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Some characterizations on totally geodesic transversal hypersurfaces of a nearly β–Kenmotsu manifold

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Abstract

In this paper, we study some results of transversal hypersurfaces with $(f, \tilde{g}, u, v, \lambda)$–structure of a nearly β–Kenmotsu manifold. Some more results on totally geodesic or totally umbilical transversal hypersurface with $(f, \tilde{g}, u, v, \lambda)$–structure of a nearly β–Kenmotsu manifold have also been studied.

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Key words: Totally geodesic; hypersurface; Kenmotsu manifold.

1 Introduction

Transversal hypersurfaces of an almost contact manifold was studied by Yano, Sang-Seup Eum and U-Hang Ki[7] in 1972. In 2003, transversal hypersurfaces of Kenmotsu manifold have been studied by Prasad and Tripathi[8]. Prasad *et.al.*[10] have studied on quasi-conformally semi- symmetric Kenmotsu manifold in 2010. In 1970, Yano and Okumura[1] have defined a new structure in an even dimensional manifolds called an (f, U, V, u, v, λ)-structure. There always exists a $(f, \tilde{g}, u, v, \lambda)$"structure and gave the results that there does not exist an invariant hy- persurface of a contact manifold. In 1967, Okumura[2] has deduced certain hypersurfaces of an odd-dimensional sphere. Later on it was studied by Blair and Ludden[3], Blair *et.al.*[4], Yano and Ki[5,6] and others. If a manifold with (f, U, V, u, v, λ)-structure has a positive def- inite Riemannian metric $\tilde{g}$, under certain conditions, then we call such a manifold has a metric

(f, $\tilde{g}$, u, v, λ)-structure or (f, $\tilde{g}$, u, v, λ)-structure. Any submanifold of codimension 2 immersed in an almost hermitian manifold and any hypersurface immersed in an almost contact metric manifold admit an (f, $\tilde{g}$, u, v, λ)-structure.

The paper is organized as follows. The Section 2 contains preliminaries. In Section 3, we study some properties of transversal hypersurfaces of nearly β–Kenmotsu manifold. Also, we obtain a necessary and sufficient condition for M, transversal hypersurface with (f, $\tilde{g}$, u, v, λ)-structure of a nearly β-Kenmotsu manifold, to be totally geodesic and totally umbilical.

2 Preliminaries :

Let $\widetilde{M}$ f be an almost contact metric manifold endowed with almost contact metric structure (φ, ξ, η, $\tilde{g}$) that is ϕ is (1, 1) tensor field, ξ is a vector field, η is 1–form and $\tilde{g}$ is a Riemannian metric such that

(2.1) $$\eta(\xi) = 1, \quad \varphi(\xi) = 0, \quad \eta o \varphi = 0, \quad \varphi^2 = -identity + \eta \otimes \xi,$$

(2.2) $$\tilde{g}(\varphi X, \varphi Y) = \tilde{g}(X, Y) - \eta(X)\eta(Y),$$

(2.3) $$\tilde{g}(X, \varphi Y) = -\tilde{g}(\varphi X, Y), \ \eta(X) = \tilde{g}(X, \xi)$$

for any $X, Y \in T\widetilde{M}$.

An almost contact metric manifold $\widetilde{M}$ with almost contact metric structure $(\varphi, \xi, \eta, \tilde{g})$ is said to be a nearly β —Kenmotsu manifold if

(2.4) $$(\tilde{\nabla}_X \varphi)Y + (\tilde{\nabla}_Y \varphi)X = -\beta(\eta(Y)\varphi X + \eta(X)\varphi Y)$$

for all vector fields X, Y on $\widetilde{M}$, where β is smooth functions on $\widetilde{M}$ and $\tilde{\nabla}$ is the operator of covariant diûerentiation with respect to $\tilde{g}$.

