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A Fuzzy type Backlogging Production Inventory Model for Perishable Items with Time Dependent Exponential Demand Rate

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Abstract

Production inventory models have an important role in production planning and scheduling. In any economic production quantity (EPQ) model, the production rate is dependent on demand. In this paper we have established a production inventory model for perishable items with partial backlogging and time dependent exponential demand rate. Allowing shortage, it is partially backlogged. The unsatisfied demand is backlogged and it is considered a function of waiting time. The aim of our study is to optimizing the total profit during a given cycle. A numerical example is given in showing the applicability of the developed model.

Key words : Inventory, Perishability, Production, Backlogging and Time Dependent Exponential Demand Rate.

Subject Classification- MSC 2010 (90B05)

Introduction

In crisp inventory models all the parameters have their definite values. But in reality, they will not be so certain. Therefore, we need to consider the fuzzy inventory models. In this scenario, scholars have payee their attention towards the time dependent demand. Because the demand of newly launched products such as fashionable garments, digital products, smart phones, motor vehicles, life care medicines and vaccines etc. increases with time and later it becomes constant.

Some scholars in this area have worth mentioning. Ghare and Schrader¹ developed an exponentially decaying inventory model for deteriorating items. Sarker *et al.*² worked on an order level inventory model for

decaying items with order level dependent demand rate. Teng *et al.*³ analyzed a deterministic lot-size inventory model for perishable item with fluctuating demand. They were also considered shortages in their inventory model. Sana *et al.*⁴ constructed a production inventory model for perishable items with linear trend in demand and allowing shortages. Cardenas established two inventory models⁵ and ⁷. In the first model he was considered an optimal manufacturing batch size and rework for a single stage production system. And in the second inventory model he was constructed an EPQ model with rework process at a single stage manufacturing and allowing planned backorders. Cheng and Wang⁶ were presented a note on inventory model of perishable items with fuzzy type trapezoidal demand rate. Cheng *et al.*⁸ presented an optimal replenishment policy for an inventory model of perishable items with partial backlogging and fuzzy type trapezoidal demand rate. Sarkar *et al.*⁹ analyzed an optimal replenishment policy for perishable items with time varying quadratic demand and shortage. Wee *et al.*¹⁰ constructed an EPQ model for deteriorating items having finite time horizon and back orders. Wee *et al.*¹¹ were presented an alternative solution procedure for an EPQ model with rework process at a single stage manufacturing system along with shortages. Entezari *et al.*¹² studied a manufacturing plan and optimal production control policy for an unreliable flexible manufacturing system with two machines. Mousavi *et al.*¹³ worked on an inventory control problem for multi products with multi period and shortages. Sivashankari and Panayappan¹⁴ analyzed two level production inventory model for deteriorating items with shortages. Behera and Tripathy were developed two inventory models¹⁵ and ²¹. First is a fuzzy EOQ model for time varying deteriorating items with penalty cost. And in the second inventory model they were presented a replenishment policy for decaying items with time and reliability varying demand. Arora¹⁶ analyzed a study on inventory models for perishable items with shortages. Hossen *et al.*¹⁷ constructed an inventory model for deteriorating items with time and price varying demand under inflation and considering fuzzy inventory costs. Sekar and Uthayakumar¹⁸ focused on a multi-production inventory model for perishable items assuming penalty and environmental pollution cost with failure rework. Maragatham and Palani¹⁹ worked on an inventory model for deteriorating items with lead time and price dependent demand allowing shortages. Naik and Patel²⁰ established an inventory model for repairable items with imperfect quality and different deterioration rates under price and time dependent demand. Haughton and Isotupa²² analyzed a continuous review inventory system with emergency orders and lost sales. Bhayana *et al.*²³ constructed an integrated supplier selection approach in supply chain system under fuzzy environment. Istam *et al.*²⁴ worked on an inventory model with three stage supply chain and random capacities under disruptions and supplier reliability.

Khanna *et al.*²⁵ worked on inventory and pricing decisions for an imperfect production with quality inspection, rework and carbon emission. Sen and Saha²⁶ developed an inventory model for deteriorating items with negative exponential demand, probabilistic deterioration and fuzzy lead time under partial backlogging. Mohammed and Azam²⁷ presented an algorithm for fuzzy soft set based decision making approach. Uddin *et al.*²⁸ established a production inventory model with level dependent demand allowing few defective items. Das *et al.*²⁹ constructed a production inventory model with partial trade credit policy and reliability.

2. Definitions :

Fuzzy Set- A fuzzy set \tilde{A} on the given universal set X is a set of ordered pairs *i.e.*

$$\tilde{A} = \left\{ \left(x, \lambda_{\tilde{A}}(x) \right) : x \in X \right\}$$

Where $\lambda_{\tilde{A}} : X \rightarrow [0, 1]$ is called a membership function.

Fuzzy Number – A fuzzy number \tilde{A} is a fuzzy set on the real line, if its membership function $\lambda_{\tilde{A}}$ satisfying the following three properties,

- 1- $\lambda_{\tilde{A}}(x)$ is upper semi continuous.
- 2- $\lambda_{\tilde{A}}(x) = 0$, outside some interval $[a_1, a_4]$.
- 3- \exists real numbers a_2 & a_3 , $a_1 \leq a_2 \leq a_3 \leq a_4$ such that $\lambda_{\tilde{A}}(x)$ is increasing on $[a_1, a_2]$, decreasing on $[a_3, a_4]$, and $\lambda_{\tilde{A}}(x) = 1$, for each x in $[a_2, a_3]$.

