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A Mixed Quadrature Rule using Clenshaw-Curtis five point Rule Modified by Richardson Extrapolation

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Abstract

A mixed quadrature rule of precision nine for approximate evaluation of real definite integrals has been constructed by blending Clenshaw-Curtis five point rule modified by Richardson Extrapolation and Gauss-Legendre four point rule. An error analysis for this mixed rule is provided. The efficiency of this rule is highlighted through numerical evaluation of some definite integrals at the end.

Key words : Clenshaw-Curtis quadrature rule, Gauss-Legendre 4-point rule, Richardson Extrapolation, mixed quadratue rule, Adaptive quadrature, $SM_7(f), GL_4(f), CC_5(f), RCC_5(f)$.

Subject classification: 65D32

1. Introduction

A mixed quadrature rule higher precision is formed by a suitable linear combination of two or more rules of lower precision. The first paper in which a mixed rule has been designed is due to R.N. Das and G. Pradhan[1]. After this paper many authors have worked in this area. S.R. Jena and R. B. Dash[2] formed a mixed rule blending Gauss-Legendre four point transformed rule and modified Birkhoff-Young rule using Richardson Extrapolation.

Recently, A.K.Tripathy *et al.*[5] used Lobatto rule and Gauss-Legendre three point rules to form a mixed rule for approximation of real definite integrals. R.B. Dash and D. Das[4] applied a mixed quadrature formed out of Clenshaw-curtis and Gauss-Legendre quadratures for evaluation of real definite integrals in adaptive environment. S. Jena and D. Nayak[6] used mixed quadrature rule Real for numerical solution of Fredholm integral equations. S. Jena *et al*.[7] applied mixed quadrature for settling some electromagnetic field problems.

In this paper, we modified the Clenshaw –curtis 5-point rule $CC_5(f)$ of precision -5 using Richardson Extrapolation to form a rule of precision -7. Then we mixed this rule with Gauss-Legendre-4-point rule to form a new mixed quadrature rule of precision-9. Then we theoretically proved and numerically verified that this mixed rule is more efficient than its constituent rules.

I. *The Clenshaw-Curtis 5-point Quadrature Rule :*

From the literature[8,9,4] we know that in Clenshaw-Curtis method the integrand function $f(t)$ over any interval $[a-h, \quad a+h]$ *is* approximated using Chebyshev polynomials $T_r(x)$ of degree r.
That means

$$f(t) = F(x) = {\sum_{r=0}^{n}}' a_r\, T_r(x) \qquad (-1 \le x \le 1) \tag{1.1}$$

$$= \frac{1}{2} a_0 T_0(x) + a_1 T_1(x) + a_2 T_2(x) + \cdots a_n T_n(x) \tag{1.2}$$

Using this we obtain

$$I_n = h {\sum_{r=0}^{n}}'' w_i f(a + h x_i) \tag{1.3}$$

$$\text{where } w_i = -\frac{4}{n} {\sum_{r=0}^{n}}'' T_r(x_i) \frac{1}{r^2-1} \qquad i=0,1,2\ldots n$$

(1.3) is known as Clenshaw-Curtis $(n+1)$ quadrature rule.

For n=4, we can obtain the Clenshaw-Curtis 5-point rule ($CC_5(f)$) as follows

$$I_4 = CC_5(f) = \frac{h}{15}\left[f(a+h) + 8f\left(a + \frac{h}{\sqrt{2}}\right) + 12f(a) + 8f\left(a - \frac{h}{\sqrt{2}}\right) + f(a-h)\right] \tag{1.4}$$

Consider a real valued function $f(x)$ and $[a-h, a+h] \subseteq dom(f)$

$$\text{Let } I(f) = \int_{a-h}^{a+h} f(x)dx \tag{1.5}$$

Using Clenshaw-Curtis 5-point rule ($CC_5(f)$) for evaluation of the integral (1.5), we have

$$I(f) \approx CC_5(f) = \frac{h}{15}\left[f(a+h) + 8f\left(a + \frac{h}{\sqrt{2}}\right) + 12f(a) + 8f\left(a - \frac{h}{\sqrt{2}}\right) + f(a-h)\right] \tag{1.6}$$

Applying Taylor's Theorem, after simplification we obtain,

$$CC_5(f) = 2h\left[f(a) + \frac{h^2}{3!} f^{ii}(a) + \frac{h^4}{5!} f^{iv}(a) + \frac{2}{15}\frac{h^6}{6!} f^{vi}(a) + \frac{1}{10}\frac{h^8}{8!} f^{viii}(a) + \frac{1}{12}\frac{h^{10}}{10!} f^{x}(a) + \frac{3}{40}\frac{h^{12}}{12!} f^{xii}(a) + \cdots\right] \tag{1.7}$$