From (2.4), we have

(2.5) $$\tilde{\nabla}_X \xi = \beta(X - \eta(X)\xi) - \varphi(\tilde{\nabla}_\xi \varphi)X.$$

The Gauss and Weingarten formulae are given by

(2.6) $$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y)\hat{N},$$

(2.7) $$\tilde{\nabla}_X \hat{N} = -A_{\hat{N}} X$$

for any $X, Y \in TM$; where $\tilde{\nabla}$ and ∇ are the Riemannian and induced Riemannian connections in $\widetilde{M}$ and M respectively and $\hat{N}$ is the unit normal vector in the normal bundle $T^{\perp}M$.

The second fundamental form σ on M related to $A_{\hat{N}}$ is given by

(2.8) $$\sigma(X, Y) = \tilde{g}(A_{\hat{N}} X, Y).$$

Let M be a hypersurface of an almost contact metric manifold $\widetilde{M}$, then we deûne the following

(2.9) $$\varphi X = fX + u(X)\hat{N},$$

(2.10) $$\varphi \hat{N} = -U,$$

(2.11) $$\xi = V + \lambda \hat{N}; \ \lambda = \eta(\hat{N}),$$

(2.12) $$\eta(X) = v(X)$$

for $X \in TM$.

We get an induced $(f, \tilde{g}, u, v, \lambda)$ −structure[2,7] on the transversal hypersurface satisfying

(2.13) $$f^2 = -I + u \otimes U + v \otimes V,$$

(2.14) $$fU = -\lambda V, \qquad fV = \lambda U,$$

(2.15) $$uof = \lambda v, \qquad vof = -\lambda u,$$

(2.16) $$u(U) = 1 - \lambda^2, \qquad u(V) = v(U) = 0, \qquad v(V) = 1 - \lambda^2,$$

(2.17) $$\tilde{g}(fX, fY) = \tilde{g}(X, Y) - u(X)u(Y) - v(X)v(Y),$$

(2.18) $$\tilde{g}(X, fY) = -\tilde{g}(fX, Y), \qquad \tilde{g}(X, U) = u(X), \qquad \tilde{g}(X, V) = v(X),$$

for any $X, Y \in TM;\ \lambda = \eta(\widehat{N})$.

Therefore every transversal hypersurface immersed in an almost contact Riemannian manifold admits an $(f, \tilde{g}, u, v, \lambda)$ −structure.

3 *Transversal hypersurfaces with* $(f, \tilde{g}, u, v, \lambda)$ *−structure :*

A hypersurface of an almost contact manifold does not in general possess an almost complex structure. It is well known that a hypersurface (real codimension 1) of an almost complex manifold admits an almost contact structure. A transversal hypersurface M of an almost contact manifold $\widetilde{M}$ equipped with an almost contact structure (φ, ξ, η) is a hypersurface such that there exists a structure vector field ξ never belong to tangent hyperplane of the hypersurface M [7]. In this case structure vector field can be taken as an affine normal to the hypersurface. For $X \in TM$, since X and ξ are linearly independent, then ϕX defined as

(3.1) $$\varphi X = FX + \epsilon(X)\xi,$$

where F is (1,1) tensor field and is form on M.

Theorem 3.1 Let M be a transversal hypersurfaces with $(f, \tilde{g}, u, v, \lambda)$ −structure of a nearly β–Kenmotsu manifold $\widetilde{M}$, and if M equipped with contact metric structure, then

(3.2) $$\nabla_X V = \lambda A_{\widehat{N}} X + \beta(X - v(X)V),$$

(3.3) $$\sigma(X, V) = -\beta\lambda v(X) - X\lambda$$

For all $X, Y \in TM$.

Proof. From (2.3) and (2.8), we have

(3.4) $$\tilde{\nabla}_X \xi = \nabla_X V - \lambda A_{\widehat{N}} + (\sigma(X, V) + X\lambda)\widehat{N}$$

Using (2.2) and (2.6), we have the following

$$\nabla_X V - \lambda A_{\widehat{N}} X + (\sigma(X,V) + X\lambda)\widehat{N}$$

(3.5) $$= \beta X - \beta v(X)V - \beta\lambda v(X)\,\widehat{N} - f((\tilde{\nabla}_\xi \varphi)X) - u((\tilde{\nabla}_\xi \varphi)X)\,\widehat{N}.$$

Since manifold M equipped with contact metric structure, the above equation reduces to

$$\nabla_X V - \lambda A_{\widehat{N}} X\, X + (\sigma(X,V) + X\lambda)\,\widehat{N}$$

(3.6) $$= \beta X - \beta v(X)V - \beta\lambda v(X)\,\widehat{N}.$$

Equating tangential and normal parts, we have respectively (3.2) and (3.3).