Triangular Fuzzy Number : Triangular fuzzy number (TFN) is specified by the triplet (a_1, a_2, a_3)

where $a_1 < a_2 < a_3$ and defined by its continuous membership function $\lambda_{\tilde{A}} : X \rightarrow [0, 1]$, as follows,

$$\lambda_{\tilde{A}}(x) = \begin{cases} \frac{x - a_1}{a_2 - a_1}, & a_1 \leq x \leq a_2 \\ \frac{a_3 - x}{a_3 - a_2}, & a_2 \leq x \leq a_3 \\ 0, & \text{otherwise} \end{cases}$$

Signed Distance : If $\tilde{A} = (a, b, c)$ is a triangular fuzzy number, the signed distance of \tilde{A} is defined as $d(\tilde{A}, 0) = \frac{(a + 2b + c)}{4}$.

3. Notations and Assumptions :

We consider the following notations and assumptions corresponding to the prescribed model

1. The demand rate is $R(t) = e^{a-bt}$, $a, b > 0$.
2. The production rate is $P(t) = k R(t)$.
3. δ is the backlogging parameter.
4. θ is the deterioration parameter.
5. O_C is the ordering cost per order.
6. h_C is the holding cost per unit per cycle.
7. S_C is the shortage cost per cycle.

8. S is an inventory level at time $t = T_0$.
9. is the time of zero inventory level.
10. T is the cycle length.
11. The infinite replenishment rate.
12. The zero lead time.
13. $TP(T_0, T_1, T)$ is the total profit per cycle.

4. Mathematical Derivation :

Graphically, it has been observed that the production starts at time $t = 0$ and stops at the maximum level S at time $t = T_0$. In the interval $[0, T_0]$ the production grows due to demand and deterioration. And in the interval $[T_0, T_1]$ the maximum production level S , decreases due to demand and deterioration and becomes zero at $t = T_1$. The interval $[T_1, T]$ is the shortage interval in which the unsatisfied demand is backlogged at a rate of $B(t) = \delta(T - t)$, where δ is the backlogging parameter and t is the waiting time.

In the given cycle, the inventory level at any time t is given by the following differential equations,

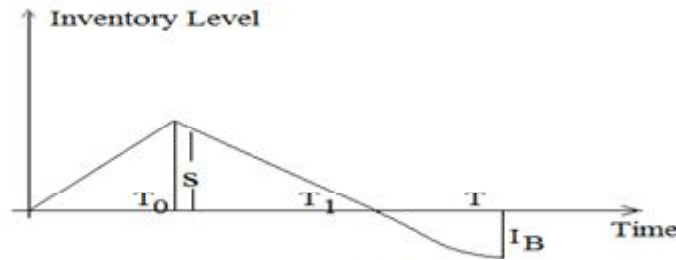


Figure 1, Inventory Model

$$\frac{dI}{dt} + \theta I = -P(t) \quad 0 \leq t \leq T_0$$

Putting the value of $P(t)$, we have

$$\frac{dI}{dt} + \theta I = -k e^{a-bt} \quad 0 \leq t \leq T_0 \quad (1)$$

With boundary condition

$$I(0) = 0$$

$$\frac{dI}{dt} + \theta I = -e^{a-bt} \quad T_0 \leq t \leq T_1 \quad (2)$$

With boundary condition

$$I(T_0) = S$$

$$\frac{dI}{dt} = -\delta(T - t) e^{a-bt} \quad (3)$$

With boundary condition

$$I(T_1) = 0$$

The solution of differential equation (1) is given by the equations (4) respectively.

$$I = \left(\frac{k e^a}{\theta - b} \right) \left[e^{-\theta t} - e^{-b t} \right] \quad 0 \leq t \leq T_0 \quad (4)$$

The value of S is obtained by putting $I(T_0) = S$ in equation (4), so we have

$$S = \left(\frac{k e^a}{\theta - b} \right) \left[e^{-\theta T_0} - e^{-b T_0} \right] \quad (5)$$

The solution of differential equations (2) and (3) are given by the equations (6) and (7) respectively.

$$I = - \left(\frac{e^a e^{-b t}}{\theta - b} \right) + \left(\frac{e^{-\theta t} e^{\theta T_0}}{(\theta - b)^2} \right) \left[k e^a e^{-\theta T_0} + (1 - k) e^a e^{-b T_0} \right] \quad (6)$$

$$I = \left(\frac{\delta e^a}{b} \right) \left[T(e^{-b t} - e^{-b T_1}) - t e^{-b t} + T_1 e^{-b T_1} \right] + \left(\frac{\delta e^a}{b^2} \right) (e^{-b T_1} - e^{-b t}) \quad (7)$$

In the cycle, the ordering cost is

$$O_C = o_C \quad (8)$$

In the cycle, the holding cost is

$$H_C = h_C \left[\int_0^{T_0} I(t) dt + \int_{T_0}^{T_1} I(t) dt \right]$$

Or

$$H_C = h_C \left[- \left(\frac{k e^a}{b \theta} \right) + \left(\frac{k e^a}{\theta - b} \right) \left(\frac{e^{-b T_0}}{b} - \frac{e^{-\theta T_0}}{\theta} \right) + \left(\frac{e^a}{b(\theta - b)} \right) (e^{-b T_1} - e^{-b T_0}) - \left(\frac{e^{\theta T_0}}{\theta(\theta - b)} \right) (e^{-\theta T_1} - e^{-\theta T_0}) \right. \\ \left. \cdot \left(\frac{1}{\theta - b} \right) \left\{ k e^a e^{-\theta T_0} - (k - 1) e^a e^{-b T_0} \right\} \right] \quad (9)$$

