Ultra Scientist Vol. 26(1)A, 97-103 (2014).

Banach Contraction Principle on Cone Hexagonal Metric Space

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(Acceptance Date 22nd February, 2014)

Abstract

We introduce the notion of cone hexagonal metric space and prove Banach contraction mapping principle in cone hexagonal metric space. Our result extends recent known results.

Key words: cone metric space; fixed points; contraction mapping principle; Cone rectangular metric space; Cone pentagonal metric space.

2000 Subject Mathematical Classification: 47H10, 54H25

1 Introduction

Huang and Zhang[4] introduced the concept of cone metric space and established some fixed point theorems for contractive mappings in normal cone metric spaces. Subsequently, several other authors[3,5,6,7,8] studied the existence of fixed points and common fixed points of mappings satisfying a contractive type condition on a normal cone metric space. Recently Garg and Agarwal[9] introduced the concept of cone pentagonal metric spaces and proved Banach contraction mapping principle in cone pentagonal metric space.

In the paper we introduce cone hexagonal metric spaces and prove Banach contraction mapping principle in a complete normal cone hexagonal metric space.

2 Preliminaries :

The following notions have been used to prove the main result.

Definition 2.1: Let E be a real Banach Space. A subset P of E is called cone[4] if and only if

(i) P is closed, non empty and $P \neq \{0\}$.

(ii) $0 \leq a, b \in R$ and $x, y \in P \Rightarrow ax + by \in P$.

(iii) $P \cap (-P) = \{0\}$.

Definition 2.2: The partial ordering[4] $\leq$ with respect to $P \subseteq E$ is defined by $x \leq y$ if and only if $y - x \in P$. $x < y$ shows that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in int(P)$, int(P) denotes the interior of P.

Definition 2.3: A cone P is called normal[4] if there is a number $k \geq 1$ such that for all $x, y \in E$, the inequality

$$0 \leq x \leq y \Rightarrow \|x\| \leq k \|y\|.$$

The least positive number k satisfying the above inequality is called the normal constant of P.

In this paper we always suppose that E is a real Banach space and P is a cone in E with $int(P) \neq \Phi$ and $\leq$ is a partial ordering with respect to P.

Definition 2.4: Let X be a non empty set. Suppose that the mapping $\rho : X \times X \to E$ satisfies:

(i) $0 < \rho(x, y)$ for all $x, y \in X$ and $\rho(x, y) = 0$ if and only if $x = y$.

(ii) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$.

(iii) $\rho(x, y) \leq \rho(x, z) + \rho(z, w) + \rho(w, u) + \rho(u, y)$ for all $x, y, z, w, u \in X$ and for all distinct points $z, w, u \in X - \{x, y\}$.

Then ρ is called a cone pentagonal metric[9] on X and (X, ρ) is called a cone pentagonal metric space.

3 *Fixed point theorem :*

In this section we shell define hexagonal cone metric space and prove a fixed point theorem of contractive mapping.

Definition 3.1 Let X be a non empty set. Suppose that the mapping $d: X \times X \to E$ satisfies:

(d_1) $0 < d(x, y)$ for all $x, y \in X$ and $d(x, y)=0$ if and only if $x = y$.

(d_2) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(d_3) $d(x, y) \leq d(x, z) + d(z, w) + d(w, u) + d(u, v) + d(v, y)$ for all x, y, z, w, u, v X and for all distinct points $z, w, u, v \in X - \{x, y\}$ [hexagonal property].

Then d is called a cone hexagonal metric on X, and (X, d) is called a cone hexagonal metric space.

Definition 3.2 Let $\{x_n\}$ be a sequence in a cone hexagonal metric space (X, d) and $x \in X$. If for every $c \in E$, with $0 << c$ there exist $n_0 \in N$ and that for all $n > n_0$, $d(x_n, x) << c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x and x is the limit of $\{x_n\}$. It is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$, as $n \to \infty$.

Definition 3.3 If for every $c \in E$, with $0 << c$ there exist $n_0 \in N$ such that for all $n > n_0$, $d(x_n, x) << c$, then $\{x_n\}$ is called Cauchy sequence in a cone hexagonal metric space X.

Definition 3.4 If every Cauchy sequence is convergent in a cone hexagonal metric space (X, d), then (X, d) is called a complete cone hexagonal metric space.

Before proving our main theorem we present some theorems.

