Ultra Scientist Vol. 26(1)A, 104-110 (2014).

vg-Lindeloff Space

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(Acceptance Date 8th March, 2014)

Abstract

The object of the present paper is to introduce vg-lindeloff spaces and study its basic properties.

Key words: v-open sets, v-continuity, v-lindeloff space; vg-open sets, vg-closed sets, vg-continuity, vg-lindeloff space.

AMS Subject Classification numbers: 54D05, 54D20, 54D30, 54D45.

1. Introduction

In this paper *vg*-lindeloffness in topological space are introduced. Some of their basic properties are obtained and interrelations are verified with other types of lindeloffness.

2. Preliminaries:

Definition 2.1: A subset $A\subset X$ is said to be

(i) regular open if $A= int(cl(A))$
(ii) semi-open[*v*-open] if there exists an open[r-open]set *s* such that $U\subset A\subset cl(U)$.
(iii) regular closed[semi-closed; *v*-closed] of its complement is regular open[semi-open *v*-open].
(iv) g-closed [rg-closed] if $cl(A)\subset U$ [$rcl(A)\subset U$] whenever $A\subset U$ and U is open [r-open].
(v) *v*g-closed if $vcl(A)\subset U$ whenever $A\subset U$ and U is *v*-open
(vi) g-open[rg-open; *v*g-open] if its complement is g-closed[rg-closed; *v*g-closed].

Definition 2.2: Let $A\subset X$. A point $x\in X$ is said to be ω-accumulation [*v*-accumulation; *rg*-accumulation] point of *A* if every regular-open [*v*-open; rg-open] neighborhood of *x* intersects *A* and the union of *A* and the set of all ω-accumulation [*v*-accumulation; *rg*-accumulation] points of *A* is called ω-closed [*v*-closed; *rg*-closed] set[4-10].

Definition 2.3: A⊂ X is said to be
(i) Lindeloff[nearly-lindeloff; *v*-lindeloff] if every open[regular-open; *v*-open] cover of A has a countable subcover.

(ii) Countably lindeloff[countably nearly-lindeloff; countably *v*-lindeloff] if every countable open[countable regular-open; countable *v*-open]

in A has a countable sub cover.
(iii) σ-lindeloff[σ-nearly-lindeloff; σ-*v*-lindeloff] if A is the countable union of lindeloff[nearly-lindeloff; *v*-lindeloff] spaces
(iv) Weak almost regular [Almost regular] iff for any point $a \in A$ and any regular-open set U containing a, there exist a regular-open [an open] set V such that $a \in V \subset cl\ V \subset U$.

3. Properties of vg-Lindeloff spaces:

Definition 3.1: $A \subset X$ is said to be
(i) *vg*-lindeloff if every *vg*-open cover of A has a countable subcover.
(ii) Countably *vg*-lindeloff if every countable *vg*-open cover in A has a countable sub cover.
(iii) σ-*vg*-lindeloff if A is the countable union of *vg*-lindeloff spaces[11-14]

Example: Any closed and bounded subset of $\Re$ with usual topology is *vg*-lindeloff and $\Re$ with usual topology is not *vg*-lindeloff.

Definition 3.2: Let $S \subset X$. A point $x \in X$ is said to be *vg*-accumulation point of S if every *vg*-open neighborhood of x intersects S. The union of S and the set of all *vg*-accumulation points of S is called *vg*-closed set.

Remark 1: Every ω-closed set is *vg*-closed.

Theorem 3.1: Let A be r-open. $A \subseteq X$ is a *vg*-lindeloff subset of X iff the subspace (A, τ_A) is *vg*-lindeloff.

Theorem 3.2:
(i) *vg*-closed subset of a (countably) *vg*-lindeloff space is (countably) *vg*-lindeloff
(ii) *vg*-irresolute image of a (countably) *vg*-lindeloff space is (countably) *vg*-lindeloff
(iii)countable product and countable union of (countably) *vg*-lindeloff spaces is (countably) *vg*-lindeloff

Proof: Let $\{X_n\}$ be a countable family of *vg*-lindeloff spaces and let $X = \Pi_n X_n$. Let $\{U_j = \Pi_{\alpha \neq \alpha ij} X_\alpha \times U_{\alpha 1j} \times \ldots \times U_{\alpha nj}: U_{\alpha ij}$ is *vg*-open in $X_{\alpha ij}$ for each i = 1 to n, $j \in I\}$ be a *vg*-open cover of $\Pi_\alpha X_\alpha$. Then $\{\Pi_i(u_j):j \in I\}$ is a *vg*-open cover of X_i. By Assumption, there exists a countable subfamily $\{\Pi_i(u_j):j= 1 \text{ to } n\}$ such that $X_i = \cup\ \Pi_i(U_j)$.