Theorem 3.2 If M is a transversal hypersurfaces with $(f,\tilde{g},u,v,\lambda)$ –structure of a nearly β–Kenmotsu manifold $\widetilde{M}$ equipped with contact metric structure, then

(3.7) $$\nabla_X U = \beta\lambda f X + f(A_{\widehat{N}}X) + (\nabla_{\widehat{N}} f)X$$

for all $X, Y \in TM$.

Proof. Consider

$$(\nabla_X \varphi)\ \widehat{N} = \tilde{\nabla}_X \varphi \widehat{N} - \varphi e\, \tilde{\nabla}_X \widehat{N}$$

$$= -\nabla_X U - \sigma(X,U)\,\widehat{N} + f(A_{\widehat{N}}X) + u(A_{\widehat{N}}X)\,\widehat{N}$$

(3.8) $$= -\nabla_X U + f(A_{\widehat{N}}X)$$

using (2.4)-(2.9).
From (2.1), (2.6) and (2.8), we obtain

$$(\tilde{\nabla}_X \varphi)Y + (\tilde{\nabla}_Y \varphi)X$$

(3.9) $$= -\beta\{\eta(Y)fX + \eta(X)fY\} - \{\beta\eta(Y)u(X) + \beta\eta(X)u(Y)\}\,\widehat{N}.$$

Replacing $Y = \widehat{N}$ in (3.9), we have

$$(\tilde{\nabla}_X \varphi)\,\widehat{N} + (\tilde{\nabla}_{\widehat{N}} \varphi)X = -\beta\lambda f X - \beta v(X) f\,\widehat{N}$$

(3.10) $$-\{\beta\lambda u(X) + \beta v(X)u(\widehat{N})\}\widehat{N}.$$

From (3.8) and (3.10), we get

$$\nabla_X U - f(A_{\widehat{N}}X) - (\tilde{\nabla}_{\widehat{N}} \varphi)X = \beta\lambda f X + \beta v(X) f \widehat{N}$$

(3.11) $$+ \{\beta\lambda u(X) + \beta v(X)u(\widehat{N})\}\widehat{N}.$$

Now equating tangential part, we have (3.7).

Proposition 3.3 Let M be a transversal hypersurfaces with $(f,\tilde{g},u,v,\lambda)$ –structure of a nearly β–Kenmotsu manifold $\widetilde{M}$. Then we have

$$(\nabla_X f)Y + (\nabla_Y f)X = 2\sigma(X,Y)U - \beta(v(Y)fX + v(X)fY)$$

(3.12) $$+u(Y)A_{\widehat{N}}X - u(X)A_{\widehat{N}}Y.$$

(3.13) $$(\nabla_X u)Y + (\nabla_Y u)X = -\beta v(Y)u(X) - \beta v(X)u(Y) - \sigma(X,fY) - \sigma(Y,fX)$$

for all $X, Y \in TM.$

Proof. Consider

$$(\tilde{\nabla}_X \varphi)Y = (\tilde{\nabla}_X \varphi Y) - \varphi(\tilde{\nabla}_X Y)$$
$$= \tilde{\nabla}_X (fY + u(Y)\hat{N}) - \varphi(\nabla_X Y + \sigma(X,Y)\hat{N})$$
$$= (\nabla_X f)Y - u(Y)A_{\hat{N}}X + \sigma(X,Y)U$$

(3.14)
$$+((\nabla_X u)Y + \sigma(X, fY))\hat{N}.$$

Similarly

$$(\tilde{\nabla}_Y \varphi)X = (\nabla_Y f)X - u(X)A_{\hat{N}}Y + \sigma(X,Y)U$$