Theorem 3.1 Let (X, d) be a hexagonal cone metric space and P be a normal cone with normal constant k. Let $\{x_n\}$ be a sequence in X, then $\{x_n\}$ converges to x if and only if $\|d(x_n, x)\| \to 0$ as $n \to \infty$.

Theorem 3.2 Let (X, d) be a hexagonal cone metric space and P be a normal cone with normal constant k. Let $\{x_n\}$ be a sequence in X, then $\{x_n\}$ is a Cauchy sequence if and only if $\|d(x_n, x_{n+m})\| \to 0$ as $n \to \infty$.

The proof of above theorems is similar to Huang and Zhang [4, lemmas 1 and 4].

Theorem 3.3 Every cone (or rectangular or pentagonal) metric space is cone hexagonal metric space.

Proof: Since every cone metric is cone rectangular metric and every cone rectangular metric is cone pentagonal metric and every cone pentagonal metric is cone hexagonal metric so the proof is obvious.

The converse of the above theorem is not necessarily true as it can be seen from the following example.

Example 3.1 Let $X = N$, $E = R^2$ and $P = \{(x, y): x, y \geq 0\}$. Define $d: X \times X \to E$ as follows:
$d(x, y) = (0, 0)$ if $x = y$; $d(x, y) = (9, 15)$ if x and y are in $\{3, 4\}$, $x \neq y$; $d(x, y) = (3, 5)$ if x and y cannot both at a time in $\{3, 4\}$, $x \neq y$.

Then (X, d) is a cone hexagonal (or pentagonal or rectangular) metric space but not a cone metric space because it lacks the triangular property:

$(9, 15) = d(3, 4) > d(3, 5) + d(5, 4) = (3, 5) + (3, 5) = (6, 10)$
as $(9, 15) - (6, 10) = (3, 5) \in P$.

Theorem 3.4 Every rectangular and pentagonal (resp. complete rectangular and complete pentagonal) cone metric space is hexagonal (resp. complete hexagonal) cone metric space.

The converse of the above theorem is not necessarily true as it can be seen from the following example.

Example 3.2 Let $X = \{1, 2, 3, 4, 5, 6\}$, $E = R^2$ and $P = \{(x, y) : x, y \;\; 0\}$ is a normal cone in E. Define $d: X \times X \to E$ as follows:

$d(1, 2) = d(2, 1) = (5, 10)$.
$d(1, 3) = d(3, 1) = d(1, 4) = d(4, 1) = d(1, 5) = d(5, 1) = d(2, 3) = d(3, 2) = d(2, 4) = d(4, 2) = d(2, 5) = d(5, 2) = d(3, 4) = d(4, 3) = d(3, 5) = d(5, 3) = d(4, 5) = d(5, 4) = (1, 2)$.
$d(1, 6) = d(6, 1) = d(2, 6) = d(6, 2) = d(3, 6) = d(6, 3) = d(4, 6) = d(6, 4) = (4, 8)$.

Then (X, d) is a cone hexagonal (resp. complete hexagonal) metric space but not a cone pentagonal (resp. complete pentagonal) and so cone rectangular (resp. complete rectangular) metric space because it lacks the pentagonal and rectangular property:
$(5, 10) = d(1, 2) \;\; d(1, 3) + d(3, 4) + d(4, 5) + d(5, 2) = (1, 2) + (1, 2) + (1, 2) + (1, 2) = (4, 8)$
as $(5, 10) - (4, 8) = (1, 2) \in P$.

The main theorem of this paper is as follows.

Theorem 3.5 Let (X, d) be a hexagonal cone metric space and P be a normal cone with normal constant k and the mapping $T: X \to X$ satisfies the contractive condition $d(Tx, Ty) \leq \lambda\, d(x, y)$ for all $x \in X$, where $\lambda \in [0, 1)$ is a constant. Then T has a unique fixed point in X.

Proof: Let $x_0 \in X$. Define a sequence of points in X as follows,
$x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, \ldots\ldots, x_{n+1} = Tx_n = T^{n+1}x_0.$

We can suppose that x_0 is not a periodic point,

if $x_n = x_0$ then,
$d(x_0, Tx_0) = d(x_n, Tx_n) = d(T^n x_0, T^{n+1}x_0) \leq \lambda d(T^{n-1}x_0, T^n x_0) \leq \lambda^2 d(T^{n-2}x_0, T^{n-1}x_0) \leq \ldots\ldots\ldots\ldots \leq \lambda^n d(x_0, Tx_0)$.
This shows that $[\lambda^n - 1]\, d(x_0, Tx_0) \in P$.