Case 1: If $\Pi_i X_i = \cup_{j=1}{}^n \Pi_i(U_j)$ then $\Pi_i X_i$ is *vg*-lindeloff.

Case 2: If not, there exists atmost countable $l_1, l_2, \ldots, l_n$ such that $X_{ls} = \cup_{k=1s}{}^m \Pi_{ls}(U_{ks})$ for each $l_s \in \{l_1, l_2, \ldots, l_n\}$. Therefore $\Pi_\alpha X_\alpha = \cup_{k=1}{}^n (U_{jk}) \cup \cup_{k1=1}{}^n (U_{jk}) \cup \ldots \ldots \ldots \cup \cup_{ks=1}{}^n (U_{jk})$. Hence ΠX_α is *vg*-lindeloff.

Remark 2: (countably) *vg*-lindeloffness is a weakly hereditary property

Theorem 3.3:
(i) *vg*-continuous image of a (countably) *vg*-lindeloff space is (countably)lindeloff
(ii) *vg*-continuous image of a (countably) *vg*-lindeloff space is (countably)nearly-lindeloff

Definition 3.4: X is said to be locally *vg*-lindeloff space if every $x \in X$ has a *vg*-neighborhood whose closure is *vg*-lindeloff.

Note 1: *v*-compact space $\Rightarrow$ *vg*-compact space locally *vg*-compact space $\Leftarrow$ locally *v*-compact space

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v-lindeloff space $\Rightarrow$ *vg*-lindeloff space $\Rightarrow$ locally *vg*-lindeloff space $\Leftarrow$ locally *v*-lindeloff space

Theorem 3.4: If *f*: X $\rightarrow$ Y is *vg*-irresolute, *vg*-open and X is locally *vg*-lindeloff, then so is Y.

Proof: For y$\in$ Y, $\exists$ x$\in$ X $\ni$ *f*(x) = y. Since X is locally *vg*-lindeloff x has a *vg*-lindeloff neighborhood V. By *vg*-irresolute, *vg*-open of *f*, *f*(V) is a *vg*-lindeloff neighborhood of y. Hence Y is locally *vg*-lindeloff.

Corollary 3.1: If *f*: X $\rightarrow$ Y is *vg*-irresolute, *vg*-open and X is *vg*-lindeloff, then Y is locally *vg*-lindeloff.

Theorem 3.5: A$\subseteq$ X be r-open. Then A is locally *vg*-lindeloff subset of X iff the subspace (A, τ_A) is locally *vg*-lindeloff.

Theorem 3.6: (i) *vg*-closed subset of a locally *vg*-Lindeloff space is locally *vg*-Lindeloff.
(ii) countable product and countable union of locally *vg*-Lindeloff spaces is locally *vg*-Lindeloff.

Theorem 3.7: The following are equivalent:
(i) X is *vg*-lindeloff.
(ii) For every family of *vg*-closed sets in X with empty intersection, there is countable subfamily whose intersection is empty.
(iii) Every family of *vg*-closed sets with countable intersection property has a non-empty intersection.

Theorem 3.8: ***Alexandroff's Subbase theorem for vg-lindeloff spaces:*** ΠX_i is *vg*-lindeloff if and only if every X_i is *vg*-lindeloff.

Proof: Assume $\Pi_i X_i$ is *vg*-lindeloff and Fix j$\in$ I. Let P_j: $\Pi_i X_i \rightarrow X_j$ be a projection and $\{V_j^{\,i} : i\in I\}$ be a *vg*-open cover of X_j. Then $\{\Pi_{i\neq j} X_i \times V_j^{\,i} : i\in I\}$ is a *vg*-open cover of $\Pi_i X_i$. Since $\Pi_\alpha X_\alpha$ is *vg*-lindeloff, there exist a countable subfamily such that $\Pi_i X_i = \cup_{=1}{}^{n}(\Pi_{i\neq j} X_i \times V_j)$. By projection P_j:$X_j = \cup_{i=1}{}^{n} V_j^{\,i}$. Therefore X_j is *vg*-lindeloff.

Converse part follows from theorem 3.2(iii)

Theorem 3.9: If A is an arbitrary *vg*-lindeloff subset of X, then every infinite subset of A has a *vg*-accumulation point.

Proof: Let A be an infinite subset of a *vg*-lindeloff space X such that $D_{vg}(A) = \phi$ and so A is *vg*-closed in X. If B is any infinite subset of A, then for each b$\in$ B there exists a *vg*-open set V_b containing b such that V_b contains no point of A other than b. Now the family $\{V_b : b\in B\}$ forms a *vg*-open cover of B. By Theorem 3.2, B itself is *vg*-lindeloff, but any countable subfamily of $\{V_b : b\in B\}$ do not cover B, which is a contradiction. Therefore the infinite subset A of X has a *vg*-accumulation point.[3-9]

Theorem 3.10: If S is an arbitrary *vg*-lindeloff subset of X, then every infinite subset of S has a ωg-accumulation point.

