Ultra Scientist Vol. 26(1)A, 79-87 (2014).

Study of Influential nodes of Fuzzy Graphs in Fuzzy Models

W.B. VASANTHA KANDASAMY[1] R.VASUKI[2] and P.NEDUMARAN[3]

[1]Department of Maths IIT(M) Chennai-600036 (INDIA)
[2]Department of Maths SIVET College Chennai-600073 (INDIA)
[3]Research Scholar University of Madras Chennai-600005 (INDIA)

(Acceptance Date 10th January, 2014)

Abstract

In this paper we give the role of fuzzy graphs in fuzzy models like Fuzzy Cognitive Maps (FCMs) and Fuzzy Relational Maps (FRMs). Our study of these fuzzy graphs is different from the usual study as we have the nodes or edges of a fuzzy graph to be dependent on the fuzzy model in which they are used.

In this paper we define different types of influential nodes of fuzzy graphs related with the fuzzy models like Fuzzy Cognitive maps (FCMs) models and Fuzzy Relational Maps (FRMs) models. This study of labeling the nodes as most influential, more influential, just influential, influential, less influential and least influential nodes of the fuzzy graphs associated with these fuzzy models is carried out in this paper. This study is new and innovative leading to several important observations on the nodes used in the fuzzy models. This answers the natural question whether the node which has the maximum number of edges incident to it is the most influential node in a fuzzy graph associated with fuzzy models. This is answered in the affirmative.

This paper is organized into three sections. The first section is introductory in nature. In section two we define the types of nodes based on the fuzzy models defined, we also prove some results in this direction. The final section gives the conclusions based on our study.

Keywords: Fuzzy Cognitive Maps (FCMs) model, Directed fuzzy graph, Fuzzy Relational Maps (FRMs) model, most influential nodes, less influential nodes and least influential nodes.

1. Introduction

In this section we just briefly describe the Fuzzy Cognitive Maps (FCMs) model and the Fuzzy Relational Maps (FRMs) model, the associated fuzzy graph and the functioning of these models.

Definition[1-4]: *An FCM is a directed graph with concepts like policies, events etc; as nodes and causalities as edges. It represents causal relationship between concepts.*

If increase (or decrease) in one concept leads to increase (or decrease) in another, then we give the value 1. If there exists no relation between two concepts the value 0 is given. If increase (or decrease) in one concept decreases (or increases) another, then we give the value −1. Thus FCMs are described in this way.

Definition[1-4]: *When the nodes of the FCM are fuzzy sets then they are called as fuzzy nodes.*

Definition[1-4]: *FCMs with edge weights or causalities from the set {−1, 0, 1} are called simple FCMs.*

Definition[1-4]: *Consider the nodes / concepts $C_1, \ldots, C_n$ of the FCM. Suppose the directed graph is drawn using edge weight $e_{ij} \in \{0, 1, -1\}$. The matrix E be defined by $E = (e_{ij})$ where e_{ij} is the weight of the directed edge $C_i C_j$. E is called the adjacency matrix of the FCM, also known as the connection matrix of the FCM.*

It is important to note that all matrices associated with an FCM are always square matrices with diagonal entries as zero.

Definition[1-4] : *Let $C_1, C_2, \ldots, C_n$ be the nodes of an FCM. $A=(a_1, a_2, \ldots, a_n)$ where $a_i \in \{0, 1\}$. A is called the instantaneous state vector and it denotes the on-off position of the node at an instant.*

$$a_i = 0 \text{ if } a_i \text{ is off and}$$
$$a_i = 1 \text{ if } a_i \text{ is on}$$

for $i = 1, 2, \ldots, n$.

