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Further Results on Maximal Domination in Graphs

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Abstract

In[11], Kulli and Janakiram initiate the concept of maximal domination in graphs. In this paper, we obtained some bounds and characterizations. Also, we estimate the value of the maximal domination number of some graph products such as join of graphs, corona product, cartesian product and strong product.

Key words : Graph product; Maximal dominating set; Maximal domination number.

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1. Introduction

Let $G = (V, E)$ be a finite and undirected graph with no loops and multiple edges of vertex set V and edge set E. As usual $p = |V|$ and $q = |E|$ denote the number of vertices and edges of a graph G respectively. For a vertex $v \in V(G)$, the open neighbourhood $N(v)$ is the set $\{u \in V : uv \in E\}$ and the closed neighbourhood $N[v]$ is the set $N(v) \cup \{v\}$. For graph theoretic terminologies which are not defined here, we follow[5].

The concept of maximal domination was introduced by Kulli and Janakiram[11] and studied by Kulli and Kattimani[12] in the following sense. A dominating set D of a graph G is a maximal dominating set if $V-D$ is not a dominating set of G. The minimum cardinality of a maximal dominating set of G is the maximal domination number $\gamma_m(G)$ of G. The minimal dominating set D with $|D|=\gamma_m(G)$ is called $\gamma_m(G)$-set of G. For comprehensive details on domination and its related parameters, we refer to [1,6,7,8,9,10,14].

1.1 Existing and preliminary Results

First, we start with couple of observation.

Observation 1.1. For any graph G with $p \geq 3$ vertices,

$$\gamma(G) \leq \gamma_m(G).$$

Further, the bound attains if and only if G has an isolated vertex.

Observation 1.2. For any tree T with vertices,

(i) $p-q-1 \leq \gamma_m(T) \leq \gamma(T)+1.$

(ii) $(p-l-2)/3 \leq \gamma_m(T) \leq p-l+1.$

Where l is the number of pendant vertices of a tree T.

The following results are used in the sequel.

Theorem 1.1.[11] For any graph G with $p \geq 3$ vertices,

$$\delta(G)+1 \leq \gamma_m(G) \leq \gamma(G)+\delta(G).$$

2. Bounds and Characterization :

Theorem 2.1. Let G be a graph with $p \geq 3$ vertices. Then D is a γ_m- set of G if and only if there exists atleast one vertex $v \in V$ such that $N[v] \subset D$.

Proof. Let G be a graph with $p \geq 3$. If D is a γ_m-set of G, then there exists atleast one vertex $v \in D$ is not adjacent to any vertex $u \in V-D$. Therefore D has atleast one element in $v \in V(G)$ such that $N_G[v]$ is contained in D. The converse is obvious.

Theorem 2.2. For any connected graph G with $G \doteqdot K_p$

$$2 \leq \gamma_m(G) \leq p-1.$$

Further, the lower bound attains if and only if the graph G having $d_G(u)=p-1$ with the vertex $u \in V(G)$ is adjacent to a pendant vertex $v \in V(G)$ and an upper bound attains if and only if $G \doteqdot K_p - e$; $e \in E(K_p)$ for $p \geq 3$.

Proof. Let G be connected graph $G \ncong K_p$. By the definition of $\gamma_m(G)$, we have γ_m-set has atleast 2 and atmost $(p-1)$ vertices for $p \geq 3$. Then $2 \leq \gamma_m(G) \leq p-1$ holds.

Now, we prove the lower bound equality of $\gamma_m(G)$.

If connected graph G having $d_G(u) = p-1$ with u is adjacent to a pendant vertex v, then $\{u,v\}$ is a γ_m-set of G. Therefore $\gamma_m(G) = 2$.

Conversely, suppose $\gamma_m(G) = 2$ holds but the graph G is not having $d_G(u) = p-1$ with u is adjacent to pendent vertex v. Then there exist atleast four vertices x, y, u and v such that x is adjacent y, y is adjacent to u, u is adjacent v and x is not adjacent to v. (i.e., $x-v$ path). This implies that $V - \{x\}$ is γ_m-set of a graph G, which is a contradiction. Thus the lower bound of γ_m attained.

Next, we prove an upper bound equality of $\gamma_m(G)$.