Proof: Consequence of theorem 3.9.

Theorem 3.11: Let X be a *vg*-lindeloff space, and let $\{S_i\}$ be a descending chain of ωg-closed subsets of X, then $\cap_{n \geq 1} S_n \neq \phi$.

Proof: Choose a point $x_n \in S_n$ for each n = 1, 2, ... then x_n will have a ωg-accumulation point x_0 in X, since X is *vg*-lindeloff, X is *rg*-lindeloff and hence it is ωg-closed. On the other hand, for each n = 1,2, ...; x_0 becomes a ωg-accumulation point of $\{x_k, x_{k+1}, \ldots \ldots\}$ also, hence of S_k. Since each S_k is ωg-closed, we know that $x_0 \in S_k$ for each k, hence the intersection of all S_k is not empty

Theorem 3.12: If *f*: X → Y is almost continuous, X is *v*-lindeloff and Y = *f*(X) then Y is *vg*-lindeloff.

Proof: Follows from Thm[2] 3.7, and Note 1.

Theorem 3.14: The *vg*-irresolute image of a *vg*-lindeloff space in any *vg*-Hausdorff space is *vg*-closed.

Corollary 3.2: The *v*-irresolute image of a *v*-lindeloff space in any *v*-Hausdorff space is *vg*-closed.

4. *Relation between vg-lindeloff and lindeloff spaces:*

Definition 4.0: An open base is said to be a regular open base if the elements of the base are regular open sets.

Definition 4.1: A⊂ X is said to be Almost *vg*-regular iff for any point a∈ A and any *vg*-open set U containing a, there exist a *vg*-open set V such that a∈ V⊂ cl V⊂ U.

Lemma 4.1: If X is *vg*-lindeloff and semiregular then X is lindeloff.

Proof: Let $\{O_i : i \in I\}$ be an open cover of X. Since X is semiregular, there is a regular open base $\mathscr{B} \Rightarrow$ we have *v*-open cover $\{B^j_i : O_i = \cup_i B^j_i$ for each i, where $B^j_i \in \mathscr{B}\}$, which in turn a *vg*-open cover. By *vg*-lindeloffness of X, $X \subset \cup^{n}_{k=1} B^{j}_{ik} \Rightarrow \cup^{n}_{k=1} O_k$. Therefore X is lindeloff.

Lemma 4.2: If X is *vg*-compact and semiregular then X is lindeloff.

Theorem 4.1: (i) If A⊂ X is Almost *vg*-regular and X is *v*-lindeloff, then cl(A) is *vg*-lindeloff.
(ii) If A⊂ X is Almost *v*-regular and lindeloff, then cl(A) is *vg*-lindeloff.

Proof: (i) Let $\{U_i\}$ be any *v*-open cover of A and let x A be any point. For x∈ A there exists a *vg*-open set U_x containing x ⇒ by almost *vg*-regularity there exists a vg-open set V_x such that $x \in V_x \subset cl\{V_x\} \subset U$. For $\{V_x\}$ forms a *vg*-open cover and X is *vg*-lindeloff, $X = \cup^{n}_{i=1} V_{xi}$. Hence $cl\{A\} \subseteq cl\{(\cup^{n}_{i=1} V_{xi})\} = \cup^{n}_{i=1} cl\{(V_{xi})\} \subseteq \cup^{n}_{i=1} U_{xi}$, which implies that cl(A) is *vg*-lindeloff.

(ii) From theorem[3] 4.1 and Note 1 cl(A) is *vg*-lindeloff.

Corollary 4.1: (i) If A⊂ X is Almost *vg*-regular and X is *v*-compact, then cl(A) is *vg*-lindeloff.
(ii) If A⊂ X is Almost *v*-regular and compact, then cl(A) is *vg*-lindeloff.

Theorem 4.2: Every almost v-regular and almost lindeloff subset A of X is *vg*-lindeloff.

Proof: By theorem[3] 4.5 and Note 1, A is *vg*-lindeloff.

Theorem 4.3: Every weak almost regular and nearly lindeloff subset A of X is *vg*-lindeloff.

Proof: Follows from theorem[3] 4.7 and Note 1, A is *vg*-lindeloff.

Corollary 4.2: (i) Every almost v-regular and almost compact subset A of X is *vg*-lindeloff.
(ii) Every weak almost regular and nearly compact subset A of X is *vg*-lindeloff.
(iii) If $A \subset X$ is Almost regular and nearly lindeloff, then cl(A) is *vg*-lindeloff.
(iv) If $A \subset X$ is weak almost regular and nearly lindeloff, then cl(A) is *vg*-lindeloff.