(3.15)
$$+((\nabla_Y u)X + \sigma(Y, fX))\hat{N}$$

using (2.6)-(2.9). From (3.14) and (3.15), we have

$$(\tilde{\nabla}_X \varphi)Y + (\tilde{\nabla}_Y \varphi)X = ((\nabla_X u)Y + (\nabla_Y u)X + \sigma(X, fY) + \sigma(Y, fX))\hat{N}$$

(3.16)
$$+(\nabla_X f)Y + (\nabla_Y f)X + 2\sigma(X,Y)U - u(Y)A_{\hat{N}}X - u(X)A_{\hat{N}}Y.$$

From (3.9) and (3.16), we have

$$(\nabla_X f)Y + (\nabla_Y f)X - u(Y)A_{\hat{N}}X - u(X)A_{\hat{N}}Y + 2\sigma(X,Y)U$$
$$+\{(\nabla_X u)Y + (\nabla_Y u)X + \sigma(X, fY) + \sigma(Y, fX)\}\hat{N}$$
$$= -\beta\{\eta(Y)fX + \eta(X)fY\}$$
$$-\{\beta\eta(Y)u(X) + \beta\eta(X)u(Y)\}\hat{N}.$$

Equating tangential and normal components of above, we get respectively (3.12) and (3.13).

Theorem 3.4 In order that a totally umbilical transversal hypersurfaces with $(f, \tilde{g}, u, v, \lambda)$ *−structure of a nearly* β*–Kenmotsu manifold* $\widetilde{M}$ *equipped with contact metric structure be totally geodesic it is necessary and suûcient that*

(3.17)
$$\beta\lambda v(X) + X\lambda = 0.$$

Proof. From Theorem 3.1, we have

(3.18)
$$\sigma(X, V) = -\beta\lambda v(X) - X\lambda.$$

If M is totally umbilical, then $A_{\hat{N}} = \zeta I$, where ζ is Kahlerian metric[9] and we know the relation of σ on M related to $A_{\hat{N}}$ by

(3.19)
$$\sigma(X,Y) = \tilde{g}(A_{\hat{N}}X, Y) = \tilde{g}(\zeta X, Y) = \zeta\tilde{g}(X,Y).$$

Therefore $\sigma(X, V) = \zeta\tilde{g}(X, V) = \zeta v(X),$ then (3.18) gives

(3.20) $$\beta\lambda v(X) + X\lambda + \zeta v(X) = 0.$$

If M is totally geodesic that is $\zeta = 0$, then (3.20) implies

(3.21) $$\beta v + d(log\lambda) = 0.$$

The converse can easily be verified.

Theorem 3.5 Let M be a transversal hypersurfaces with $(f, \tilde{g}, u, v, \lambda)$ −structure of a nearly β–Kenmotsu manifold with parallel tensor field f of type (1, 1) equipped with contact structure. Then M is totally geodesic if

(3.22) $$2\beta\lambda v + d(log\lambda) = 0.$$

Proof. Since f is parallel, (3.12) reduces to

(3.23) $$2\sigma(X,Y)U = \beta(v(Y)fX + v(X)fY) - u(Y)A_{\hat{N}}X - u(X)A_{\hat{N}}Y.$$

Applying u on (3.23) and using (2.12), we obtain

(3.24) $$2(1-\lambda^2)\sigma(X,Y) = 2\beta\lambda v(X)v(Y) - u(Y)u(A_{\hat{N}}X) - u(X)u(A_{\hat{N}}Y).$$

In view of (3.24), we have

(3.25) $$u(A_{\hat{N}}X) = -\frac{\sigma(U,U)}{3(1-\lambda^2)}u(X).$$

In the similar way, we have

(3.26) $$u(A_{\hat{N}}Y) = -\frac{\sigma(U,U)}{3(1-\lambda^2)}u(Y).$$

With the help of (3.24), (3.25) and (3.26), we have

(3.27) $$\sigma(X,Y) = \frac{\beta\lambda}{3(1-\lambda^2)}v(X)v(Y) + \frac{\sigma(U,U)}{3(1-\lambda^2)^2}u(X)u(Y),$$

where $\sigma(U,U) = g(A_{\hat{N}}U,U) = u(A_{\hat{N}}U)$.

