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Complete set of Genera of Compact Riemann Surfaces with reference to the point Group of Sulphur (S_8) Molecule

RAFIQUL ISLAM[1] and CHANDRA CHUTIA[2]

[1]Department of Mathematics, Jorhat Engineering College, Jorhat-785007 (India)
[2]Department of Mathematics, Jorhat Institute of Science and Technology, Jorhat (India)
Corresponding Author E-mail: rafiqul.jec@gmail.com

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Abstract

In this paper we consider the group of symmetries of the Sulphur molecule (S_8) which is a finite point group of order 16 denote by D_{16} generated by two elements having the presentation $\{u, v / u^2 = v^8 = (uv)^2 = 1\}$ and find the complete set of genera $(g \geq 2)$ of Compact Riemann surfaces on which D_{16} acts as a group of automorphisms as follows:

D_{16}, the group of symmetries of the sulphur (S_8) molecule of order 16 acts as an automorphism group of a compact Riemann surfaces of genus $g \geq 2$ if and only if there are integers λ and μ such that $\lambda \leq 1$ and $\mu \geq 1$ and $g = \lambda + 8\mu \ (\geq 2)$, $\mu \geq |\lambda|$

Key words : Symmetries, Fuchsian group, Smooth quotient, Riemann surface, Automorphism group and Genus.

1991 Mathematics subject classification: 20H10, 30F10

Introduction

Every finite group acts as group of automorphisms on a compact Riemann surface of some genus $g \geq 2$. The same group may act as an automorphism group of many surfaces of different genera. The problem of finding minimum g such that there is a surface of genus g admitting a given finite group G as an automorphism group has been studied extensively since 1960s. Finding the complete set of genera of the surfaces admitting a given finite group G as an automorphism group seems to be a interesting problem. The problem has however been solved for some simple classes of groups during the last few years[2,3,4,5]. In this paper we find the solutions of the problem for the group of symmetries D_{16} of the Sulphur molecule (S_8) which is a finite group of order 16.

A finite group G acts as a group of automorphisms of compact Riemann surface of genus $g \geq 2$ if and only if there is a smooth epimorphism of some Fuchian group Γ to G with surface group of genus g as its kernel.

A Fuchsian group Γ is an infinite group having presentation of the form

$$\langle x_1, x_2, \ldots x_k ; b_1, c_1, b_2, c_2 \ldots b_\gamma, c_\gamma \; ; x_1^{m_1} = x_2^{m_2} = \cdots = x_k^{m_k} = \prod_{i=1}^{k} x_i \prod_{j=1}^{\gamma} [b_j, c_j] = 1 \rangle \quad (1)$$

Where $[b_j, c_j] = b_j^{-1} c_j^{-1} b_j c_j$ and $x_1, x_2, x_3, \ldots\ldots\ldots\ldots\ldots . x_k$ are of finite order generators of order $m_1, m_2, m_3, \ldots\ldots\ldots\ldots\ldots . m_k$ respectively and

$b_1, c_1, b_2, c_2 \ldots\ldots\ldots .. b_\gamma, c_\gamma$ are of infinite order generators of Γ, and the measure of Γ is given by

$$\delta(\Gamma) = 2\gamma - 2 + \sum_{i=1}^{k} \left(1 - \frac{1}{m_i}\right) > 0 \quad (2)$$

Such a Fuchsian group is also denoted by $\Delta(\gamma;\ m_1, m_2, , \ldots\ldots\ldots\ldots\ldots . m_k)$ the non negative integer γ is called the genus Γ and the integer $m_i (\geq 2)$ are called the periods of Γ.