Definition[1-4]: *Let $C_1, C_2, \ldots, C_n$ be the nodes of an FCM. Let $\overrightarrow{C_1C_2}$, $\overrightarrow{C_2C_3}$, $\overrightarrow{C_3C_4}, \ldots, \overrightarrow{C_iC_j}$ be the edges of the FCM ($i \neq j$). Then the edges form a directed cycle. An FCM is said to be cyclic if it possesses a directed cycle. An FCM is said to be acyclic if it does not possess any directed cycle.*

Definition[1-4]: *An FCM with cycles is said to have a feedback.*

Definition[1-4]: *When there is a feedback in an FCM, i.e., when the causal relations flow through a cycle in a revolutionary way, the FCM is called a dynamical system.*

Definition[1-4]: *Let $\overline{C_1C_2}, \overline{C_2C_3}, \ldots, \overrightarrow{C_{n-1}C_n}$ be a cycle. When C_i is switched on and if the causality flows through the edges of a cycle and if it again causes C_i,* *we say that the dynamical system goes round and round. This is true for any node C_i, for i = 1, 2, ... , n. The equilibrium state for this dynamical system is called the hidden pattern.*

Definition[1-4]*: If the equilibrium state of a dynamical system is a unique state vector, then it is called a fixed point.*

Definition[1-4]*: If the FCM settles down with a state vector repeating in the form*

$$A_1 \rightarrow A_2 \rightarrow \dots \rightarrow A_i \rightarrow A_1 \rightarrow$$

Given any A_i or any bit vector A that sufficiently resembles A_i; then this equilibrium is called a limit cycle.

Now we proceed on to recall the definition of Fuzzy Relational Maps (FRMs).

Definition[7-8,10]*: A FRM is a directed graph or a map from D to R with concepts like policies or events etc, as nodes and causalities as edges. It represents causal relations between spaces D and R.*

Let D_i and R_j denote that the two nodes of an FRM. The directed edge from D_i to R_j denotes the causality of D_i on R_j called relations. Every edge in the FRM is weighted with a number in the set {0, ±1}. Let e_{ij} be the weight of the edge D_iR_j, $e_{ij} \in \{0, \pm 1\}$. The weight of the edge $D_i R_j$ is positive if increase in D_i implies increase in R_j or decrease in D_i implies decrease in R_j ie causality of D_i on R_j is 1. If $e_{ij} = 0$, then D_i does not have any effect on R_j. We do not discuss the cases when increase in D_i implies decrease in R_j or decrease in D_i implies increase in R_j.

Definition[7-8,10]*: When the nodes of the FRM are fuzzy sets then they are called fuzzy nodes. FRMs with edge weights {0, ±1} are called simple FRMs.*

Definition[7-8,10]*: Let $D_1, \dots, D_n$ be the nodes of the domain space D of an FRM and $R_1, \dots, R_m$ be the nodes of the range space R of an FRM. Let the matrix E be defined as $E = (e_{ij})$ where e_{ij} is the weight of the directed edge D_iR_j (or R_jD_i), E is called the relational matrix of the FRM.*

Note: It is pertinent to mention here that unlike the FCMs the FRMs can be a rectangular matrix with rows corresponding to the domain space and columns corresponding to the range space. This is one of the marked[5,6] differences between FRMs and FCMs.

Definition[7-8,10]*: Let $D_1, \dots, D_n$ and $R_1, \dots, R_m$ denote the nodes of the FRM. Let $A = (a_1, \dots, a_n)$, $a_i \in \{0, 1\}$. A is called the instantaneous state vector of the domain space and it denotes the on-off position of the nodes at any instant. Similarly let $B = (b_1, \dots, b_m)$ $b_i \in \{0, 1\}$. B is called instantaneous state vector of the range space and it denotes the on-off position of the nodes at any instant $a_i = 0$ if a_i is off and $a_i = 1$ if a_i is on for i= 1, 2,..., n Similarly, $b_i = 0$ if b_i is off and $b_i = 1$ if b_i is on, for i= 1, 2,..., m.*

Definition[7-8,10]*: Let $D_1, \ldots, D_n$ and $R_1, \ldots, R_m$ be the nodes of an FRM. Let D_iR_j (or $R_j D_i$) be the edges of an FRM, $j = 1, 2, \ldots, m$ and $i = 1, 2, \ldots, n$. Let the edges form a directed cycle. An FRM is said to be a cycle if it possesses a directed cycle. An FRM is said to be acyclic if it does not possesses any directed cycle.*