Again from (3.3) and (3.27), we obtain

(3.28) $$2\beta\lambda v(X) + X\lambda = 0.$$

Theorem 3.6 If M is a transversal hypersurfaces with $(f, \tilde{g}, u, v, \lambda)$ −structure of a nearly β–Kenmotsu manifold with a parallel vector field U, then M is totally geodesic if

(3.29) $$\beta\lambda fX + (\nabla_{\hat{N}}f)X = 0.$$

Proof. Since we have seen that

(3.30) $$(\tilde{\nabla}_X\varphi)\hat{N} = -\nabla_X U + f(A_{\hat{N}}X),$$

and

(3.31) $$(\tilde{\nabla}_X\varphi)\hat{N} + (\tilde{\nabla}_{\hat{N}}\varphi)X = -\beta\lambda fX - \beta v(X)f\hat{N} - \{\beta\lambda u(X) + \beta v(X)u(\hat{N})\}\hat{N}.$$

Now from (3.15), we get

(3.32) $(\tilde{\nabla}_{\hat{N}}\varphi)X = (\nabla_{\hat{N}}f)X - u(X)A_{\hat{N}}\hat{N} + \{(\nabla_{\hat{N}}u)X + \sigma(\hat{N}, fX)\}\hat{N}.$

Using (3.30), (3.31) and (3.32), we obtain

$$\nabla_X U - f(A_{\hat{N}}X) - (\tilde{\nabla}_{\hat{N}}\varphi)X = \beta\lambda fX + \beta v(X)f\,\hat{N}$$

(3.33) $$-\{\beta\lambda u(X) + \beta v(X)u(\hat{N})\}\hat{N}.$$

Equating tangential part, we have

(3.34) $$\nabla_X U = \beta\lambda fX + f(A_{\hat{N}}X) + (\nabla_{\hat{N}}f)X.$$

Since U is parallel, then (3.34) implies

(3.35) $$\beta\lambda fX + f(A_{\hat{N}}X) + (\nabla_{\hat{N}}f)X = 0.$$

Now, if M is totally geodesic then $\zeta = 0$, that is, $A_{\hat{N}} = 0$, then from (3.35), we have (3.29).

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References

1. K. Yano and M. Okumura, On $(f, \tilde{g}, u, v, \lambda)$ −structures, Kodai Math. Sem. Rep. *22*, 401- 423 (1970).
2. M. Okumura, Certain hypersurfaces of an odd- dimensional sphere, Tohoku Math. J. *19*, 381- 395 (1967).
3. D. E. Blair, and g. D. Ludden, Hypersurfaces in almost contact manifold, Tohoku Math. J. *22*, 354- 362 (1969).
4. D. E. Blair, and g. D. Ludden and K. Yano, Induced structures on submanifolds, Kodai Math. Sem. Rep. *22*, 188- 198 (1970).
5. K. Yano and U-Hang Ki, On quasi-normal $(f, \tilde{g}, u, v, \lambda)$ −structures, Kodai Math. Sem. Rep. *24*, 121- 130 (1972).
6. K. Yano and U-Hang Ki, Submanifolds of codimension 2 in an even- dimensional Euclidean space, Kodai Math. Sem. Rep. *24*, 315-330 (1972).
7. K. Yano, Sang-Seup Eum and U-Hang Ki, On transversal hypersurfaces of an almost contact manifold, Kodai Math. Sem. Rep. *24*, 459- 470 (1972).
8. R. Prasad and M. M. Trapathi, Transversal hypersurfaces of Kenmotsu manifold, Indian J. pure appl. Math. *34(3)*, 443-452 (2003).
9. S. I. goldberg, Conformal transformation of Kähler manifolds Bull. Amer. Math. Soc., *66*, 54-58 (1960).
10. R. Prasad, S. Kishor and S. P. Yadav, On quasi-conformally semi-symmetric Kenmotsu manifold, *Ultra Scientist 22(1)M*, 267-270 (2010).