In the cycle, the shortage cost is

$$S_C = -s_C \int_{T_1}^T I(t) dt$$

Or

$$S_C = - \left(\frac{\delta s_C e^a}{b} \right) \left[\frac{T(e^{-b T_1} - e^{-b T})}{b} - T^2 e^{-b T_1} + T T_1 e^{-b T_1} + \frac{(T e^{-b T} - T_1 e^{-b T_1})}{b} + T T_1 e^{-b T_1} - T_1^2 e^{-b T_1} \right]$$

$$+ \frac{(e^{-bT} - e^{-bT_1})}{b^2} \left] - \left(\frac{\delta s_C e^a}{b^2} \right) \left[(T - T_1)e^{-bT_1} + \frac{(e^{-bT} - e^{-bT_1})}{b} \right] \quad (10)$$

The maximum back-ordered I_B is obtained by putting $t = T$ in equation (7), so we have

$$I_B = \left(\frac{\delta e^a}{b} \right) \left[T(e^{-bT} - e^{-bT_1}) - T e^{-bT} + T_1 e^{-bT_1} \right] + \left(\frac{\delta e^a}{b^2} \right) (e^{-bT_1} - e^{-bT}) \quad (11)$$

The maximum ordered quantity Q is given by

$$Q = S + I_B$$

Or

$$Q = \left(\frac{k e^a}{\theta - b} \right) (e^{-\theta T_0} - e^{-bT_0}) - \left(\frac{\delta e^a}{b} \right) \{ T(e^{-bT} - e^{-bT_1}) - T e^{-bT} + T_1 e^{-bT_1} \} - \left(\frac{\delta e^a}{b^2} \right) (e^{-bT_1} - e^{-bT}) \quad (12)$$

In the cycle, the purchasing cost is

$$P_C = p_C \times Q$$

Or

$$P_C = p_C \left[\left(\frac{k e^a}{\theta - b} \right) (e^{-\theta T_0} - e^{-bT_0}) - \left(\frac{\delta e^a}{b} \right) \{ T(e^{-bT} - e^{-bT_1}) - T e^{-bT} + T_1 e^{-bT_1} \} - \left(\frac{\delta e^a}{b^2} \right) (e^{-bT_1} - e^{-bT}) \right] \quad (13)$$

In the cycle, the opportunity cost is

$$OP_C = op_C \int_{T_1}^T e^{a-bt} \{ 1 - \delta(T-t) \} dt$$

Or

$$OP_C = op_C e^a \left[\frac{(e^{-bT_1} - e^{-bT})}{b} - \frac{\delta(T - T_1)e^{-bT_1}}{b} + \frac{\delta(e^{-bT} - e^{-bT_1})}{b^2} \right] \quad (14)$$

In the cycle, the sales revenue is

$$S_R = s_R \left[\int_{T_0}^{T_1} e^{a-bt} dt + \int_{T_1}^T \delta(T-t) e^{a-bt} dt \right]$$

Or

$$S_R = \left(\frac{s_R e^a}{b} \right) \left[(e^{-bT_0} - e^{-bT_1}) + \delta(T - T_1)e^{-bT_1} + \frac{(e^{-bT} - e^{-bT_1})}{b^2} \right] \quad (15)$$

In the cycle, the total profit is

$$TP(T_0, T_1, T) = \left(\frac{1}{T} \right) [S_R - (O_C + H_C + S_C + P_C + OP_C)]$$

Or

$$\begin{aligned}
 TP(T_0, T_1, T) = & \left(\frac{1}{T} \right) \left[-o_C + e^a \left\{ k p_C - s_R - \frac{h_C(2b+1)}{b(\theta-b)} \right\} T_0 + e^a \left\{ \frac{s_R(b^2-b\delta+1)}{b^2} + \frac{h_C(2b+1)}{b(\theta-b)} - \frac{\delta s_C(1-b)}{b^2} \right. \right. \\
 & - \frac{op_C(2\delta-b)}{b} \left. \right\} T_1 + e^a \left\{ \frac{s_R(b\delta-1)}{b^2} + \frac{s_C\delta(1-b)}{b^2} - \frac{op_C b(1-2\delta)}{b^2} \right\} T + \frac{h_C(k-1)e^a}{b} T_0^2 - \frac{s_C\delta e^a}{b} T^2 \\
 & + e^a \delta \left\{ s_R - p_C + op_C + \frac{s_C}{b} \right\} T_1^2 + e^a \delta \{ p_C - s_R - op_C \} T T_1 - \frac{h_C(k-1)e^a}{b} T_0 T_1 + \frac{s_C\delta e^a}{b} T_1^3 \\
 & \left. - \frac{h_C(k-1)\theta e^a}{(\theta-b)} T_0^3 - 2s_C\delta e^a T T_1^2 + s_C\delta e^a T^2 T_1 + \frac{h_C(k-1)\theta e^a}{(\theta-b)} T_0^2 T_1^2 \right] \quad (16)
 \end{aligned}$$

(By considering first degree terms of θ and δ in equation (16))

The necessary conditions for $TP(T_0, T_1, T)$ to be maximum are

$$\frac{\partial TP(T_0, T_1, T)}{\partial T_0} = 0, \quad \frac{\partial TP(T_0, T_1, T)}{\partial T_1} = 0, \quad \frac{\partial TP(T_0, T_1, T)}{\partial T} = 0 \quad (17)$$

After solving the equations in equation (17), we find the optimum values of T_0 , T_1 and T for which the total profit is maximum.

