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On generating functions of modified Gegenbauer polynomials

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Abstract

In this note, we have obtained some novel bilateral generating functions involving modified Gegenbauer polynomials, $C_n^{\lambda+n}(x)$ which is converted into trilateral generating functions with Tchebycheff polynomials by group theoretic method.

Key words: Gegenbauer polynomials, generating functions.

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1. Introduction

The Gegenbauer polynomial, $C_n^\lambda(x)$ is defined by[2]

$$C_n^\lambda(x) = \sum_{p=0}^{\left[\frac{n}{2}\right]} (-1)^p \frac{(\lambda)_{n-p}(2x)^{n-2p}}{p!\,(n-2p)!},$$

where $(a)_n$ is the pochhammer symbol[3].

The aim at presenting this paper is to obtain the trilateral generating functions for the modified Gegenbauer polynomials, $C_n^{\lambda+n}(x)$ with Tchebycheff polynomials by the group-theoretic method. At first we shall obtain the following theorem on bilateral generating functions.

Theorem 1. If there exists a unilateral generating relation of the form[1]

$$G(x,w) = \sum_{n=0}^{\infty} a_n C_n^{\lambda+n}(x)\, w^n \tag{1.1}$$

then

$$\frac{(1-w)^{\lambda+\frac{1}{2}}}{\{1-w+wx^2\}^{\lambda}} G\left(\frac{x}{\{1-w+wx^2\}^{\frac{1}{2}}}, \frac{wv(1-w)}{\{1-w+wx^2\}^{\frac{3}{2}}}\right)$$

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$$= \sum_{n=0}^{\infty} w^n \sigma_n(x, v), \qquad (1.2)$$

where

$$\sigma_n(x, v) = \sum_{k=0}^{n} a_k \frac{\left(\frac{k+1}{2}\right)_{n-k} \left(\frac{k+2}{2}\right)_{n-k}}{k!\,(1-\lambda-k)_{n-k}} C_{2n-k}^{\lambda-n+2k}(x)\, v^k.$$

2. Operator and extended form of the group:

Now we consider following linear partial differential operator[4]:

$$R = x(1-x^2)\frac{y^2}{z^3}\frac{\partial}{\partial x} + (1-3x^2)\frac{y^3}{z^3}\frac{\partial}{\partial y} - \frac{2x^2y^2}{z^2}\frac{\partial}{\partial z} + \frac{y^2}{z^3}$$

such that

$$R\left(C_n^{\lambda+n}(x) y^n z^\lambda\right) = \frac{(n+1)(n+2)}{2(1-\lambda-n)} C_{n+2}^{\lambda+n-1}(x) y^{n+2} z^{\lambda-3}.$$

The extended form of the group generated by R is given by

$$e^{wR} f(x, y, z) = \left\{1 - 2w\frac{y^2}{z^3}\right\}^{\frac{1}{2}} \times f(X, Y, Z), \qquad (2.2)$$

$$\times f\left(\frac{x}{\left\{1-2w(1-x^2)\frac{y^2}{z^3}\right\}^{\frac{1}{2}}}, \frac{y\left(1-2w\frac{y^2}{z^3}\right)}{\left\{1-2w(1-x^2)\frac{y^2}{z^3}\right\}^{\frac{3}{2}}}, \frac{z\left(1-2w\frac{y^2}{z^3}\right)}{\left\{1-2w(1-x^2)\frac{y^2}{z^3}\right\}}\right), \qquad (2.2)$$

where $f(x,y,z)$ is an arbitrary function and w is an arbitrary constant.