If $\gamma = 0$ then Γ has the signature of the form $\Gamma = \Delta(m_1, m_2, , \ldots\ldots\ldots\ldots\ldots . m_k)$ and if there is no finite order element except identity in Γ then $\Gamma = (\gamma\ ; 0)$, called a surface group. It is known that if Γ_1 is a subgroup of Γ of finite index then Γ_1 is a Fuchsian group and

$$[\Gamma : \Gamma_1] = \frac{\delta(\Gamma_1)}{\delta(\Gamma)} \quad (3)$$

A homomorphism ϕ from a Fuchsian group Γ to a finite group G is called smooth if the kernel is a surface subgroup of Γ. The factor group $\Gamma/\ker\varphi (\cong \varphi(\Gamma))$ is called a smooth quotient of genus g. It is found that[5] there is a set a set of necessary and sufficient conditions for the existence of a smooth epimorphism from a Fuschian group Γ to the finite group D_{16} of order 16 generated by two elements *u* and *v* of order 2 and 8 respectively and whose product has order 2, has a presentation

$$D_{16} = \{u, v / u^2 = v^8 = (uv)^2 = 1\} \quad (4)$$

as follows:

Theorem 1:[5]

There is a smooth epimorphism $\phi : \Gamma \to D_{16}$ where Γ and D_{16} are defined as (1) and (4) respectively if and only if

(1) When $k = 0$ then $\Gamma = \Delta(\gamma; -)$ is a surface group and $\gamma \geq 2$

(2) When $k \neq 0$ then $O(\phi(x_i)) = m_i$; m_i divides 16

Moreover,

(i) if all $\phi(x_i) \in < v >$ then m_i divides 8 and $\gamma \geq 1$

(ii) if all $\phi(x_i) \in D_{16} - < v >$ then $m_i = 2$, k is even and $\gamma \geq 1$

(iii) If $(x_i) \in \langle v \rangle$, $i = 1,2, \ldots \ldots s$; $s \geq k$ and
$\phi(x_{s+j}) \in D_{16} - \langle v \rangle, j = 1,2, \ldots \ldots t$; then $s + t = k$
t is even and also $sl \equiv 0 (mod\ 8)$ and $1 \leq l \leq 7$.

Now using the conditions of these theorem we calculate the Complete set of genera.

Complete Set of Genera :

In this section we first find all the values of $g \geq 2$ for which a compact Riemann surface of genus g admits D_{16} as an automorphism group. The conditions of the above theorem enable us to do so as follows:

Theorem 2: Let D_{16} be the group of symmetries of the Sulphur (S_8) molecule of order 16. Then D_{16} acts as an automorphism group of a compact Riemann surface of genus $g \geq 2$ if and only if there are integer λ and μ such that $\lambda \leq 1$ and $\mu \geq 1$ and $g = \lambda + 8\mu (\geq 2)$. Furthermore $\mu \geq |\lambda|$

Proof: Let g be the genus of a compact Riemann surface on which D_{16} acts as an automorphism group. Then there exist a Fuschian group

$$\Gamma = \langle x_1, x_2, \ldots x_k ; b_1, c_1, b_2, c_2 \ldots b_\gamma, c_\gamma \ ; x_1^{m_1} = x_2^{m_2} = \cdots = x_k^{m_k} = \prod_{i=1}^{k} x_i \prod_{j=1}^{\gamma} [b_j, c_j] = 1 \rangle$$

and an epimorphism $\phi : \Gamma \to D_{16}$ such that the kernel of ϕ is a surface group of genus g.

Now

$$O(D_{16}) = O(\Gamma / \ker\phi) = \frac{\delta(\ker\phi)}{\delta(\Gamma)}$$

i.e. $16 = \dfrac{2(g-1)}{\delta(\Gamma)}$

i.e. $g = 1 + 8\delta(\Gamma)$

We know that for the existence of a smooth epimorphism $\phi : \Gamma \to D_{16}$, the periods of Γ and the genus of Γ must satisfy the conditions of the theorem(1) therefore the periods of Γ can take the values 2, 4 and 8 only.