The exact value of the integral

$$I(f) = 2h\left[f(a) + \frac{h^2}{3!} f^{ii}(a) + \frac{h^4}{5!} f^{iv}(a) + \frac{h^6}{7!} f^{vi}(a) + \frac{h^8}{9!} f^{viii}(a) + \frac{h^{10}}{11!} f^{x}(a) + \frac{h^{12}}{13!} f^{xii}(a) + \cdots\right] \tag{1.8}$$

Error of the $CC_5(f)$ rule

Let us denote the truncation error due to Clenshaw-Curtis 5-point rule for approximating the integral (1.5) by $ECC_5(f)$

Thus $I(f) = CC_5(f) + ECC_5(f)$

$$\Rightarrow ECC_5(f) = I(f) - CC_5(f) \tag{1.9}$$

Using the (1.7) and (1.8) on (1.9), after simplification we obtain

$$ECC_5(f) = \frac{2}{15}\frac{h^7}{7!}f^{vi}(a) + \frac{1}{5}\frac{h^8}{9!}f^{viii}(a) + \frac{1}{6}\frac{h^{11}}{11!}f^{x}(a) + \frac{1}{20}\frac{h^{13}}{13!}f^{xii}(a) + \cdots \tag{1.10}$$

From the error expression it is clear that the degree of precision of the Clenshaw-Curtis 5-point rule is five.

2. *Modified Clenshaw-Curtis 5 point rule due to Richardson Extrapolation* $\boldsymbol{RCC_5(f)}$ *:*

We have

$$CC_5(f) = \frac{h}{15}\left[f(a+h) + 8f\left(a + \frac{h}{\sqrt{2}}\right) + 12f(a) + 8f\left(a - \frac{h}{\sqrt{2}}\right) + f(a-h)\right]$$

Changing the step length as in[2] and [3] we have

$$CC_{5\frac{h}{2}}(f) = \frac{2h}{15}\left[f(a+2h) + 8f\left(a+\sqrt{2}h\right) + 12f(a) + 8f\left(a-\sqrt{2}h\right) + f(a-2h)\right] \tag{2.1}$$

Applying Taylor's Theorem

$$CC_{5\frac{h}{2}}(f) = 4h\left[f(a) + \frac{(2h)^2}{3!}f^{ii}(a) + \frac{(2h)^4}{5!}f^{iv}(a) + \frac{128}{15}\frac{h^6}{6!}f^{vi}(a) + \frac{384}{15}\frac{h^8}{8!}f^{viii}(a)\right.$$
$$\left. + \frac{1280}{15}\frac{h^{10}}{10!}f^{x}(a) + \frac{4608}{15}\frac{h^{12}}{12!}f^{xii}(a) + \cdots\right] \tag{2.2}$$

Error associated due to change of the step length $ECC_{5\frac{h}{2}}(f)$ is given by

$$ECC_{5\frac{h}{2}}(f) = I(f) - CC_{5\frac{h}{2}}(f)$$

Using (1.8) and (2.2)

$$ECC_{5\frac{h}{2}}(f) = \frac{256}{15}\frac{h^7}{7!}f^{vi}(a) + \frac{1536}{15}\frac{h^8}{9!}f^{viii}(a) + \frac{5120}{15}\frac{h^{11}}{11!}f^{x}(a) + \frac{6144}{15}\frac{h^{13}}{13!}f^{xii}(a) + \cdots \tag{2.3}$$

Now $I(f) = CC_5(f) + ECC_5(f)$ (2.4)

and $I(f) = CC_{5\frac{h}{2}}(f) + ECC_{5\frac{h}{2}}(f)$ (2.5)

Subtracting (2.5) from 128 times of (2.4), we get

$$(128-1)I(f) = \left[128CC_5(f) - CC_{5\frac{h}{2}}(f)\right] + \left[128ECC_5(f) - ECC_{5\frac{h}{2}}(f)\right]$$

$$\Rightarrow I(f) = \frac{1}{127}\left[128CC_5(f) - CC_{5\frac{h}{2}}(f)\right] + \frac{1}{127}\left[128ECC_5(f) - ECC_{5\frac{h}{2}}(f)\right]$$

$$= RCC_5(f) + ERCC_5(f)$$

Where

$$RCC_5(f) = \frac{1}{127}\left[128CC_5(f) - CC_{5\frac{h}{2}}(f)\right] \tag{2.6}$$

and $$ERCC_5(f) = \frac{1}{127}\left[128ECC_5(f) - ECC_{5\frac{h}{2}}(f)\right]$$