On contrary, suppose $\gamma_m(G) = p-1$ holds, but that $G \ncong K_p - e$, with $e \in E(K_p)$, $p \geq 3$. Then G has atleast three vertices such that every pair of vertices are adjacent. This implies that V is a γ_m-set of G. Therefore $\gamma_m(G) = p$ for $p \geq 3$, a contradiction. This prove the necessity. Sufficiency is obvious.

Lemma 2.1. Let D_1 and D_2 be a γ-set and γ_m-set of a graph G, respectively. Then $D_1 \cap D_2 \neq \phi$.

γ

Proof. Let D_1 be a γ-set and D_2 be a γ_m-set of a graph G. Suppose that $D_1 \cap D_2 \neq \phi$, then $D_1 \subset V - D_2$ and hence $V - D_2$ contains a dominating set D_1. Therefore $V - D_2$ itself is a dominating set, which is a contradiction.

Lemma 2.2 Let G be a connected graph with $\gamma(G) = \frac{p}{2}$. Then $\gamma_m(G) = \frac{p}{2} + 1$.

Proof. Let G be a connected graph. We have

Case 1. $\gamma_m(G) < \frac{p}{2}$, then γ-set D satisfies atmost $\frac{p}{2}$. Therefore $V - D$ contain a dominating set of a graph G, which is a contradiction to the fact of γ_m-set of a graph G.

Case 2. $\gamma_m(G) > \frac{p}{2}$, then γ-set D satisfies atmost p and the graph G has only isolated vertices. Therefore $V - D$ not contain a dominating set of G. This leads to the concept of γ_m-set of a graph G.

Thus the result follows.

Theorem 2.2. [3,16] Let G be a graph with even order and no isolated vertices. Then $\gamma_m(G) = \frac{p}{2}$ if and only if the components of the graph $G \cong H \circ K_1$ or C_4, where H is any connected graph.

Theorem 2.4. Let G be a graph with $p \geq 3$ vertices. Then $\gamma_m(G) = \frac{p}{2}+1$ if and only if the components of the graph $G \cong H \circ K_1$ or C_4, where H is any connected graph.

Proof. Suppose $\gamma_m(G) = \frac{p}{2}+1$ holds on the contrary $G \not\cong H \circ K_1$ or C_4. Then there exist the graph G has atleast three vertices u_1, u_2 and u_3 such that u_1 is adjacent to u_2, u_2 is adjacent to u_3 and u_1 is not adjacent to u_3. This implies that $V - u_1$ is a γ_m-set of a graph G Therefore $\gamma_m(G) < \frac{p}{2}+1$ which is a contradiction. Further, u_2 and u_3 are adjacent to the new vertex u_3, which is not adjacent to u_1. This implies that $V - \{u_3, u_4\}$ is γ_m-set of a graph $G + u_4$. Therefore $\gamma_m(G) < \frac{p}{2}+1$, again a contradiction. This proves necessity.
Sufficiency follows from lemma 2.1, lemma 2.2 and theorem 2.3.

To prove our next result, we make use of the following definition.

A subset $D \subset V(G)$ is an independent dominating set of a graph G if $\langle D \rangle$ is totally disconnected, see[2].

Lemma 2.3. Let G be a connected graph with $p \geq 3$ vertices. If D is a γ_m-set of G then $\langle D \rangle$ contains at least one edge.

Proof. Let G be a connected graph with $p \geq 3$ vertices. If D is a γ_m-set of G and $\langle D \rangle$ is an independent set. Then $V - D$ is a dominating set of G. Therefore D is not a γ_m-set of G, which is a contradiction to our assumption.
Thus the result follows.
By lemma 2.3, we arrive at

Theorem 2.5. Let G be a connected graph with $p \geq 3$ vertices. If D is a γ_m-set of G then $\langle D \rangle$ is not an independent dominating set.

Theorem 2.6. Let $G_1, G_2, G_3, \ldots, G_k$ are the connected components of a graph G. Then

$$\gamma_m(G) = \gamma_m(G_s) + \sum_{j=1\, j \neq s}^{k} \gamma_m(G_j),$$

where $\gamma_m(G_s)=\min\{\gamma_m(G_1),\gamma_m(G_2),...,\gamma_m(G_{i-1}),\gamma_m(G_{i+1}),...\gamma_m(G_k)\}$.