Definition[7-8,10]*: An FRM with cycles is said to be an FRM with feedback.*

Definition[7-8,10]*: When there is a feedback in the FRM, i.e. when the causal relations flow through a cycle in a revolutionary manner, the FRM is called a dynamical system.*

Definition[7-8,10]*: Let $D_i R_j$ (or $R_j D_i$), $1 \le j \le m$, $1 \le i \le n$. When R_i (or D_j) is switched on and if causality flows through edges of the cycle and if it again causes R_i (or D_j), we say that the dynamical system goes round and round. This is true for any node R_j (or D_i) for $1 \le i \le n$, (or $1 \le j \le m$). The equilibrium state of this dynamical system is called the hidden pattern.*

Definition[7-8,10]*: If the equilibrium state of a dynamical system is a unique state vector, then it is called a fixed point. Consider an FRM with $R_1, R_2, \ldots, R_m$ and $D_1, D_2, \ldots, D_n$ as nodes. For example, let us start the dynamical system by switching on R_1 (or D_1). Let us assume that the FRM settles down with R_1 and R_m (or D_1 and D_n) on, i.e. the state vector remains as $(1, 0, \ldots, 0, 1)$ in R (or $1, 0, 0, \ldots, 0, 1)$ in D). This state vector is called the fixed point.*

Definition[7-8,10]*: If the FRM settles down with a state vector repeating in the form*

$A_1 \to A_2 \to A_3 \to \ldots \to A_i \to A_1$ (or $B_1 \to B_2 \to \ldots \to B_i \to B_1$)

then this equilibrium is called a limit cycle.

2. Definition of types of nodes of the fuzzy graphs of the fuzzy models:

Here we define the notion of types of nodes of fuzzy graphs associated with fuzzy models.

Let $C_1, C_2, \ldots, C_n$ be the nodes of a FCM. We call a node C_i to be a most influential node if the on state of that node alone in the state vector makes ON state the greatest number of nodes in the resultant state vector given by the fixed point or the limit cycle, then we define the node C_i to be most influential node. If the node C_i alone is in the ON state of a state vector and if it gives maximum numbers of ON state of nodes in the resultant state vector which is a limit or a fixed point and if it is not the most influential node then we define it as a more influential node. So a most influential node has the maximum number of ON states in the resultant state vector than the more influential node. Likewise we define just influential node, influential node and so on[9].

(The number of ON state of nodes in the resultant state vector of the most influential node) > (The number of ON state of nodes in the resultant state vector of the more influential node) > (The number of ON state of nodes in the resultant state

vector of the just influential node) > (The number of ON state of nodes in the state vector of the influential node) > (The number of ON state of nodes in the state vector of the less influential node) > (The number of ON state of nodes in the state vector of the least influential node).

Here we prove the long standing discussions among the graph theorist who feel that the most powerful node would be the node which has the maximum number of edges incident towards it. But however such claim is not true in case of the fuzzy graphs associated with the fuzzy models FCMs and FRMs.

We give some examples of the most influential, more influential nodes and so on of a fuzzy graph associated with a FCMs model[11].

Example 2.1: We describe the attributes related to the problem given by an expert. To build the model we use the following 9 nodes of the FCMs which are labeled as follows for more refer[12]

C_1 - Lung disease like cold, breathlessness etc
C_2 - Malaria
C_3 - Elephantiasis
C_4 - Stagnant water near by the dumped waste
C_5 - Diarrhea
C_6 - Dengue
C_7 - Hypertension and headache
C_8 - Burning solid waste so acute environmental pollutions
C_9 - Unbearable foul smell due to the decaying waste

The directed graph given by the local experts is as follows:

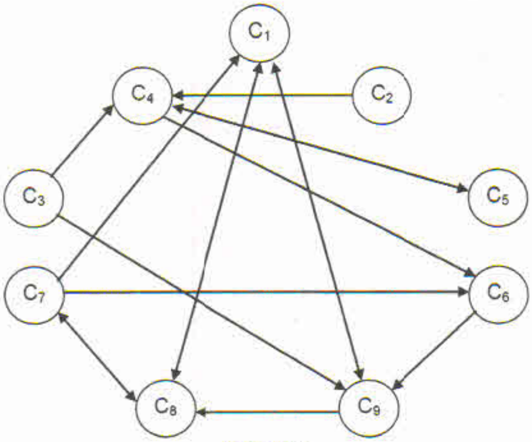

FIGURE 1

The related connection matrix is given below:

$$E = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The working of the dynamical system using state vector, $A_1 = (1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$ that is the only node C_1, that is lung disease is on and all other nodes are in the off state given by an expert is as follows.