The sufficient conditions for $TP(T_0, T_1, T)$ to be maximum are that the principal minors of the Hessian matrix or H matrix are negative definite. The Hessian matrix is defined as follows

$$H = \begin{bmatrix} \frac{\partial^2 TP(T_0, T_1, T)}{\partial T_0^2} & \frac{\partial^2 TP(T_0, T_1, T)}{\partial T_0 \partial T_1} & \frac{\partial^2 TP(T_0, T_1, T)}{\partial T_0 \partial T} \\ \frac{\partial^2 TP(T_0, T_1, T)}{\partial T_1 \partial T_0} & \frac{\partial^2 TP(T_0, T_1, T)}{\partial T_1^2} & \frac{\partial^2 TP(T_0, T_1, T)}{\partial T_1 \partial T} \\ \frac{\partial^2 TP(T_0, T_1, T)}{\partial T \partial T_0} & \frac{\partial^2 TP(T_0, T_1, T)}{\partial T_0 \partial T_1} & \frac{\partial^2 TP(T_0, T_1, T)}{\partial T^2} \end{bmatrix}$$

On differentiating equation (17), we obtain the following

$$\begin{aligned}
 \frac{\partial TP(T_0, T_1, T)}{\partial T_0} = & \left(\frac{1}{T} \right) \left[e^a \left\{ k p_C - s_R - \frac{h_C(2b+1)}{b(\theta-b)} \right\} + e^a \left\{ \frac{2h_C(k-1)}{b} \right\} T_0 - e^a \left\{ \frac{h_C(k-1)}{b} \right\} T_1 \right. \\
 & \left. - e^a \left\{ \frac{3h_C(k-1)\theta}{\theta-b} \right\} T_0^2 + e^a \left\{ \frac{2h_C(k-1)\theta}{\theta-b} \right\} T_0 T_1^2 \right] \quad (18)
 \end{aligned}$$

$$\begin{aligned} \frac{\partial TP(T_0, T_1, T)}{\partial T_1} = & \left(\frac{1}{T} \right) \left[e^a \left\{ \frac{s_R(b^2 - b\delta + 1)}{b^2} + \frac{h_C(2b + 1)}{b(\theta - b)} - \frac{s_C\delta(1 - b)}{b^2} - \frac{op_C(2\delta - b)}{b} \right\} + 2\delta e^a \left\{ s_R + \frac{s_C}{b} \right\} T_1 \right. \\ & \left. - s_R\delta e^a T - e^a \left(\frac{h_C(k - 1)}{b} \right) T_0 - 4s_C\delta e^a TT_1 + s_C\delta e^a T^2 + e^a \left\{ \frac{2h_C(k - 1)\theta}{\theta - b} \right\} T_0^2 T_1 \right] \quad (19) \end{aligned}$$

$$\begin{aligned} \frac{\partial TP(T_0, T_1, T)}{\partial T} = & \left(\frac{1}{T} \right) \left[e^a \left\{ \frac{s_R(b\delta - 1)}{b^2} + \frac{s_C\delta(1 - b)}{b^2} - \frac{op_C(1 - 2\delta)}{b} \right\} - 2e^a s_C\delta T_1^2 + e^a \delta(p_C - s_R - op_C)T_1 \right. \\ & \left. - e^a \left(\frac{2s_C\delta}{b} \right) T + 2e^a s_C\delta TT_1 \right] - \left(\frac{1}{T^2} \right) \left[-o_C + e^a \left\{ kp_C - s_R - \frac{h_C(2b + 1)}{b(\theta - b)} \right\} T_0 + e^a \left\{ \frac{s_R(b^2 - b\delta + 1)}{b^2} \right. \right. \\ & \left. \left. + \frac{h_C(2b + 1)}{b(\theta - b)} - \frac{s_C\delta(1 - b)}{b^2} - \frac{op_C(2\delta - b)}{b} \right\} T_1 + \left(\frac{e^a}{b^2} \right) \{ s_R(b\delta - 1) + s_C\delta(1 - b) - op_C b(1 - 2\delta) \} T \right. \\ & \left. + e^a \left(\frac{h_C(k - 1)}{b} \right) T_0^2 + e^a \delta \left(s_R + op_C - p_C + \frac{s_C}{\delta} \right) T_1^2 - e^a \left(\frac{s_C\delta}{b} \right) T^2 + e^a \delta(p_C - op_C - s_R)TT_1 \right. \\ & \left. - e^a \left(\frac{h_C(k - 1)}{b} \right) T_0 T_1 - e^a \left(\frac{h_C\theta(k - 1)}{\theta - b} \right) T_0^3 + e^a \left(\frac{s_C\delta}{b} \right) T_1^3 - 2s_C\delta e^a TT_1^2 + s_C\delta e^a T^2 T_1 \right. \\ & \left. + e^a \left(\frac{h_C\theta(k - 1)}{\theta - b} \right) T_0^2 T_1^2 \right] \quad (20) \end{aligned}$$

$$\frac{\partial^2 TP(T_0, T_1, T)}{\partial T_0^2} = \left(\frac{1}{T} \right) \left[2e^a \left\{ \frac{h_C(k - 1)}{b} - \frac{3h_C(k - 1)\theta}{\theta - b} T_0 + \frac{h_C(k - 1)\theta}{\theta - b} T_1^2 \right\} \right] \quad (21)$$