Proof. Let D_i and $D_i^{'}$ are γ -set and γ_m-set of G_i for $1\le i\le k$ respectively. Let. $D_s=\min\{D_1^{'},D_2^{'},D_3^{'},...,D_k^{'}\}$. Then $D_s^{'}\cup\{D_1,D_2,D_3,...,D_{s-1},D_{s+1},...,D_k\}$ is maximal dominating set of G. Therefore $\gamma_m(G)\le\gamma_m(G_s)+\sum_{j=1\, j\neq s}^{k}\gamma_m(G_j)$. Also $D_i^{'}\cup\{D_1,D_2,D_3,...,D_{i-1},D_{i+1},...,D_k\}$ for some $1\le i\le k$ is a γ_m-set. Therefore $\gamma_m(G)\ge\gamma_m(G_s)+\sum_{j=1\, j\neq s}^{k}\gamma_m(G_j)$ and hence the theorem.

Corollary 2.1. Let $G_1,G_2,G_3,...,G_k$ are the connected components of a graph *G*. Then $\gamma_m(G)=\sum_{j=1}^{k}\gamma(G_j)$ if and only if $\gamma(G_i)=\gamma_m(G_i)$ for atleast some $1\le i\le k$.

Observation 2.1. The graph $K_p+\{v\}$ is the graph obtained by adding an edge uv to the complete graph K_p for some $u\in V(K_p)$ and $v\notin V(K_p)$. Clearly $\gamma_m(K_p+\{v\})\le\gamma_m(K_p)$. In general, the connected graph G without triangle, we have $\gamma_m(G)\le\gamma_m(G+v)$.

By Theorem 2.2, we have the relations of Nordhaus-Gaddum type for maximal domination in graphs. For more details, we refer to [15].

Theorem 2.7. Let G and $\overline{G}$ be two connected graphs with $p\ge 4$. Then

(i) $4\le\gamma_m(G)+\gamma_m(\overline{G})\le 2p-2,$

(ii) $4\le\gamma_m(G)\gamma_m(\overline{G})\le(p-1)^2.$

3. Some Specific Families of Graph Products :

3.1 Join of graphs :

The join of two graphs G_1 and G_2 is the graph G_1+G_2, with the vertex set $V(G_1+G_2)=V(G_1)\cup V(G_2)$ and the edge set $E(G_1+G_2)=E(G_1)\cup E(G_2)\cup\{uv:u\in V(G_1),v\in V(G_2)\}$, The join of graphs was defined by Zykoc[20].

Theorem 3.1. Let G_1 and G_2 be two non-trivial graphs of order p_1 and p_2, Then

$$\gamma_m(G_1+G_2)=\min\{p_1+\delta(G_2)+1,p_2+\delta(G_1)+1\}.$$

Proof: Let $V(G_1)=\{u_1,u_2,u_3.....u_{p_1}\}$ and $V(G_2)=\{v_1,v_2,v_3......v_{p_2}\}$. Let D_1 and D_2 are dominating set of G_1 and G_2 respectively. Then D_1 as well as D_2 is dominating set of D_1+D_2 Therefore $D_1 \cup N_{(G_1+G_2)}(v_i)$ as well as $D_2 \cup N_{(G_1+G_2)}(u_j)$ for some $u_j \in V(G_1)$ and $v_i \in V(G_2)$ is maximal dominating set of G_1+G_2. But $N_{(G_1+G_2)}(v_i)=V(G_1)\cup N_{G_2}[v_i]$ and hence $D_1 \cup V(G_1)\cup N_{G_2}[v_i]=V(G_1)\cup N_{G_2}[v_i]$ is the maximal dominating set of G_1+G_2. Similarly $D_2 \cup V(G_2)\cup N_{G_1}[u_j]=V(G_2)\cup N_{G_1}[u_j]$ is also the maximal dominating set of G_1+G_2. Hence the result is proved.

3.2 Corona product :

The corona product of two graphs G_1 and G_2, denoted by G_1oG_2, is the graph obtained by taking one copy of G_1 of order m and n copies of G_2, and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2. It was introduced by Fruncht and Harary[4].

