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Extending the Aryabhatan algorithm for the extraction of higher order roots

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Abstract

In schools, children are taught algorithm for extracting the square and cube roots of a number. However, they are not informed that Aryabhata (b. 476 CE) was the first Indian mathematician to present a systematic algorithm for the extraction of the square and cube root of a number. This algorithm has been succinctly presented by Aryabhata in two successive verses in the Ganitapada of his Aryabhatiyam composed in 499 CE.

A careful study of the algorithm given by Aryabhata shows that it can be easily extended to find higher order roots. It also clearly reveals that the decimal place value system should have been in vogue in India for quite a long time before him, because this algorithm is heavily based on a thorough understanding of the decimal place value system. In our paper, we will discuss some of these aspects as well as outline how his algorithm can be extended to find the higher order roots.

Key Words : Root of a Number, Digits, Algorithm, Decimal System)

Introduction

An explicit algorithm for finding the cube root of a number is given for the first time in verse 5 of the section on Gaṇita in Âryabhamṭīya [c. 499 CE] [Shukla and Sarma[1]]

Aghanād bhajet dvitīyāt triguṇena ghanasya mūlavargeṇa |

vargastripūrvaguṇitah ṣodhyah prathamād ghanaṣca ghanāt ||
[Having subtracted the greatest possible cube from the last cube place and then having written down the cube root of the number subtracted in the line of the cube root], divide the second non-cube place [standing on the right of the last cube place] by thrice the square of the cube root [already obtained]; [then] subtract from the first non-cube place [standing on the right of the second non-cube place] the square of the quotient multiplied by thrice the previous [cube root]; and [then subtract] the cube [of the quotient] from the cube place [standing on the right of the first non-cube place] [and write down the quotient on the right of the previous cube root in the line of the cube root, and treat this as the new cube root. Repeat the process if there are still digits on the right].
[Translation by Shukla and Sarma[1]]

In order to explain and apply the Aryabhatan algorithm for extracting the cube root of a number, we adopt the following terminology and procedure:

First of all, we consider the given number in groups of three digits starting from the right.

To make the matter simpler to understand as well as to explain, we quite often name the digits in the groups in the following manner;

$$\overbrace{12}^{b_1 b_0}\ \overbrace{167}^{b_2 b_1 b_0},$$

We call, following[2], the units place the cube place, that is the"ghana" place, the 10-place the first non-cube place, "the first aghana", the 10^2-place the second non-cube place, the "second aghana".

Therefore, b_0 denotes the cube place, b_1 the first non-cube place and b_2 the second non-cube place. This process is continued until all the digits are exhausted. (We have arbitrarily chosen the number 12167). In this example, the digit 2 is at the last cube place, that is the b_0 place. Following the steps as described in the Aryabhatiya, we easily find the cube root to be 23 of the number 12167.

Extending the algorithm for higher order roots :

This algorithm can be extended to finding the fourth, fifth and other higher order root of a number. Let us explain the procedure with the help of an example. Let us take the number $N = 33243864241$ in order to find $\sqrt[4]{N}$. First of all, we consider the given number in groups of four, say $b_3b_2b_1b_0$ from the right and express it as

$$\overbrace{332}^{b_2 b_1 b_0}\ \overbrace{4386}^{b_3 b_2 b_1 b_0}\ \overbrace{4241}^{b_3 b_2 b_1 b_0}.$$

As a next step, we follow the following procedure-

1. Subtract the greatest possible fourth power of a number, say f_1, from the last b_0 place starting from right. In the present case, it simply means $(f_1)^4 \leq 332 < (f_1+1)^4$.
2. Divide the b_3 place by four times the cube of the fourth order root already obtained in the step

1, i.e., by $4f_1{}^3$. Let the quotient be q_1.

3. Subtract from the b_2 place six times the product of the squares of f_1 and the quotient q_1, i.e., $6f_1{}^2q_1{}^2$.
4. Subtract from b_1 place four times the product of f_1 and the cube of the quotient b_1, i.e., $4f_1q_1{}^3$.
5. Subtract from the next b_0 place, lying on the right to the b_1 place, fourth power of the quotient, i.e., $q_1{}^4$.
6. If the remainder is zero, and no digits remain, we say that the number, following[3], $f_1q_1 = 10f_1 + q_1$ is the required fourth order root of the given number.
7. Otherwise, following[4], repeat the process until all the digits are exhausted.

Let us now follow the steps mentioned above for the number $N = 33243864241$ in order to extract its fourth order root.

$$\overbrace{332}^{b_2b_1b_0}\quad \overbrace{4386}^{b_3b_2b_1b_0}\quad \overbrace{4241}^{b_3b_2b_1b_0}$$

$$\underline{-256} \qquad\qquad\qquad f_1 = 4$$

$$= 76$$

Here, we have taken $f_1 = 4$, since $4^4 < 332 < 5^4$. Now we bring down the first number $b_3 = 4$ of the next group of numbers and divide by 4.4^3 and note that the quotient is 2. We write $q_1 = 2$. In the subsequent steps, we bring down $b_2, b_1,\ and\ b_0$ of this group successively and divide the resulting numbers by $6.\,4^2.\,2^2$, $4.4.\,2^3$ and 2^4

At this step, if the remainder comes out to be zero, then we conclude that the fourth order root is 42. But we see that the remainder is not zero, therefore, we bring down the next digit and continue the process. Thus, we bring down the number 4 and divide the resulting number by $4(42)^3$ and note that the quotient is 7. We write $q_2 = 7$. In the next three steps, we bring down successively 2,4 and 1 and divide the resulting numbers by $6.\,(42)^2.\,7^2$, $4.\,(42).\,7^3$, and 7^4 respectively and the remainder becomes 0.