It again implies that $\left[\dfrac{\lambda^n - 1}{1-\lambda^n}\right] d(x_0, Tx_0) \in P$.

Thus $-d(x_0, Tx_0) \in P$ and $d(x_0, Tx_0) = 0$ i. e. x_0 is the fixed point of T.

Now we suppose that $x_m \neq x_n \ \forall\ m, n \in N$. Using hexagonal property for all $y \in X$, we have

$$\begin{aligned} d(y, T^5y) &\leq d(y, Ty) + d(Ty, T^2y) + d(T^2y, T^3y) + d(T^3y, T^4y) + d(T^4y, T^5y) \\ &\leq d(y, Ty) + \lambda d(y, Ty) + \lambda^2 d(y, Ty) + \lambda^3 d(y, Ty) + \lambda^4 d(y, Ty) \\ &\leq \sum_{i=0}^{4} \lambda^i d(y, Ty) \end{aligned}$$

Similarly, $d(y, T^9y) \leq d(y, Ty) + d(Ty, T^2y) + d(T^2y, T^3y) + d(T^3y, T^4y) - d(T^4y, T^5y) + d(T^5y, T^6y) + d(T^6y, T^7y) + d(T^7y, T^8y) + d(T^8y, T^9y)$.

$$\leq \sum_{i=0}^{8} \lambda^i d(y, Ty)$$

Now by induction, we obtain for each k = 1, 2, 3 …

$$d(y, T^{4k+1}y) \leq \sum_{i=0}^{4k} \lambda^i d(y, Ty) \qquad (1)$$

Again for all $y \in X$,

$$\begin{aligned} d(y, T^6y) &\leq d(y, Ty) + d(Ty, T^2y) + d(T^2y, T^3y) + d(T^3y, T^4y) + d(T^4y, T^6y) \\ &\leq d(y, Ty) + \lambda d(y, Ty) + \lambda^2 d(y, Ty) + \lambda^3 d(y, Ty) + \lambda^4 d(y, T^2y) \\ &\leq \sum_{i=0}^{3} \lambda^i d(y, Ty) + \lambda^4 d(y, T^2y) \end{aligned}$$

Similarly,

$$\begin{aligned} d(y, T^{10}y) &\leq d(y, Ty) + d(Ty, T^2y) + d(T^2y, T^3y) + d(T^3y, T^4y) + d(T^4y, T^5y) + d(T^5y, T^6y) \\ &\quad + d(T^6y, T^7y) + d(T^7y, T^8y) + d(T^8y, T^{10}y). \\ &\leq d(y, Ty) + \lambda d(y, Ty) + \lambda^2 d(y, Ty) + \lambda^3 d(y, Ty) + \lambda^4 d(y, Ty) + \lambda^5 d(y, Ty) + \lambda^6 d(y, T^2y) \\ &\quad + \lambda^7 d(y, Ty) + \lambda^8 d(y, T^2y). \\ &\leq \sum_{i=0}^{7} \lambda^i d(y, Ty) + \lambda^8 d(y, T^2y). \end{aligned}$$

By induction, we obtain for each k = 1, 2, 3……..

$$d(y, T^{4k+2}y) \leq \sum_{i=0}^{4k-1} \lambda^i d(y, Ty) + \lambda^{4k} d(y, T^2y). \qquad (2)$$

Again, for all $y \in X$,

$$\begin{aligned} d(y, T^7y) &\leq d(y, Ty) + d(Ty, T^2y) + d(T^2y, T^3y) + d(T^3y, T^4y) + d(T^4y, T^7y) \\ &\leq d(y, Ty) + \lambda d(y, Ty) + \lambda^2 d(y, Ty) + \lambda^3 d(y, Ty) + \lambda^4 d(y, T^3y) \\ &\leq \sum_{i=0}^{3} \lambda^i d(y, Ty) + \lambda^4 d(y, T^3y). \end{aligned}$$

Similarly,

$$\begin{aligned} d(y, T^{11}y) &\leq d(y, Ty) + d(Ty, T^2y) + d(T^2y, T^3y) + d(T^3y, T^4y) + d(T^4y, T^5y) + d(T^5y, T^6y) + \\ &\quad d(T^6y, T^7y) + d(T^7y, T^8y) + d(T^8y, T^{11}y). \\ &\leq d(y, Ty) + \lambda d(y, Ty) + \lambda^2 d(y, Ty) \end{aligned}$$