Theorem 3.2. Let G_1 and G_2 be two connected graph of order p_1 and p_2 respectively. Then

$$\gamma_m(G_1oG_2)=p_1+\delta(G_2)+1$$

Proof : Let G_2^v be the copy of G_2 whose vertices are joined or attached to $v \in V(G_1)$, and $v+G_2^v$ is the subgraph of the corona G_1oG_2 corresponding to the join $\langle\{v\}\rangle+G_2^v$. Let D be the γ_m-set of $G_1 \circ G_2$. Then $D \cap V(v+G_2^v)=\{v\}$ for every $v \in V(G_1)$ and $D \cap V(v+G_2^v)=N_{G_1\circ G_2}[u]$ for atleast one $v \in V(G_1)$ and $u \in V(G_2)$. Therefore $V(G_1)\bigcup N_{G_1oG_2}[u]=V(G_1)\bigcup N_{G_2}[u]$ for some $u \in V(G_2)$ is the maximal domination set of $G_1 \circ G_2$. and hence $\gamma(G_1oG_2)=p_1+\delta(G_2)+1$.

3.3 Complementary prisms :

The complementary prism graph of G, denoted by $G\overline{G}$, is the graph formed from the disjoint union of G and $\overline{G}$, the complement of G, by adding the edges of perfect matching between the corresponding vertices of G and $\overline{G}$. For more details, we refer to[6].

Theorem 3.3. [6] For any non-trivial graph G, $\max\{\gamma(G),\gamma(\overline{G})\}\le\gamma(G\overline{G})\le\gamma(G)+\gamma(\overline{G})$.

Theorem 3.4. For any non-trivial graph G,

$$\gamma_m(G\overline{G})\le\min\{\gamma_m(G)+\gamma(\overline{G}),\gamma(G)+\gamma_m(\overline{G})\}+1.$$

Proof : From the definition of complementary prism graph, $G\overline{G}$ is connected graph with 2p - vertices, As $G\overline{G}$ is connected graph.by theorem 1.1 and theorem3.3, we have

$$\gamma_m(G\overline{G}) \le \gamma(G) + \gamma(\overline{G}) + \delta(G\overline{G}) = \gamma(G) + \gamma(\overline{G}) + \min\{\delta(G), \delta(\overline{G})\} + 1$$

$$\le \min\{\gamma_m(G) + \gamma\ (\overline{G}), \gamma_m(\overline{G}) + \gamma\ (G)\} + 1\ .$$

Hence the theorem is proved.

3.4. Cartesian product :

The Cartesian product of two graphs G and H is a graph, denoted as $G\square H$, whose vertex set is $V(G)\times V(H)$. Two vertices (g,h) and (g',h') are adjacent precisely if $g = g'$ and $hh' \in E(H)$ or $h = h'$ and $gg' \in E(G)$. For more details we refer to[17].

Theorem 3.5. Let G_1 and G_2 be two arbitrary graphs of order p_1 and p_2 respectively. Then

$$\delta(G_1\square G_2) = \delta(G_1) + \delta(G_2).$$

Proof. Let $u \in V(G_1)$ and $v \in V(G_2)$ such that $d_{G_1}(u) = \delta(G_1)$ and $d_{G_2}(u) = \delta(G_2)$. Then $d_{(G_1\square G_2)}(u,v) = d_{G_1}(u) + d_{G_2}(v) = \delta(G_1) + \delta(G_2) \le d_{G_1}(u') + d_{G_2}(v')$ for any $(u',v') \in (G_1\square G_2)$. Therefore $\delta(G_1\square G_2) = \delta(G_1) + \delta(G_2)$.

Theorem 3.6.[19] Let G_1 and G_2 be two arbitrary graphs of order p_1 and p_2 respectively. Then

$$\min\{\mathrm{p}_1, p_2\} \le \gamma(G_1\square G_2) \le \min\{\mathrm{p}_2\gamma(G_1), p_1\gamma(G_2)\}.$$

Theorem 3.7. Let G_1 and G_2 be two arbitrary graphs of order p_1 and p_2 respectively. Then

$$\min\{\mathrm{p}_1, p_2\} + 1 \le \gamma_m(G_1\square G_2) \le \min\{\mathrm{p}_2\gamma(G_1), p_1\gamma(G_2)\} + \delta(G_1) + \delta(G_2).$$

Proof. Proof follows from Theorem 1.1, 3.5 and 3.6.