3. Derivation of generating function:

Now writing $f(x, y, z) = C_n^{\lambda+n}(x) y^n z^\lambda$ in (2.2), we get

$$e^{wR}\left(C_n^{\lambda+n}(x) y^n z^\lambda\right) = \left\{1 - 2w(1-x^2)\frac{y^2}{z^3}\right\}^{-\left(\frac{3n}{2}+\lambda\right)}$$

$$\left(1 - 2w\frac{y^2}{z^3}\right)^{n+\lambda+\frac{1}{2}} y^n z^\lambda \times C_n^{\lambda+n}\left(\frac{x}{\left\{1-2w(1-x^2)\frac{y^2}{z^3}\right\}^{\frac{1}{2}}}\right) \qquad (3.1)$$

Again, on the other hand, with the help of (2.1) we have

$$e^{wR}\left(C_n^{\lambda+n}(x) y^n z^\lambda\right) = \sum_{k=0}^{\infty} \frac{(2w)^k}{k!} \frac{\left(\frac{n+1}{2}\right)_k \left(\frac{n+2}{2}\right)_k}{(1-\lambda-n)_k} C_{n+2k}^{\lambda+n-k}(x) \times y^{n+2k} z^{\lambda-3k}. \qquad (3.2)$$

Equating (3.1) and (3.2) and then substituting $\frac{2w\,y^2}{z^3} = t$, we get

$$\{1 - t(1-x^2)\}^{-\left(\frac{3n}{2}+\lambda\right)}$$

$$(1-t)^{n+\lambda+\frac{1}{2}} C_n^{\lambda+n}\left(\frac{x}{\{1-t(1-x^2)\}^{\frac{1}{2}}}\right)$$

$$= \sum_{k=0}^{\infty} \frac{\left(\frac{n+1}{2}\right)_k \left(\frac{n+2}{2}\right)_k}{(1-\lambda-n)_k} C_{n+2k}^{\lambda+n-k}(x) \frac{t^k}{k!}, \qquad (3.2)$$

which is found derived[4].

Now we proceed to prove the Theorem 1.

4. Proof of theorem 1 :

Let us consider the generating relation of the form:

$$G(x, w) = \sum_{n=0}^{\infty} a_n C_n^{\lambda+n}(x)\, w^n. \qquad (4.1)$$

Replacing w by wvy and multiplying both sides of (4.1) by z^{λ} and finally operating e^{wR} on both sides, we get

$$e^{wR}\left(z^{\lambda}\,G(x,wvy)\right)$$

$$= e^{wR}\left(\sum_{n=0}^{\infty} a_n\left(C_n^{\lambda+n}(x)y^n z^{\lambda}\right)(wv)^n\right). \quad (4.2)$$

Now the left member of (4.2), with the help of (2.2), reduces to

$$\left\{1-2w(1-x^2)\frac{y^2}{z^3}\right\}^{-\lambda}\left(1-2w\frac{y^2}{z^3}\right)^{\lambda+\frac{1}{2}} z^{\lambda}$$

$$\times G\left(\frac{x}{\left\{1-2w(1-x^2)\frac{y^2}{z^3}\right\}^{\frac{1}{2}}}, \frac{wvy\left(1-2w\frac{y^2}{z^3}\right)}{\left\{1-2w(1-x^2)\frac{y^2}{z^3}\right\}^{\frac{3}{2}}}\right). \quad (4.3)$$

The right member of (4.2), with the help of (2.1), becomes

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_{n-k}\frac{(2w)^k}{k\,!}\frac{\left(\frac{n-k+1}{2}\right)_k\left(\frac{n-k+2}{2}\right)_k}{(1-\lambda-n+k)_k}$$

$$\times C_{n+k}^{\lambda+n-2k}(x)\,y^{n+k}z^{\lambda-3k}(wv)^{n-k}. \quad (4.4)$$

Now equating (4.3) and (4.4) and then substituting $y = z = 1, 2w = w$ and $\frac{v}{2} = v$, we get

$$\frac{(1-w)^{\lambda+\frac{1}{2}}}{\{1-w+wx^2\}^{\lambda}} G\left(\frac{x}{\{1-w+wx^2\}^{\frac{1}{2}}}, \frac{wv(1-w)}{\{1-w+wx^2\}^{\frac{3}{2}}}\right)$$

$$= \sum_{n=0}^{\infty} w^n \sigma_n(x,v), \quad (4.5)$$

where

$$\sigma_n(x,v) = \sum_{k=0}^{n} a_k \frac{\left(\frac{k+1}{2}\right)_{n-k}\left(\frac{k+2}{2}\right)_{n-k}}{k\,!\,(1-\lambda-k)_{n-k}} C_{2n-k}^{\lambda-n+2k}(x)\,v^k,$$

which is found derived[4] by classical method.