$$\frac{\partial^2 TP(T_0, T_1, T)}{\partial T_0 \partial T_1} = \left(\frac{1}{T} \right) \left[e^a \left\{ -\frac{h_C(k - 1)}{b} + \frac{4h_C(k - 1)\theta}{\theta - b} T_0 T_1 \right\} \right] \quad (22)$$

$$\begin{aligned} \frac{\partial^2 TP(T_0, T_1, T)}{\partial T_0 \partial T} = & - \left(\frac{1}{T^2} \right) \left[e^a \left\{ kp_C - s_R - \frac{h_C(2b + 1)}{b(\theta - b)} \right\} + \frac{2h_C(k - 1)e^a}{b} T_0 - \frac{h_C(k - 1)e^a}{b} T_1 \right. \\ & \left. - \frac{3h_C(k - 1)\theta e^a}{\theta - b} T_0^2 + \frac{2h_C(k - 1)\theta e^a}{\theta - b} T_0 T_1^2 \right] \quad (23) \end{aligned}$$

$$\frac{\partial^2 TP(T_0, T_1, T)}{\partial T_1 \partial T_0} = \left(\frac{1}{T} \right) \left[e^a \left\{ -\frac{h_C(k - 1)}{b} + \frac{4h_C(k - 1)\theta}{\theta - b} T_0 T_1 \right\} \right] \quad (24)$$

$$\frac{\partial^2 TP(T_0, T_1, T)}{\partial T_1^2} = \left(\frac{1}{T} \right) \left[e^a \left\{ 2s_R \delta + \frac{s_C \delta}{b} - 4s_C \delta T + \frac{2h_C(k-1)\theta}{\theta-b} T_0^2 \right\} \right] \quad (25)$$

$$\begin{aligned} \frac{\partial^2 TP(T_0, T_1, T)}{\partial T_1 \partial T} = & \left(\frac{1}{T} \right) \left[\delta(p_C - s_R - op_C) e^a - 4s_C \delta e^a T_1 + 2s_C \delta e^a T \right] - \left(\frac{1}{T^2} \right) \left[\left\{ \frac{s_R(b^2 - b\delta + 1)}{b^2} + \frac{h_C(2b+1)}{b(\theta-b)} \right. \right. \\ & - \frac{s_C \delta(1-b)}{b^2} - \frac{op_C(2\delta-b)}{b} \left. \right\} e^a + 2\delta \left(s_R + op_C - p_C + \frac{s_C}{b} \right) e^a T_1 + \delta(p_C - op_C - s_R) e^a T \\ & \left. - \frac{h_C(k-1)e^a}{b} T_0 + \frac{3s_C \delta e^a}{b} T_1^2 - 4s_C \delta e^a T T_1 + s_C \delta e^a T^2 + \frac{2h_C(k-1)\theta e^a}{\theta-b} T_0^2 T_1 \right] \quad (26) \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 TP(T_0, T_1, T)}{\partial T \partial T_0} = & - \left(\frac{1}{T^2} \right) \left[e^a \left\{ k p_C - s_R - \frac{h_C(2b+1)}{b(\theta-b)} \right\} + \frac{2h_C(k-1)e^a}{b} T_0 - \frac{h_C(k-1)e^a}{b} T_1 \right. \\ & \left. - \frac{3h_C(k-1)\theta e^a}{\theta-b} T_0^2 + \frac{2h_C(k-1)\theta e^a}{\theta-b} T_0 T_1^2 \right] \quad (27) \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 TP(T_0, T_1, T)}{\partial T \partial T_1} = & \left(\frac{1}{T} \right) \left[\delta(p_C - s_R - op_C) e^a - 4s_C \delta e^a T_1 + 2s_C \delta e^a T \right] - \left(\frac{1}{T^2} \right) \left[\left\{ \frac{s_R(b^2 - b\delta + 1)}{b^2} + \frac{h_C(2b+1)}{b(\theta-b)} \right. \right. \\ & - \frac{s_C \delta(1-b)}{b^2} - \frac{op_C(2\delta-b)}{b} \left. \right\} e^a + 2\delta \left(s_R + \frac{s_C}{b} \right) e^a T_1 + \delta(p_C - op_C - s_R) e^a T \\ & \left. - \frac{h_C(k-1)e^a}{b} T_0 + \frac{3s_C \delta e^a}{b} T_1^2 - 4s_C \delta e^a T T_1 + s_C \delta e^a T^2 + \frac{2h_C(k-1)\theta e^a}{\theta-b} T_0^2 T_1 \right] \quad (28) \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 TP(T_0, T_1, T)}{\partial T^2} = & \left(\frac{1}{T} \right) \left[2\delta e^a \left(-\frac{s_C}{b} + s_C T_1 \right) \right] - \left(\frac{1}{T^2} \right) \left[2e^a \left\{ \frac{s_R(b\delta-1)}{b^2} + \frac{s_C \delta(1-b)}{b^2} - \frac{op_C(1-2\delta)}{b} \right\} \right. \\ & - \frac{4s_C \delta e^a}{b} T + 2\delta e^a (p_C - s_R - op_C) T_1 - 4s_C \delta e^a T_1^2 + 4s_C \delta e^a T T_1 \left. \right] + \left(\frac{2}{T^3} \right) \left[-o_C + e^a \left\{ k p_C - s_R - \frac{h_C(2b+1)}{b(\theta-b)} \right\} T_0 \right. \\ & + e^a \left\{ \frac{s_R(b^2 - b\delta + 1)}{b^2} + \frac{h_C(2b+1)}{b(\theta-b)} - \frac{s_C \delta(1-b)}{b^2} - \frac{op_C(2\delta-b)}{b} \right\} T_1 + e^a \left\{ \frac{s_R(b\delta-1)}{b^2} + \frac{s_C \delta(1-b)}{b^2} \right. \\ & \left. \left. - \frac{op_C(1-2\delta)}{b} \right\} T + \frac{h_C(k-1)e^a}{b} T_0^2 + \delta e^a \left\{ s_R + op_C - p_C + \frac{s_C}{b} \right\} T_1^2 - \frac{s_C \delta e^a}{b} T^2 - \frac{h_C(k-1)\theta e^a}{\theta-b} T_0^3 \right] \end{aligned}$$

