha^3\beta^5$

$= \beta^1.\alpha^4.\beta^5$

$= \beta^1.1.\beta^5$

$= \beta^6$

$= 1.$

Hence the condition that gives ϕ is a smooth epimorphism is sufficient .

(viii) If some $\phi(x_i) \in \langle\alpha\rangle$ for $i = 1,2,\dots,s; s < k$ and some $\phi(x_{s+j}) \in \langle\beta\rangle, j = 1,2,\dots,t$ and remaining $(x_{s+t+j}) \in \langle\alpha\beta\rangle, j = 1,2,\dots,p$. Then $s + t + p = k$. Then clearly k must be even an ≥ 6.

(a) If $k = 6$; $s = 2$, $t = 2$, $p = 2$, $\gamma \geq 0$. then let us construct ϕ as

$\phi(x_1) = \alpha^2, \quad \phi(x_2) = \alpha^3,$

$\phi(x_3) = \beta^3, \quad \phi(x_4) = \beta^5,$

$\phi(x_5) = \alpha^2\beta^5, \qquad \phi(x_6) = \alpha^3\beta^3,$

And $\phi(a_j) = 1 = \phi(b_j)$ $\forall j$ if any

Then $\prod_{k=1}^{6} \phi(x_i).1 = \alpha^2.\alpha^3.\beta^3.\beta^5.\alpha^2\beta^5.\alpha^3\beta^3$

$= \alpha^5.\beta^8.\beta^{-5}.\alpha^{-2}.\alpha^3\beta^3$

$= \alpha^5.\beta^3.\alpha^1\beta^3$

$= \alpha^5.\alpha^{-1}\beta^{-3}.\beta^3$

$= \alpha^4.1$

(b) If $k \geq 6$, say $k = 8$, then $s = 2$, $t = 4$, $p = 2$, $\gamma \geq 0$,
Then let us construct ϕ as

$\phi(x_1) = \alpha^2, \phi(x_2) = \alpha^3, \phi(x_3) = \beta^2, \phi(x_4) = \beta^3,$

$\phi(x_5) = \beta^4, \phi(x_6) = \beta^5, \phi(x_7) = (\alpha^3\beta^2\alpha^3), \phi(x_8) = (\beta^4\alpha^3\beta^4),$

And $\phi(a_j) = 1 = \phi(b_j)$ $\forall j$ if any

Then $\prod_{k=1}^{6} \phi(x_i).1 = \alpha^2.\alpha^3.\beta^2.\beta^3.\beta^4.\beta^5.(\alpha^3\beta^2\alpha).(\beta^4\alpha^3\beta^4)$

$= \alpha^5.\beta^{14}.\beta^{-2}.\alpha^{-3}.\alpha^3.\alpha^{-3}\beta^{-4}.\beta^4$

$$= \alpha^5 . \beta^{12} . 1 . \alpha^{-3} . 1$$
$$= \alpha^5 . \alpha^3 . \beta^{-12}$$
$$= \alpha^8 . \beta^{-12}$$
$$= 1$$

Hence $\langle\alpha\rangle, \langle\beta\rangle, \langle\alpha\beta\rangle \in \phi(\Gamma)$ implies ϕ is a smooth epimorphism.

This proves the sufficiency of the conditions.

4. **Conclusion**

In this paper we can successfully established the existence of smooth epimorphism from the Fuchsian group Γ to the finite group G of the group of symmetries of sulphur-hexa-flouride molecule. The next step of this topic is to find the minimum genus g and then to calculate the upper bound and lower bound, which will give us the signature of the corresponding Fuchsian group[6].

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