5. Trilateral generating functions of biorthogonal polynomials :

In this Section the above bilateral generating function has been converted into trilateral generating relation with Thcebycheff polynomial by means of the relation

$$T_n(x) = \frac{1}{2}\left[\left(x+\sqrt{x^2-1}\right)^n + \left(x-\sqrt{x^2-1}\right)^n\right],$$

utilizing the method of Chongdar and Chatterjea[1].

Now to convert the above bilateral generating relation into a trilateral generating relation with Tchebycheff polynomial as done[1], we notice that

$$\sum_{n=0}^{\infty} w^n\,\sigma_n(x,v)T_n(u)$$

$$= \frac{1}{2}\left[\frac{(1-\rho_1)^{\lambda+\frac{1}{2}}}{\{1-\rho_1+\rho_1x^2\}^{\lambda}} G\left(\frac{x}{\{1-\rho_1+\rho_1x^2\}^{\frac{1}{2}}}, \frac{\rho_1 v(1-\rho_1)}{\{1-\rho_1+\rho_1x^2\}^{\frac{3}{2}}}\right)\right.$$

$$\left.+\frac{(1-\rho_2)^{\lambda+\frac{1}{2}}}{\{1-\rho_2+\rho_2x^2\}^{\lambda}} G\left(\frac{x}{\{1-\rho_2+\rho_2x^2\}^{\frac{1}{2}}}, \frac{\rho_2 v(1-\rho_2)}{\{1-\rho_2+\rho_2x^2\}^{\frac{3}{2}}}\right)\right],$$

where $\rho_1 = w\left(u+\sqrt{u^2-1}\right)$ and $\rho_2 = w\left(u-\sqrt{u^2-1}\right)$

Thus we have the following general theorem:

Theorem 2: If there exists a generating relation of the form

$$G(x,w) = \sum_{n=0}^{\infty} a_n C_n^{\lambda+n}(x)\,w^n, \quad (5.1)$$

then

$$\sum_{n=0}^{\infty} w^n \sigma_n(x,v)T_n(u)$$

$$=\frac{1}{2}\left[\frac{(1-\rho_1)^{\lambda+\frac{1}{2}}}{\{1-\rho_1+\rho_1x^2\}^{\lambda}}G\left(\frac{x}{\{1-\rho_1+\rho_1x^2\}^{\frac{1}{2}}},\frac{\rho_1v(1-\rho_1)}{\{1-\rho_1+\rho_1x^2\}^{\frac{3}{2}}}\right)\right.$$

$$\left.+\frac{(1-\rho_2)^{\lambda+\frac{1}{2}}}{\{1-\rho_2+\rho_2x^2\}^{\lambda}}G\left(\frac{x}{\{1-\rho_2+\rho_2x^2\}^{\frac{1}{2}}},\frac{\rho_2v(1-\rho_2)}{\{1-\rho_2+\rho_2x^2\}^{\frac{3}{2}}}\right)\right], \quad (5.2)$$

where

$$\sigma_n(x,t)=\sum_{k=0}^{n} a_k \frac{\left(\frac{k+1}{2}\right)_{n-k}\left(\frac{k+2}{2}\right)_{n-k}}{k!\,(1-\lambda-k)_{n-k}} C_{2n-k}^{\lambda-n+2k}(x)\,v^k,$$

which is believed to be new.