$$+ \frac{s_c \delta e^a}{b} T_1^3 - 2s_c \delta e^a T T_1^2 + s_c \delta e^a T^2 T_1 + \frac{h_c (k-1) \theta e^a}{\theta - b} T_0^2 T_1^2 \Big] \quad (29)$$

Numerically, the Hessian matrix is obtained as follows

$$H = \begin{bmatrix} 0.1012 & -0.6739 & -0.0019 \\ -0.6739 & -655.5900 & -1.3128 \\ -0.0019 & -1.3128 & 1.0072 \end{bmatrix}$$

5. Fuzzy Model :

The real market is filled with uncertainty. The uncertainty is handled by the fuzzy set theory, because fuzziness is the closed possible approach to the reality. Therefore we consider the backlogging parameter δ in fuzzy environment. Let $\delta = (\delta_1, \delta_2, \delta_3)$ is a triangular fuzzy number. In the cycle, the total profit is given by the signed distance method

$$\tilde{TP}(T_0, T_1, T) = \frac{1}{4T} \left[\tilde{TP}_1(T_0, T_1, T) + 2\tilde{TP}_2(T_0, T_1, T) + \tilde{TP}_3(T_0, T_1, T) \right] \quad (30)$$

Where,

$$\begin{aligned} \tilde{TP}_1(T_0, T_1, T) = & \left(\frac{1}{4T} \right) \left[-o_c + e^a \left\{ k p_c - s_r - \frac{h_c(2b+1)}{b(\theta-b)} \right\} T_0 + e^a \left\{ \frac{s_r(b^2 - b\delta_1 + 1)}{b^2} + \frac{h_c(2b+1)}{b(\theta-b)} - \frac{\delta_1 s_c(1-b)}{b^2} \right. \right. \\ & \left. \left. - \frac{op_c(2\delta_1 - b)}{b} \right\} T_1 + e^a \left\{ \frac{s_r(b\delta_1 - 1)}{b^2} + \frac{s_c\delta_1(1-b)}{b^2} - \frac{op_c b(1-2\delta_1)}{b^2} \right\} T + \frac{h_c(k-1)e^a}{b} T_0^2 - \frac{s_c\delta_1 e^a}{b} T^2 \right. \\ & \left. + e^a \delta_1 \left\{ s_r - p_c + op_c + \frac{s_c}{b} \right\} T_1^2 + e^a \delta_1 \{ p_c - s_r - op_c \} T T_1 - \frac{h_c(k-1)e^a}{b} T_0 T_1 + \frac{s_c\delta_1 e^a}{b} T_1^3 \right. \\ & \left. - \frac{h_c(k-1)\theta e^a}{(\theta-b)} T_0^3 - 2s_c\delta_1 e^a T T_1^2 + s_c\delta_1 e^a T^2 T_1 + \frac{h_c(k-1)\theta e^a}{(\theta-b)} T_0^2 T_1^2 \right] \\ \tilde{TP}_2(T_0, T_1, T) = & \left(\frac{1}{4T} \right) \left[-o_c + e^a \left\{ k p_c - s_r - \frac{h_c(2b+1)}{b(\theta-b)} \right\} T_0 + e^a \left\{ \frac{s_r(b^2 - b\delta_2 + 1)}{b^2} + \frac{h_c(2b+1)}{b(\theta-b)} - \frac{\delta_2 s_c(1-b)}{b^2} \right. \right. \\ & \left. \left. - \frac{op_c(2\delta_2 - b)}{b} \right\} T_1 + e^a \left\{ \frac{s_r(b\delta_2 - 1)}{b^2} + \frac{s_c\delta_2(1-b)}{b^2} - \frac{op_c b(1-2\delta_2)}{b^2} \right\} T + \frac{h_c(k-1)e^a}{b} T_0^2 - \frac{s_c\delta_2 e^a}{b} T^2 \right. \\ & \left. + e^a \delta_2 \left\{ s_r - p_c + op_c + \frac{s_c}{b} \right\} T_1^2 + e^a \delta_2 \{ p_c - s_r - op_c \} T T_1 - \frac{h_c(k-1)e^a}{b} T_0 T_1 + \frac{s_c\delta_2 e^a}{b} T_1^3 \right. \\ & \left. - \frac{h_c(k-1)\theta e^a}{(\theta-b)} T_0^3 - 2s_c\delta_2 e^a T T_1^2 + s_c\delta_2 e^a T^2 T_1 + \frac{h_c(k-1)\theta e^a}{(\theta-b)} T_0^2 T_1^2 \right] \end{aligned}$$