Thus the fourth order root of the number is = 427, which may be written as

$$10^2 f_1 + 10q_1 + q_2.$$

In order to generalise the procedure, and apply it for any higher order root, let us consider a number given by

$$N = a_0.\,10^m + a_1.\,10^{m-1} + a_2.\,10^{m-2} \dots\dots\dots + a_{m-1}.\,10 + a_m$$

where, $0 < a_0 < 10^n$ and $0 \le a_1, a_2, \dots\dots a_m \le 9$ are integers. We are interested in finding the nth order root of the number ($m \ge n$).

Then as a first step, we consider the number in groups of n digits, say $b_{n-1} \dots\dots b_3b_2b_1b_0$, from right. Then we follow the following steps in order to get the desired root:

1. Subtract the greatest possible n^{th} power of a number, say f_1 from the last b_0 place.
2. Divide the b_{n-1} place, which lies on the right to the last b_0 place, by n times the $(n-1)^{th}$ power of f_1, obtained in the step 1, *i.e.*, by $\binom{n}{1} f_1^{(n-1)}$. Let the quotient be denoted by q_1.
3. Subtract from the b_{n-2} place $\frac{n(n-1)}{2}$ times the product of the $(n-2)^{th}$ power of f_1 and the square of the quotient q_1, obtained in the preceding step, i.e., $\binom{n}{2} f_1^{(n-2)} q_1^{2}$.
4. Subtract from the b_{n-3} place $\left(\frac{n(n-1)(n-2)}{6}\right)$ times the product of the $(n-3)^{th}$ power of f_1 and the cube of the quotient q_1, obtained in the second step, i.e., $\binom{n}{3} f_1^{(n-3)} q_1^{3}$.
5. Continuing in this manner, we successively subtract $\binom{n}{4} f_1^{(n-4)} q_1^{4}$, $\binom{n}{5} f_1^{(n-5)} q_1^{5}$,... ...., $\binom{n}{n-2} f_1^{2} q_1^{(n-2)}$ and $\binom{n}{n-1} f_1\, q_1^{(n-1)}$ from the b_{n-4}, b_{n-5} ,....., b_2 and the b_1 places, respectively.
6. Subtract from the next b_0 place, lying on the right to the b_1 of the preceding step the n^{th} power of the quotient, i.e., q_1^{n}.
7. If the remainder is zero, and no digits remain, we say that the number
$$f_1 q_1 = 10 f_1 + q_1$$
is the required n^{th} order root of the given number.
8. Otherwise, repeat the process. The repetition of the process starts from the step 2. We divide the b_{n-1} place, which lies on the right to the b_0 place of the step 6, by n times the $(n-1)^{th}$ power of $f_1 q_1 (= 10 f_1 + q_1)$, i.e., by $\binom{n}{1}(10 f_1 + q_1)^{(n-1)}$. We denote the quotient by q_2. After this, we follow the steps 3-6 in order. If the remainder is zero, and no digits remain, we say that the number
$$f_1 q_1 q_2 = 10^2 f_1 + 10 q_1 + q_2$$
is the required n^{th} order root of the given number.
9. We repeat the process until all the digits are exhausted.

Conclusion

Thus, we see that the Aryabhatan algorithm for the square and cube roots can be extended to any higher order root of a positive integer. This is possible on a proper understanding of the decimal place value system. Zero was invented in India about the beginning of the Christian era to help the writing of numbers in the decimal scale[5]. In this manner, we have been able to add a little to what we have received from our ancient *scientific knowledge systems*.

Scope for Future Work and Applications :

In the ancient Indian mathematics, mathematical propositions were stated in the form of

Sutras and *Shlokas.* For their proper understanding, elaborate proofs with examples must be provided to our younger generation. Therefore, the propositions stated in *Vedic Mathematics, Jain and Buddhist texts, Brahma Gupta, Bhaskara II, Madhava, Narayan Pandit and Nilakantha* etc dealing with arithmetic, algebra and geometry etc may be elaborated upon in modern mathematical language with examples[6,7]. It is worth mentioning that *Nilakantha* wrote a commentary on the *Aryabhatiya*[8]. It is need of the hour to include the aspects of Indian Knowledge System in the curricula at various levels of studies as its inclusion and applications has much potential in promoting interdisciplinary researches because all of the systematised disciplines of knowledge in India were developed to a higher degree of sophistication in synchronization with each other from ancient times. There are purely mathematical problems which can be solved using intuitions coming from physics[9]. The modern aspects of this interaction are called mathematical physics. There are several examples of such fruitful synergies between the subjects.

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