Let the number of periods equal to 2 be r, the number of periods equal to 4 be s and that of periods equal to 8 be t then it is clear from the conditions of the theorem(1) that r must be even. Also from (2) it follows that

$$g = 1 + 8[\,2(\gamma - 1) + s\left(1 - \frac{1}{4}\right) + t\left(1 - \frac{1}{8}\right) + \frac{1}{2}r]$$

$$g = 1 + 8[\,2(\gamma - 1) + \frac{3s}{4} + \frac{7t}{8} + \frac{1}{2}r]$$

i.e. $= 1 - 2s - t + 8[2\gamma + s + t + m - 2]$, where $m = \dfrac{r}{2}$, an integer

Let us put $\lambda = 1 - 2s - t$ and $\mu = 2\gamma + s + t + m - 2$ then $g = \lambda + 8\mu$ so that $\lambda \leq 1$, $\mu \geq 1$, because subject to the conditions mentioned in the previous theorem1 on γ, r, t and s we get $\lambda \leq 1$, $\mu \geq 1$.

Now we see that under those conditions $g \geq 2$. It is observe that $g \geq 2$ for $\gamma \geq 2$. If $\gamma = 0$ then $r = 2m \neq 0,2,4$ when $s = 0$ and $t = 0$ which gives $g \geq 2$. If $\gamma = 1$ then r, s and t are not zero simultaneously. This also give $g \geq 2$. Thus it follows that $g \geq 2$ for all possible (admissible) values of γ, r, s and t under the conditions of the theorem (1). It is also noted that in the above cases $\lambda = 1 - 2s - t$ and $\mu = 2\gamma + s + t + \mathrm{m} - 2$, $\mu \geq |\lambda|$ also. Thus the conditions are necessary.

Now we show that the conditions are sufficient:

Let g be an integer ≥ 2 such that $g = \lambda + 8\mu$ where λ and μ are integers with $\lambda \leq 1$, $\mu \geq 1$ and $\mu \geq |\lambda|$. It is sufficient to show that there is a Fuschian group Γ satisfies the conditions of theorem(1) and that there is a smooth epimorphism $\phi: \Gamma \to D_{16}$ such that kerϕ has genus $g \geq 2$.

Set

$$\lambda = 1 - p - q \tag{1}$$

$$\mu = 2\gamma + p + q + \frac{z}{2} - 2 \tag{2}$$

For any given integer $\lambda \leq 1$ we can find at least one non-zero positive integral value p_0, q_0 of $p \; and \; q$ of which satisfy the equation (1). Putting these values of $p \; and \; q$ in (2) we get an equation in $\mu, \gamma \; and \; z$

$$\mu = 2\gamma + \frac{z}{2} + p_0 + q_0 - 2$$

This equation in μ, γ and z has a solution (μ_0, γ_0, z_0) where γ_0 and z_0 satisfy the conditions of the previous Theorem and μ_0 is the integer ≥ 1. Therefore we can always find a set of values $p_0, q_0 \; and \; z_0, \gamma_0$ of p, q, z and γ respectively which give $\lambda \leq 1$, $\mu \geq 1$ so that

$\Gamma = \Delta\,(\gamma_0: \underbrace{2,2,\ldots\ldots 2}_{z_0}, \underbrace{4,4\ldots\ldots .4}_{p_0}, \underbrace{8,8\ldots\ldots .8}_{q_0},\ )$ is a Fuschian group satisfying conditions of the previous Theorem. Hence there is a smooth epimorphism from Γ to D_{16} and consequently D_{16} acts as a group of automorphisms of a compact Riemann surface of genus g, given by

$$\begin{aligned} g &= 1 + 8[\,2\gamma_0 - 2 + p_0\left(1 - \frac{1}{4}\right) + q_0\left(1 - \frac{1}{8}\right) + z_0(1 - \frac{1}{2}) \\ &= 1 - 2p_0 - q_0 + 8[\,2\gamma_0 + p_0 + q_0 + \frac{z_0}{2} - 2\,] \\ &= \lambda + 8\mu \text{ from (1) and (2)} \end{aligned}$$

Thus D_{16} acts as an automorphism group of a compact Riemann surface of genus g.

This completes the proof.

Scope of research:

It is observed that the molecular group of symmetries is a rich field of various research work in different dimensions The topic considered in this paper has tremendous potential to carry out future research

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