$$\begin{aligned}
& -\frac{h_c(k-1)\theta e^a}{(\theta-b)}T_0^3 - 2s_c\delta_2 e^a T T_1^2 + s_c\delta_2 e^a T^2 T_1 + \frac{h_c(k-1)\theta e^a}{(\theta-b)}T_0^2 T_1^2 \Bigg] \\
\tilde{TP}_3(T_0, T_1, T) = & \left(\frac{1}{4T}\right) \left[-o_c + e^a \left\{ k p_c - s_R - \frac{h_c(2b+1)}{b(\theta-b)} \right\} T_0 + e^a \left\{ \frac{s_R(b^2 - b\delta_3 + 1)}{b^2} + \frac{h_c(2b+1)}{b(\theta-b)} - \frac{\delta_3 s_c(1-b)}{b^2} \right. \right. \\
& \left. \left. - \frac{op_c(2\delta_3 - b)}{b} \right\} T_1 + e^a \left\{ \frac{s_R(b\delta_3 - 1)}{b^2} + \frac{s_c\delta_3(1-b)}{b^2} - \frac{op_c b(1-2\delta_3)}{b^2} \right\} T + \frac{h_c(k-1)e^a}{b} T_0^2 - \frac{s_c\delta_3 e^a}{b} T^2 \right. \\
& \left. + e^a \delta_3 \left\{ s_R - p_c + op_c + \frac{s_c}{b} \right\} T_1^2 + e^a \delta_3 \{ p_c - s_R - op_c \} T T_1 - \frac{h_c(k-1)e^a}{b} T_0 T_1 + \frac{s_c\delta_3 e^a}{b} T_1^3 \right. \\
& \left. \left. - \frac{h_c(k-1)\theta e^a}{(\theta-b)} T_0^3 - 2s_c\delta_3 e^a T T_1^2 + s_c\delta_3 e^a T^2 T_1 + \frac{h_c(k-1)\theta e^a}{(\theta-b)} T_0^2 T_1^2 \right] \right]
\end{aligned}$$

Putting the values of $\tilde{TP}_1(T_0, T_1, T)$, $\tilde{TP}_2(T_0, T_1, T)$, and $\tilde{TP}_3(T_0, T_1, T)$ in equation (21), we have

$$\begin{aligned}
\tilde{TP}(T_0, T_1, T) = & \left(\frac{1}{4T}\right) \left[-4o_c + 4e^a \left\{ k p_c - s_R - \frac{h_c(2b+1)}{b(\theta-b)} \right\} T_0 + e^a \left\{ \frac{s_R(b^2 - b(\delta_1 + 2\delta_2 + \delta_3) + 1)}{b^2} + \frac{4h_c(2b+1)}{b(\theta-b)} \right. \right. \\
& \left. \left. - \frac{s_c(1-b)(\delta_1 + 2\delta_2 + \delta_3)}{b^2} - \frac{op_c(2(\delta_1 + 2\delta_2 + \delta_3) - b)}{b} \right\} T_1 + e^a \left\{ \frac{s_R(b(\delta_1 + 2\delta_2 + \delta_3) - 1)}{b^2} \right. \right. \\
& \left. \left. + \frac{s_c(1-b)(\delta_1 + 2\delta_2 + \delta_3)}{b^2} - \frac{op_c(1 - 2(\delta_1 + 2\delta_2 + \delta_3))}{b} \right\} T + 4e^a \left(\frac{h_c(k-1)}{b} \right) T_0^2 \right. \\
& \left. + e^a (\delta_1 + 2\delta_2 + \delta_3) \left(s_R + \frac{s_c}{b} - p_c + op_c \right) T_1^2 - e^a \left(\frac{s_c(\delta_1 + 2\delta_2 + \delta_3)}{b} \right) T^2 \right. \\
& \left. + e^a (\delta_1 + 2\delta_2 + \delta_3) \{ p_c - s_R - op_c \} T T_1 - \frac{4h_c(k-1)e^a}{b} T_0 T_1 + \frac{s_c(\delta_1 + \delta_2 + \delta_3)e^a}{b} T_1^3 \right. \\
& \left. \left. - \frac{4h_c(k-1)\theta e^a}{(\theta-b)} T_0^3 - 2s_c(\delta_1 + 2\delta_2 + \delta_3)e^a T T_1^2 + s_c(\delta_1 + 2\delta_2 + \delta_3)e^a T^2 T_1 + \frac{4h_c(k-1)\theta e^a}{(\theta-b)} T_0^2 T_1^2 \right] \right] \quad (31)
\end{aligned}$$

6. Numerical Example with Analysis :

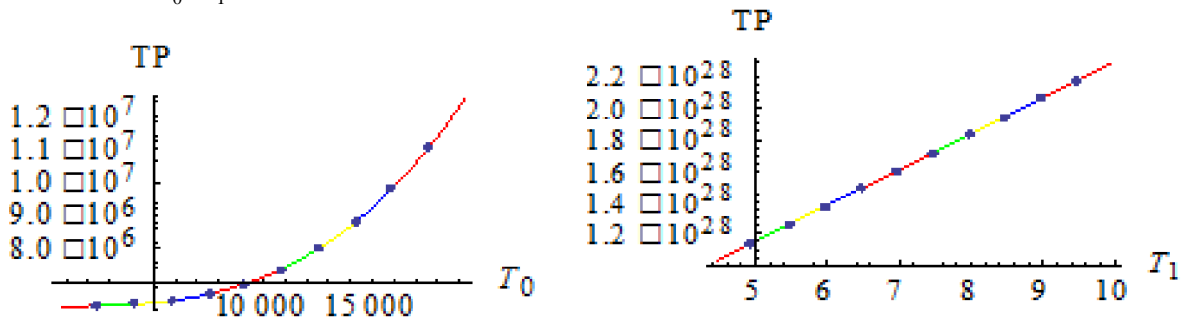
Let us assuming the following numerical data for parameters of the model in appropriate units as follows

$$a = 0.1, \quad b = 3, \quad k = 30, \quad \theta = 0.01, \quad \delta = 0.2, \quad o_c = 20, \quad h_c = 4, \quad s_c = 6, \quad p_c = 8, \quad op_c = 5, \quad s_R = 15$$