Theorem 3.8. Let G be a connected graph with $G \doteqdot K_p$ or $\overline{K_p}$. Then

$$\gamma_m(G\square\overline{\mathrm{G}}) \le p\min\{\gamma(G), \gamma(\overline{\mathrm{G}})\} + \delta(G) + \delta(\overline{\mathrm{G}}).$$

Proof. Let G be a connected graph with $G \doteqdot K_p$ or $\overline{K_p}$. By Theorem 3.7, we have

$$\gamma_m(G\square\overline{\mathrm{G}}) \le \min\left\{|V(G)|\gamma(\overline{\mathrm{G}}), |V(\overline{\mathrm{G}})|\gamma(\mathrm{G})\right\} + \delta(G\square\overline{\mathrm{G}}).$$

$$= \min\left\{p\gamma(\overline{\mathrm{G}}), p\gamma(\mathrm{G})\right\} + \delta(G) + \delta(\overline{\mathrm{G}})$$

$$= p\min\left\{\gamma(\overline{\mathrm{G}}), \gamma(\mathrm{G})\right\} + \delta(G) + \delta(\overline{\mathrm{G}})$$

Hence the proof.

Theorem 3.9. Let G be a connected graph. Then

$$\gamma_m(G\square\mathrm{K}_2) \le 2\gamma(G) + \delta(G) + 2.$$

Further, an equality holds if the graph G is isomorphic with star $K_{1,p-1}$.

Proof. Let $\{v_1, v_2\}$ be vertices of K_2. Suppose D_1 and D_2 are the γ-sets of $G \times v_1$ and $G \times v_2$ respectively. Then $D_1 \cup D_2 \cup N_{G\square K_2}(u)$ for some $u \in V(G\square K_2)$ is the γ_m-set of $G\square K_2$ and hence the proof.

3.5 Strong Product :

The strong product of G and H is a graph, denoted as $G \boxtimes H$, whose vertex set is $V(G) \times V(H)$. Two vertices (g, h) and (g', h') are adjacent precisely if $g = g'$ and $hh' \in E(H)$ or $h = h'$ and $gg' \in E(G)$ or $hh' \in E(H)$ and $gg' \in E(G)$ For more details, we refer to[17].

Lemma 3.2 Let G_1 and G_2 be two connected graph. Then

$$d_{G_1 \boxtimes G_2}(u,v) = d_{G_1}(u) + d_{G_2}(v) + d_{G_1}(u) d_{G_2}(v).$$

Theorem 3.10.[13] Let G_1 and G_2 be two connected graph. Then

$$\gamma(G_1 \boxtimes G_2) \le \min\{\gamma(G_1), \gamma(G_2)\}.$$

Theorem 3.11. Let G_1 and G_2 be two connected graph. Then

$$\gamma_m(G_1 \boxtimes G_2) \le \gamma(G_1)\gamma(G_2) + \delta_{G_1}(u) + \delta_{G_2}(v) + \delta_{G_1}(u)\delta_{G_2}(v)$$

Further the bounds are sharp.

Proof : Let G_1 and G_2 be two connected graph. Then by Lemma 3.2, Theorem 1.1 and 3.10, we have

$$\begin{aligned}
\gamma_m(G_1 \boxtimes G_2) &\le \gamma(G_1 \boxtimes G_2) + \delta(G_1 \boxtimes G_2) \\
&\le \gamma(G_1)\gamma(G_2) + \delta(G_1 \boxtimes G_2) \\
&= \gamma(G_1)\gamma(G_2) + \delta(G_1) + \delta(G_2) + \delta(G_1)\delta(G_2).
\end{aligned}$$

An equality holds if and only if $\gamma_m(G_1) = \gamma(G_1) + \delta(G_1)$ and $\gamma_m(G_2) = \gamma(G_2) + \delta(G_2)$.

5. Conclusions

In this article, we consider the possible form of restriction in complement of dominating set D (*i.e.*, outside the dominating set D or V-D). Many of these parameters could have been placed into more than one of the domination-related parameters, such as maximal domination in graphs. Here we obtain further results on this parameters. Also, yet to settle many open problems on this parameters such as critical and stability conditions. Finding the results towards some graph operations/ transformation graph/ product graphs/derived graphs are yet to settle.

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