Now using (1.10) and (2.3) we have

$$ERCC_5(f) = \frac{1}{127}\left[128\left[\frac{2}{15}\frac{h^7}{7!}f^{vi}(a) + \frac{1}{5}\frac{h^8}{9!}f^{viii}(a) + \frac{1}{6}\frac{h^{11}}{11!}f^{x}(a) + \frac{1}{20}\frac{h^{13}}{13!}f^{xii}(a) + \cdots\right] - \left[\frac{256}{15}\frac{h^7}{7!}f^{vi}(a) + \frac{1536}{15}\frac{h^8}{9!}f^{viii}(a) + \frac{5120}{15}\frac{h^{11}}{11!}f^{x}(a) + \frac{6144}{15}\frac{h^{13}}{13!}f^{xii}(a) + \cdots\right]\right]$$

$$= -\frac{384}{127\times 5}\frac{h^8}{9!}f^{viii}(a) - \frac{320}{127}\frac{h^{11}}{11!}f^{x}(a) - \frac{2016}{127\times 5}\frac{h^{13}}{13!}f^{xii}(a) + \cdots \quad (2.7)$$

(2.6) and (2.7) are respectively called Clenshaw-Curtis rule and Error in Modified Clenshaw-Curtis rule due to Richardson extrapolation.
From (2.7), we see that the degree of precision of the rule is 7.

3. Gauss-Legendre 4 point rule :

The Gauss Legendre-4 point rule is given by

$$GL_4(f) = \frac{h}{36}\left[(18+\sqrt{30})\{f(a-\alpha h) + f(a+\alpha h)\} + (18-\sqrt{30})\{f(a-\beta h) + f(a+\beta h)\}\right] \quad (3.1)$$

where $\alpha = \sqrt{\frac{3-2\sqrt{\frac{6}{5}}}{7}}$, $\beta = \sqrt{\frac{3+2\sqrt{\frac{6}{5}}}{7}}$ and f is infinitely differentiable in its domain.

We can write $(18+\sqrt{30}) = \frac{1}{2}(21+35\beta^2)$ and $(18-\sqrt{30}) = \frac{1}{2}(21+35\alpha^2)$. Using this and applying Taylors theorem, we have

$$GL_4(f) = 2h\left[f(a) + \frac{h^2}{3!}f^{ii}(a) + \frac{h^4}{5!}f^{iv}(a) + \frac{h^6}{7!}f^{vi}(a) + \frac{3\times 301}{7^3\times 5^2}\frac{h^8}{8!}f^{viii}(a) + \frac{3\times 1561}{7^4\times 5^2}\frac{h^{10}}{10!}f^{x}(a) + \frac{9\times 13503}{7^5\times 5^3}\frac{h^{12}}{12!}f^{xii}(a) + \cdots\right] \quad (3.2)$$

Denoting the truncation error of the Gauss-Legendre 4 point rule by $EGL_4(f)$,
Hence

$$I(f) = GL_4(f) + EGL_4(f)$$

$$\Rightarrow EGL_4(f) = I(f) - GL_4(f) \quad (3.3)$$

Using (1.8) and (3.2) on (3.3) we obtain

$$EGL_4(f) = \frac{128}{7^2 \times 5^2}\frac{h^9}{9!}f^{viii}(a) + \frac{128 \times 19}{7^3 \times 5^2}\frac{h^{11}}{11!}f^{x}(a) + \frac{128 \times 1163}{7^4 \times 5^3}\frac{h^{13}}{13!}f^{xii}(a) + \cdots \quad (3.4)$$

The error (3.4) shows that the degree of precision of $GL_4(f)$ is 7.

4. Construction of the new mixed quadrature rule of precision nine :

For the construction of proposed mixed quadrature rule we proceed as follows.
We have

$$I(f) = RCC_5(f) + ERCC_5(f) \quad (4.1)$$

$$I(f) = GL_4(f) + EGL_4(f) \quad (4.2)$$

Now adding 127 times of (4.1) with 735 times of (4.2) , we obtain

$$862I(f) = [735GL_4(f) + 127RCC_5(f)] + [735EGL_4(f) + 127ERCC_5(f)]$$

$$\Rightarrow I(f) = \frac{1}{862}[735GL_4(f) + 127RCC_5(f)] + \frac{1}{862}[735EGL_4(f) + 127ERCC_5(f)]$$

$$\Longrightarrow I(f) = SM_7(f) + ESM_7(f)$$

Where $SM_7(f) = \frac{1}{862}[735GL_4(f) + 127RCC_5(f)]$. Thus we have

$$SM_7(f) = \frac{1}{862}\left[735GL_4(f) + 128CC_5(f) - CC_{5\frac{h}{2}}(f)\right] \quad \text{[Using (2.6)]} \quad (4.3)$$