Table 1, variation in total profit with respect to parameter θ

θ	T_0	T_1	T	TP
0.01	19267.80	9.9833	489561	1.2529×10^7
0.02	4734.21	7.0482	101251	1.8032×10^6
0.03	2075.43	5.7467	40154.90	5.7645×10^5
0.04	1153.92	4.9708	20803.90	2.5582×10^5
0.05	731.12	4.4415	12484.50	1.3597×10^5

From the table 1, we observe that as we increase the deterioration parameter θ , the total profit is decreased also the values of T_0 , T_1 , T and TP are decreased.

Figure 2, variation in TP with respect to T_0 Figure 3, variation in TP with respect to T_1 Table 2, variation in total profit with respect to parameter δ

δ	T_0	T_1	T	TP
0.2	19267.80	9.9833	489561	1.2529×10^7
0.4	19268.50	9.9833	346191	1.7721×10^7
0.6	19269.00	9.9833	282675	2.1704×10^7
0.8	19269.40	9.9833	244812	2.5062×10^7
1.0	19269.90	9.9833	218974	2.8021×10^7

The table 2, shows as we increase the backlogging parameter δ , the values of T_0 and TP are increased. But the values of T are decreased.

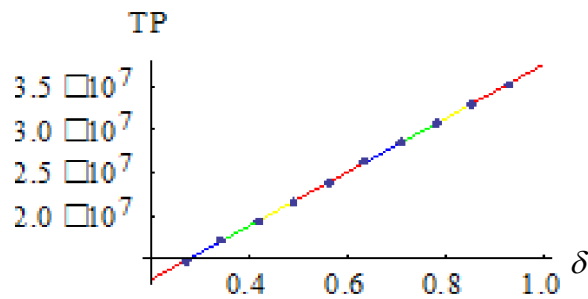
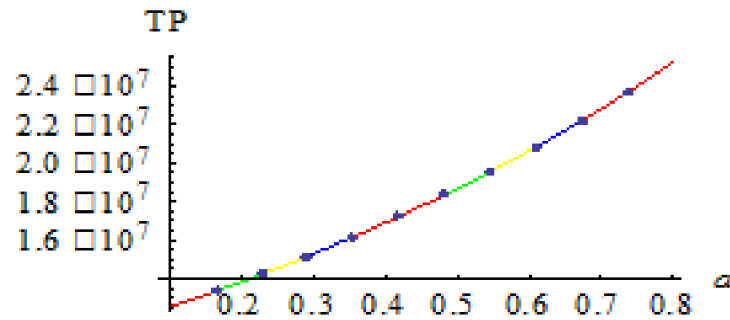
Figure 4, variation in TP with respect to δ

Table 3. variation in total profit with respect to parameter a

a	T_0	T_1	T	TP
0.1	19267.80	9.9833	489561	1.2529×10^7
0.2	19267.30	9.9833	489543	1.3847×10^7
0.4	19268.60	9.9833	489582	1.6915×10^7
0.6	19269.60	9.9833	489609	2.0661×10^7
0.8	19269.80	9.9833	489610	2.5236×10^7

From the table 3, we see that as we increase the demand parameter a , the values of T_0 , T and TP are increased.

Figure 5, variation in TP with respect to a Table 4. variation in total profit with respect to parameter b

b	T_0	T_1	T	TP
3	19267.80	9.9833	489561	1.2529×10^7
6	19638.00	9.9916	352682	9.1910×10^6
9	19764.50	9.9959	289803	7.5979×10^6
12	19827.20	9.9966	251750	6.6199×10^6
15	19879.20	10.0008	225793	5.9501×10^6

The table 4, shows as we increase the demand parameter b , the values of T and TP are decreased and the values of T_0 and T_1 are increased.

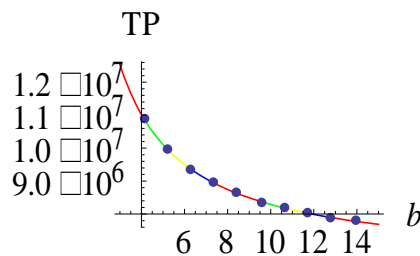
Figure 6. variation in TP with respect to b

Table 5, variation in total profit by signed distance method

$\delta = (\delta_1, \delta_2, \delta_3)$	T_0	T_1	T	TP
(0.001, 0.002, 0.003)	0.5650	6.5525	37.3076	4.0545
(0.002, 0.003, 0.004)	0.5075	6.4931	29.9875	7.0171
(0.003, 0.004, 0.005)	0.4688	6.4523	26.2036	9.2074

The table 5, shows as we increase the backlogging parameter δ , the total profit is increased.

7. Conclusion

In this paper, we have developed a profit maximization fuzzy type backlogging production inventory model for perishable items with time dependent exponential demand rate. The results of proposed model, show that the parameters θ and b are more sensitive than the parameters δ and a . Therefore the wholesaler/retailer can reduced their profit either by satisfying the maximum customers demand or backlogging the maximum number of units of the product for completing the unsatisfied demand of the customers. In future this model can be generalized by considering different assumptions and conditions on inventory costs, demand rate and backlogging rate.

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