$+ \lambda^3 d(y, Ty) + \lambda^4 d(y, Ty) + \lambda^5 d(y, Ty)$
$+ \lambda^4 d(y, Ty) + \lambda^5 d(y, Ty) + \lambda^6 d(y, Ty)$
$+ \lambda^7 d(y, Ty) + \lambda^8 d(y, T^3y)$

$$\leq \sum_{i=0}^{7} \lambda^i d(y, Ty) + \lambda^8 d(y, T^3y)$$

So by induction, we obtain for each k = 1, 2, 3………

$$d(y, T^{4k+3}y) \leq \sum_{i=0}^{4k-1} \lambda^i d(y, Ty)) + \lambda^{4k} d(y, T^3y) \quad (3)$$

In a similar way for all $y \in X$,

$d(y, T^8y) \leq d(y, Ty) + d(Ty, T^2y) + d(T^2y, T^3y) + d(T^3y, T^4y) + d(T^4y, T^8y)$

$\leq d(y, Ty) + \lambda d(y, Ty) + \lambda^2 d(y, Ty) + \lambda^3 d(y, Ty) + \lambda^4 d(y, T^4y)$

$$\leq \sum_{i=0}^{3} \lambda^i d(y, Ty) + \lambda^4 d(y, T^4y)$$

Similarly,

$d(y, T^{102}y) \leq d(y, Ty) + d(Ty, T^2y) + d(T^2y, T^3y) + d(T^3y, T^4y) + d(T^4y, T^5y) + d(T^5y, T^6y) + d(T^6y, T^7y) + d(T^7y, T^8y) + d(T^8y, T^{12}y).$

$\leq d(y, Ty) + \lambda d(y, Ty) + \lambda^2 d(y, Ty) + \lambda^3 d(y, Ty) + \lambda^4 d(y, Ty) + \lambda^5 d(y, Ty) + \lambda^6 d(y, T^2y) + \lambda^7 d(y, Ty) + \lambda^8 d(y, T^4y)$

$$\leq \sum_{i=0}^{7} \lambda^i d(y, Ty) + \lambda^8 d(y, T^4y).$$

So by induction, we obtain for each k = 1, 2, 3……..

$$d(y, T^{4k+4}y) \leq \sum_{i=0}^{4k-1} \lambda^i d(y, Ty) + \lambda^{4k} d(y, T^4y) \quad (4)$$

Using inequality (1) for k = 1, 2, 3 …………we have,

$d(T^n x_0, T^{n+4k+1} x_0) \leq \lambda^n d(Tx_0, T^{4k+1} x_0)$

$$\leq \lambda^n \sum_{i=0}^{4k} \lambda^i d(x_0, Tx_0)$$

$$\leq \frac{\lambda^n}{1-\lambda} [d(x_0, Tx_0) + d(x_0, T^2x_0) + d(x_0, T^3x_0) + d(x_0, T^4x_0)] \quad (5)$$

Similarly for k = 1, 2, 3……., inequality (2) implies that

$d(T^n x_0, T^{n+4k+2} x_0) \leq \lambda^n d(x_0, T^{4k+2} x_0)$

$$\leq \lambda^n [\sum_{i=0}^{4k-1} \lambda^i d(x_0, Tx_0) + \lambda^{4k} d(x_0, T^2x_0)]$$

$$\leq \lambda^n [\sum_{i=0}^{4k-1} [\lambda^i \{d(x_0, Tx_0) + d(x_0, T^2x_0) + d(x_0, T^3x_0) + d(x_0, T^4x_0)\} + \lambda^{4k} \{d(x_0, Tx_0) + d(x_0, T^2x_0) + d(x_0, T^3x_0) + d(x_0, T^4x_0)\}]$$

$$\leq \lambda^n [\sum_{i=0}^{4k} \lambda^i \{d(x_0, Tx_0) + d(x_0, T^2x_0) + d(x_0, T^3x_0) + d(x_0, T^4x_0)\}]$$

$$\leq \frac{\lambda^n (1-\lambda^{4k-1})}{1-\lambda} [d(x_0, Tx_0) + d(x_0, T^2x_0) + d(x_0, T^3x_0) + d(x_0, T^4x_0)]$$

$$\leq \frac{\lambda^n}{1-\lambda} [d(x_0, Tx_0) + d(x_0, T^2x_0) + d(x_0, T^3x_0) + d(x_0, T^4x_0)] \quad (6)$$