Again using the generating relation (3.2), we get

$$\sum_{k=0}^{\infty}\frac{\left(\frac{n+1}{2}\right)_k\left(\frac{n+2}{2}\right)_k}{(1-\lambda-n)_k} C_{n+2k}^{\lambda+n-k}(x)\frac{t^k}{k!}.\ T_k(y)$$

$$=\frac{1}{2}\left[\left\{1-t\left(y+\sqrt{y^2-1}\right)(1-x^2)\right\}^{-\left(\frac{3n}{2}+\lambda\right)}\left\{1-t\left(y+\sqrt{y^2-1}\right)\right\}^{n+\lambda+\frac{1}{2}}\right.$$

$$\times C_n^{\lambda+n}\left(\frac{x}{\{1-(y+\sqrt{y^2-1})(t-tx^2)\}^{\frac{1}{2}}}\right)$$

$$+\left\{1-t\left(y-\sqrt{y^2-1}\right)(1-x^2)\right\}^{-\left(\frac{3n}{2}+\lambda\right)}\left\{1-t\left(y-\sqrt{y^2-1}\right)\right\}^{n+\lambda+\frac{1}{2}}$$

$$\left.\times C_n^{\lambda+n}\left(\frac{x}{\{1-(y-\sqrt{y^2-1})(t-tx^2)\}^{\frac{1}{2}}}\right)\right], \quad (5.3)$$

which is believed to be new.

Corollary 1: Substituting λ by λ–*n* in (5.3), we get the following generating relation:

$$\sum_{k=0}^{\infty}\frac{\left(\frac{n+1}{2}\right)_k\left(\frac{n+2}{2}\right)_k}{(1-\lambda)_k} C_{n+2k}^{\lambda-k}(x)\frac{t^k}{k!}.\ T_k(y)$$

$$=\frac{1}{2}\left[\left\{1-t\left(y+\sqrt{y^2-1}\right)(1-x^2)\right\}^{-\left(\frac{n}{2}+\lambda\right)}\left\{1-t\left(y+\sqrt{y^2-1}\right)\right\}^{\lambda+\frac{1}{2}}\right.$$

$$\times C_n^{\lambda}\left(\frac{x}{\{1-(y+\sqrt{y^2-1})(t-tx^2)\}^{\frac{1}{2}}}\right)$$

$$+\left\{1-t\left(y-\sqrt{y^2-1}\right)(1-x^2)\right\}^{-\left(\frac{n}{2}+\lambda\right)}\left\{1-t\left(y-\sqrt{y^2-1}\right)\right\}^{\lambda+\frac{1}{2}}$$

$$\left.\times C_n^{\lambda}\left(\frac{x}{\{1-(y-\sqrt{y^2-1})(t-tx^2)\}^{\frac{1}{2}}}\right)\right], \quad (5.4)$$

which is believed to be new.

Corollary 2: Putting $n=0$ in (5.3), we get the following generating relation:

$$\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k}{(1-\lambda)_k} C_{2k}^{\lambda-k}(x)t^kT_k(y)$$

$$= \frac{1}{2}\left[\left\{1 - t\left(y + \sqrt{y^2 - 1}\right)(1 - x^2)\right\}^{-\lambda}\left\{1 - t\left(y + \sqrt{y^2 - 1}\right)\right\}^{\lambda + \frac{1}{2}} + \left\{1 - t\left(y - \sqrt{y^2 - 1}\right)(1 - x^2)\right\}^{-\lambda}\left\{1 - t\left(y - \sqrt{y^2 - 1}\right)\right\}^{\lambda + \frac{1}{2}}\right], \quad (5.5)$$

which is believed to be new.

1. Conclusion

From the above discussion, it is clear that whenever one knows a generating relation of the form (1.1, 5.1) then the corresponding bilateral andtrilateral generating function can at once be written down from (1.2, 5.2). So one can get a large number of bilateral andtrilateral generating functions by attributing different suitable values to a_n in (1.1, 5.1).

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