Input the vector

$$A_1 = (1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0);$$

that is find

$$A_1E = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1)$$
$$\rightarrow (1\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1) = A_2 \text{ (say)}$$

($\rightarrow$ denotes the state vector has been updated and thresholded)

Now

$$
\begin{aligned}
A_2E &= (2\ 0\ 0\ 0\ 0\ 0\ 1\ 2\ 1)\\
&\rightarrow (1\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1)\\
&= A_3 \text{ (say)}\\
A_3E &= (3\ 0\ 0\ 0\ 0\ 1\ 1\ 3\ 1)\\
&\rightarrow (1\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1)\\
&= A_4 \text{ (say)}\\
A_4E &= (3\ 0\ 0\ 0\ 0\ 1\ 1\ 3\ 2)\\
&\rightarrow (1\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1)\\
&= A_5 = A_4.
\end{aligned}
$$

Thus the hidden pattern of the state vector A_1 is a fixed point.

The effect of C_i $(1 \leq i \leq 9)$ state vectors on the dynamical system is calculated in the similar way is as follows:

$C_1=(1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$;(3) $A_1=(1\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1)$
$C_2=(0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$;(1) $A_2=(1\ 1\ 0\ 1\ 1\ 1\ 1\ 1\ 1)$
$C_3=(0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0)$;(2) $A_3=(1\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1)$
$C_4=(0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0)$;(4) $A_4=(1\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1)$
$C_5=(0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0)$;(1) $A_5=(1\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1)$
$C_6=(0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0)$;(3) $A_6=(1\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1)$
$C_7=(0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0)$;(3) $A_7=(1\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1)$
$C_8=(0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0)$;(3) $A_8=(1\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1)$
$C_9=(0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1)$;(4) $A_9=(1\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1)$

The number the bracket indicates the number of edges incident to the respective nodes; 3 edges are incident to node C_1, one edge is incident to C_2 and so on.

From observing $C_1, C_2, \ldots, C_9$ we make the following conclusions. C_2 and C_3 are the most influential nodes, C_2 has only one edge incident to it and C_3 has two edges incident on to it, C_4 and C_5 are more influential nodes, they have four and one edges respectively incident to them C_1, C_6, C_7, C_8 and C_9 are just influential modes and the edges incident to them are given in the bracket for easy comparison.

Thus the nodes with 1 edge incident to it happens to be the most influential node C_2 and C_5 the more influential node which has only one node incident to it.

Now we give an example for the fuzzy graph associated with FRMs model.

Here we describe the FRMs model from[7-8,10] which studies the child labour problem.

Here the attributes with the children working as child labourers is taken as domain space of the FRMs. For the range space the attributes associated with the public awareness is taken.[5,6]

Domain Space :

C - The attributes associated with the children working as child labourer.

C_1 - Abolition of child labour

C_2 - Uneducated parents

C_3 - School dropouts / never attended any school

C_4 - Social status of child labourers

C_5 - Poverty / source of livelihood

C_6 - Orphans, Runaways, and parents are beggars, fathers in prison.

C_7 - Habits like cinema, smoking, alcoholic etc.

Range Space :

P - The attributes associated with public awareness in support of child labour.

P_1 - Cheap and long hours of labour from children
P_2 - Children as domestic servants
P_3 - Sympathetic public
P_4 - Motivation by teachers to children to pursue education
P_5 - Perpetuating slavery and caste bias.

The related connection graph is given in figure 2.