Using(1.4), (2.1) and (3.1) on (4.3), we obtain

$$SM_7(f) = \frac{245h}{10344}\left[(18 + \sqrt{30})\{f(a - \alpha h) + f(a + \alpha h)\} + (18 - \sqrt{30})\{f(a - \beta h) + f(a + \beta h)\}\right] + \frac{64h}{6465}\left[f(a + h) + f(a - h) + 8\left\{f\left(a + \frac{h}{\sqrt{2}}\right) + f\left(a - \frac{h}{\sqrt{2}}\right)\right\}\right] + \frac{h}{6465}\left[f(a + 2h) + f(a - 2h) + 8\left\{f\left(a + \sqrt{2}\,h\right) + f\left(a - \sqrt{2}\,h\right)\right\}\right] + \frac{252h}{2155}f(a) \quad (4.4)$$

$SM_7(f)$ is the desired mixed quadrature rule.

The degree of precision of the rule is 9 which is established by the Theorem-1. The truncation error generated by $SM_7(f)$ is given by

$$ESM_7(f) = \frac{1}{862}[735EGL_4(f) + 127ERCC_5(f)] \quad (4.5)$$

Theorem-1 :

If $f(x)$ is sufficiently differentiable in the interval $[a - h, a + h]$, the degree of precision of the rule $SM_7(f)$ is 9 and $ESM_7(f) = o(h^{11})$.

Proof

From (4.5), we have

$$ESM_7(f) = \frac{1}{862}\,[735EGL_4(f) + 127ERCC_5(f)]$$

Now using (2.7) and (3.4) the truncation error becomes

$$ESM_7(f) = -\frac{37472}{431\times 49\times 25}\frac{h^{11}}{11!}f^{x}(a) - \frac{1011504}{431\times 49\times 25}\frac{h^{13}}{13!}f^{xii}(a) \qquad (4.6)$$

This established that the degree of precision of the rule $SM_7(f)$ is 9 and $ESM_7(f) = o(h^{11})$. □

Theorem-2 (*Error Analysis*) *:*

The error committed due to the mixed quadrature rule $SM_7(f)$ is less than its constituent rules

Proof:

From (1.10) and (4.6) $|ESM_7(f)| \le |ECC_5(f)|$

From (2.7) and (4.6) $|ESM_7(f)| \le |ERCC_5(f)|$

From (3.4) and (4.6) $|ESM_7(f)| \le |EGL_4(f)|$. □

5. Numerical verification :

The effectiveness of the rule is shown in the Table-5.1 by applying it and its constituent rules on different integrals.

Table-5.1

Sl no	*Integrals*	*Exact value*	*Value obtained by quadrature rules*			*Error approximated*		
			$CC_5(f)$	$GL_4(f)$	$SM_7(f)$	$\lvert ECC_5(f)\rvert$	$\lvert EGL_4(f)$	$\lvert ESM_7(f)\rvert$
1	$\int_0^1 \frac{1}{1+e^x}dx$	0.379885493041	0.3798854926	0.379885493045	0.379885493038	4.41× 10^{-10}	4× 10^{-12}	3× 10^{-12}
2	$\int_1^2 \frac{1}{1+x^4}dx$	0.2031547018	0.2031545421	0.2031548460	0.2031547983	1.597× 10^{-7}	1.442×10^{-7}	9.65×10^{-8}
3	$\int_0^{\pi/2} \frac{sinx\,dx}{(1+cosx)^3}$	0.375	0.3749998904	0.3749999636	0.3749999676	1.096× 10^{-7}	3.64× 10^{-8}	3.24× 10^{-8}
4	$\int_0^2 \frac{x}{1+x^3}dx$	0.7237976339	0.7237975821	0.7237977153	0.7237976839	5.18× 10^{-8}	8.14× 10^{-8}	5× 10^{-8}
5	$\int_0^{\pi/2} \frac{dx}{1+cosx}$	1	0.9999998240	0.9999999369	0.9999999447	1.76× 10^{-7}	6.31× 10^{-8}	5.53× 10^{-8}

6. Conclusion

From the table it is evident that the new mixed quadrature rule $SM_7(f)$ when applied each of the five integrals gives better result than that of its constituent rules $CC_5(f)$ and $GL_4(f)$. This efficient rule can be used for numerical integration of infinitely differentiable integrands in adaptive environment to achieve desired accuracy.

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