Again for k=1, 2, 3…... inequality (3) implies that

$d(T^n x_0, T^{n+4k+3} x_0) \leq \lambda^n d(x_0, T^{4k+3} x_0)$

$$\leq \lambda^n [\sum_{i=0}^{4k-1} \lambda^i d(x_0, Tx_0) + \lambda^{4k} d(x_0, T^3x_0)]$$

$$\le \lambda^n [\sum_{i=0}^{4k} \lambda^i\{d(x_0,Tx_0) + d(x_0, T^2x_0) + d(x_0, T^3x_0) + d(x_0, T^4x_0)\}]$$

$$\le \lambda^n \sum_{i=0}^{3k} \lambda^i[d(x_0,Tx_0) + d(x_0, T^2x_0) + d(x_0, T^3x_0)]$$

$$\le \frac{\lambda^n}{1-\lambda} [d(x_0, Tx_0) + d(x_0, T^2x_0) + d(x_0, T^3x_0) + d(x_0, T^4x_0)] \quad (7)$$

In a similar way by inequality (4) it can be shown that,

$$d(T^n x_0, T^{n+4k+4}x_0) \le \frac{\lambda^n}{1-\lambda} [d(x_0, Tx_0) + d(x_0, T^2x_0) + d(x_0, T^3x_0) + d(x_0, T^4x_0)] \quad (8)$$

Thus by inequality (5), (6), (7) and (8) we have,

$$d(T^n x_0, T^{n+m}x_0) \le \frac{\lambda^n}{1-\lambda} [d(x_0, Tx_0) + d(x_0, T^2x_0) + d(x_0, T^3x_0) + d(x_0, T^4x_0)]$$

Since P is a normal cone with normal constant k, therefore,

$$\| d(T^n x_0, T^{n+m}x_0) \| \le \frac{\lambda^n k}{1-\lambda} \|[d(x_0, Tx_0) + d(x_0, T^2x_0) + d(x_0, T^3x_0) + d(x_0, T^4x_0)]\|$$

i.e. $\| d(x_n, x_m) \| \to 0$ as $n \to \infty$.

Now theorem 3.2 implies that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $z \in X$ such that $x_n \to z$.

Again by theorem 3.1, we have $\| d(T^n x_0, z) \| \to 0$ as $n \to \infty$. Since $x_n \ne x_m$ for $n \ne m$, therefore by hexagonal property, we have $d(Tz, z) \le d(Tz, T^n x_0) + d(T^n x_0, T^{n+1}x_0) + d(T^{n+1}x_0, T^{n+2}x_0) + d(T^{n+2}x_0, T^{n+3}x_0) + d(T^{n+3}x_0, z)$

$$\le \lambda d(z, T^{n-1}x_0) + \lambda^n d(x_0, Tx_0) + \lambda^{n+1} d(x_0, Tx_0) + \lambda^{n+2} d(x_0, Tx_0) + d(T^{n+3}x_0, z).$$

Thus we have, $\| d(Tz, z) \| \le k\, [\lambda \| d(z, T^{n-1}x_0) \| + \lambda^n \| d(x_0, Tx_0) \| + \lambda^{n+1} \| d(x_0, Tx_0) \| + \lambda^{n+2} \| d(x_0, Tx_0) \| + \| d(T^{n+3}x_0, z)\|]$.

Letting $n \to \infty$, we have, $\| d(Tz, z) \| = 0$.

Hence $Tz = z$, i. e. z is a fixed point of T.

Now we show that z is unique. For suppose z' be another fixed point of T such that $Tz' = z'$.
$d(z, z') = d(Tz, Tz') \le \lambda d(z, z)$.
Hence $z = z'$.

This completes the proof of the theorem.

Remark 3.1 In example 3.2, Define a mapping $T: X \to X$ as follows:
$T(x) = 5$ if $x \ne 6$ and $T(x) = 2$ if $x = 6$.

Note that $d(T(1), T(2)) = d(T(1), T(3)) = d(T(1), T(4)) = d(T(1), T(5)) = d(T(2), T(3)) = d(T(2), T(4)) = d(T(2), T(5)) = d(T(3), T(4)) = d(T(3), T(5)) = 0$.
And in all other cases $d(T(x), T(y)) = (1, 2)$, $d(x, y) = (4, 8)$.

Hence, for $\lambda = 1/4$, all conditions of theorem 3.5 are satisfied and 5 is a unique fixed point of T.

Remark 3.2 In the example 3.2, results of Garg and Agarwal[9] are not applicable to obtained the fixed point of the mapping T on X. Since (X, d) is not a cone pentagonal metric space.

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