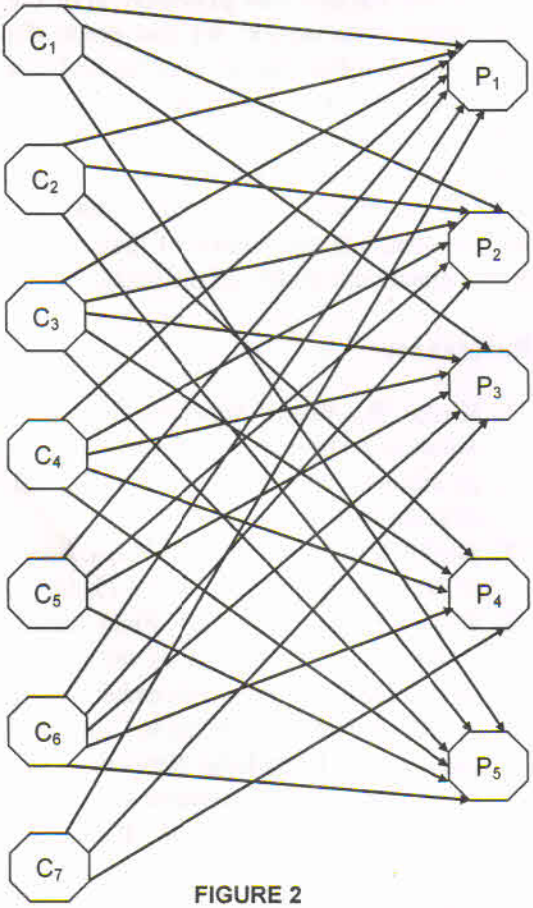

FIGURE 2

The related connection matrix is given by the following matrix E.

$$E = \begin{bmatrix} -1 & -1 & 1 & 0 & -1 \\ -1 & 1 & 0 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & -1 & -1 & 1 \\ 1 & 0 & -1 & -1 & 0 \end{bmatrix}.$$

The effect of C_i $(1 \le i \le 7)$ state vectors on the dynamical system is as follows and the number given in the bracket gives the number of edges incident to the respective nodes.

C_1=(1 0 0 0 0 0 0); (4)R_1 : (0 0 1 1 0); D_1 : (1 0 0 1 1 0 0)
C_2=(0 1 0 0 0 0 0); (4)R_2 : (1 1 0 0 1): D_2 : (1 0 0 1 1 0 0)
C_3=(0 0 1 0 0 0 0); (5)R_3 : (1 1 0 0 1):D_3 : (0 1 1 0 1 1 1)
C_4=(0 0 0 1 0 0 0); (5)R_4 : (0 0 1 1 0):D_4 : (1 0 0 1 1 0 0)
C_5=(0 0 0 0 1 0 0); (4)R_5 : (1 1 0 0 1):D_5 : (0 1 1 0 1 1 1)
C_6=(0 0 0 0 0 1 0); (5)R_6 : (1 1 0 0 1):D_6 : (0 1 1 0 1 1 1)
C_7=(0 0 0 0 0 0 1); (3)R_7 : (1 1 0 0 1); D_7 : (0 1 1 0 1 1 1)

The effect of P_i $(1 \le i \le 5)$ state vectors on the dynamical system.

P_1=(1 0 0 0 0); (7)R_1 : (1 1 0 0 1): D_1 : (0 1 1 0 1 1 1)
P_2=(0 1 0 0 0); (6)R_2 : (1 1 0 0 1): D_2 : (0 1 1 0 1 1 1)
P_3=(0 0 1 0 0); (6)R_3 : (0 0 1 1 0): D_3 : (1 0 0 1 1 0 0)
P_4=(0 0 0 1 0); (5)R_4 : (0 0 1 1 0): D_4 : (1 0 0 1 1 0 0)
P_5=(0 0 0 0 1); (6)R_5 : (1 1 0 0 1); D_5 : (0 1 1 0 1 1 1)

From this figure we see the least edges are incident to the vertex C_7, has the maximum effect on the dynamical system being the most influential node.

However certain vertices with maximum edges incident to it also have same maximum effect.

This example clearly shows that the most influential node does not have the maximum number of edges incident to it. Consequent of these observations we prove the following result.

Theorem 2.1: The influential node of the fuzzy graph related with the fuzzy model in general is not dependent on the number of edges incident to it.

Proof: Let us assume that the influential node of the fuzzy graph related with the fuzzy model is dependent on the number of edges incident to it.

Consider the fuzzy graph of the FCM model given in example 2.1. The node C_2 has only one edge incident to it but it is the most influential node. Likewise C_3 has only two edges incident to it but it is the more influential node. C_5 has only one edge incident to it and it is a more influential node. C_4 and C_9 have the most number of edges incident to it, but they are not the most influential nodes. This shows that the influential node is not dependent on the number of edges incident on it.

Likewise from the fuzzy graph of the FRMs model C_7 has the least number of edges incident to it but it is the most influential node is not the node with the maximum number of edges incident to it. This clearly proves that our assumption is wrong, and that the influential node is not dependent on the number of the edges incident to it.

3. Conclusions

Thus in this model we define those vertices which has maximum effect on the dynamical system as most influential node/vertex. Finding the influential vertices by observing the fuzzy graph of the fuzzy models is not possible in general.

From our study it is clear that fuzzy graphs of the fuzzy model cannot give the complete impact about the dynamical system or about the influential nodes. Only the connection matrix of the graph which is used in getting the hidden pattern can precisely give the influential node/vertex. We can grade the nodes as most influential, more influential, just influential, influential, less influential and least influential nodes of the fuzzy graphs associated with these fuzzy models and so on by working with all the related state vectors. From this we can conclude the impact of these nodes on the dynamical system of the fuzzy model.

References

1. Kosko, B., *Neural Networks and Fuzzy Systems, A Dynamical Systems Approach to Machine Intelligence,* Prentice Hall of India (1997).
2. Kosko, B. Fuzzy Cognitive Maps, *Int. J. of Man-Machine Studies, 24,* 65-75 (1986).
3. Kosko, B. *Heaven in a chip: Fuzzy Visions of Society and Science in the Digital Age*, Three Rivers Press, November (2000).
4. Kosko, B. Hidden Patterns in Combined and Adaptive Knowledge Networks, *Proc. of the First IEEE International Conference on Neural Networks* (ICNN-86), 2, 377-393 (1988).
5. Vasantha Kandasamy, W.B., and P. Pramod.

Parent Children Model using FCM to Study Dropouts in Primary Education, *Ultra Sci.*, *13*, 174-183 (2000).

6. Vasantha Kandasamy, W.B., and S. Uma. Fuzzy Cognitive Map of Socio-Economic Model, *Appl. Sci. Periodical*, *1*, 129-136 (1999).
7. Vasantha Kandasamy, W.B., and Yasmin Sultana, FRM to Analyse the Employee-Employer Relationship Model, *J. Bihar Math. Soc.*, *21*, 25-34 (2001).
8. Vasantha Kandasamy, W.B., and Yasmin Sultana, Knowledge Processing Using Fuzzy Relational Maps, *Ultra Sci.*, *12*, 242-245 (2000).
9. Vasantha Kandasamy, W.B., N.R. Neelakantan and S.R. Kannan. Replacement of Algebraic Linear Equations by Fuzzy Relation Equations in Chemical Engineering, In *Recent trends in Mathematical Sciences*, Proc. of Int. Conf. on Recent Advances in Mathematical Sciences held at IIT Kharagpur on Dec. 20-22, 2001, published by Narosa Publishing House, 161-168 (2001).
10. Vasantha Kandasamy, W.B. and Smarandache, F., *Fuzzy Cognitive Maps and Neutrosophic Cognitive Maps*, Xiquan, USA, (2003).
11. Vasantha Kandasamy, W.B., R. Vasuki and P. Nedumaran. Analysis of Fuzzy graphs in fuzzy models, *National Symposium on Mathematical Methods and Applications*, Dec. 22, (2013).
12. Vasantha Kandasamy, W.B., R. Vasuki and K.Thulukkanam. Simple FCMs to analyse the problems of the locals due to disposal of solid waste in Kodungaiyur, *National Symposium on Mathematical Methods and Applications*, Dec. *22*, (2013).