Proof: (i) By theorem[11] 4.2 and Note 1, A is *vg*-lindeloff.
(ii) By theorem[11] 4.3 and Note 1, A is *vg*-lindeloff.
(iii) & (iv) By theorem[3] 4.3 and Note 1 cl(A) is *vg*-lindeloff.

Corollary 4.3: Every weak almost regular and *vg*-lindeloff subset A of X is *vg*-lindeloff.

Corollary 4.4: Every weak almost regular and *vg*-compact subset A of X is *vg*-lindeloff.

Proof: Follows from Corollary[11] 4.2 and Note 1.

Theorem 4.4: Let A be any dense almost *vg*-regular subset of X such that every *vg*-open covering of A is a *vg*-open covering of X. Then X is almost lindeloff if and only if X is *vg*-lindeloff.

Theorem 4.5: Each *vg*-lindeloff metrizable space is finite.

5. Relation between vg-lindeloff and weakly lindeloff spaces:

Theorem 5.1: If X is weakly lindeloff and almost regular, then X is *vg*-lindeloff.

Proof: Let $\{V_i\}$ be any *vg*-open cover of X. For each $x \in X$, there exists $i_x \in I$ such that $x \in V_{ix}$. Since X is almost regular, there exists a regular open set G_{ix} such that $x \in G_{ix} \subset cl\{G_{ix}\} \subset V_{ix}$ and G_{ix} are open. Since X is weakly lindeloff, $X = \cup^{n}_{i=1} cl\{G_{ix}\}$. Thus $X = \cup^{n}_{i=1} V_{ix}$. Hence X is *vg*-lindeloff.

Following two corollaries are immediate consequences of above theorem and the proofs are thus omitted.

Corollary 5.1: An almost regular space X is weakly lindeloff if and only if X is *vg*-lindeloff.

Corollary 5.2: A Hausdorff space X is almost regular and weakly lindeloff if and only if X is *vg*-lindeloff.

Theorem 5.2: If X is weakly compact and almost regular, then X is *vg*-lindeloff.

Proof: By theorem[11] 5.1 and Note 1, X is *vg*-lindeloff.

6. Covering properties of Weak and Strong continuous functions:

Theorem 6.1: Almost *vg*-continuous image of a (countably) *vg*-lindeloff space is nearly (countably) lindeloff.

Note 2: Every almost *vg*-lindeloff space is *vg*-lindeloff and hence locally *vg*-lindeloff

Theorem 6.2: If f is c.*vg*.c.[resp: c.r.c] surjection and X is *vg*-lindeloff, then Y is closed lindeloff.

Proof: Let $\{G_i : i\in I\}$ be any closed cover for Y. Since f is c.*vg*.c., $\{f^{-1}(G_i)\}$ forms a *vg*-open cover for X with a countable subcover, since X is *vg*-lindeloff. For $Y = f(X) = \cup^{n}_{i=1} G_i$, Y is closed lindeloff.

Theorem 6.3: If f is c.*vg*.c., surjection. Then the following statements hold:
(i) If X is locally *vg*-lindeloff, then Y is locally closed lindeloff[locally nearly closed lindeloff; locally mildly lindeloff.]
(ii) If X is *vg*-closed[countably *vg*-closed], then Y is nearly lindeloff[nearly countably lindeloff].
(iii) If X is *vg*-lindeloff, then Y is nearly closed lindeloff; mildly closed lindeloff.[7-14]

Theorem 6.4: If f is sl.*vgc*. surjection and X is *vg*-lindeloff, then Y is lindeloff.

Proof: Let $\{G_i : i\in I\}$ be any clopen cover for Y. Since f is sl.*vgc*., $\{ f^{-1}(G_i)\}$ is a *vg*-open cover for X with countable subcover as X is *vg*-lindeloff. Since f is surjection, $Y = f(X) = \cup^{n}_{i=1} G_i$. Therefore Y is lindeloff.

Corollary 6.1: If f:X→ Y is sl.*vgc*. surjection and X is locally *vg*-compact, then Y is locally lindeloff.

Proof: Follows from theorem 6. 4 and Note 1.

Theorem 6.5: If f:X→ Y is sl.*vgc*., surjection and X is *vg*-lindeloff then Y is mildly lindeloff.

Proof: Let $\{U_i : i\in I\}$ be clopen cover for Y. For each $x\in X$, there exists $\alpha_x \in I$ such that $f(x) \in U_{\alpha x}$ and $V_x \in v GO(X, x)$ such that $f(V_x) \subset U_{\alpha x}$. Since $\{V_i : i\in I\}$ is a cover of X by *vg*-open sets, there exists a countable subset I_0 of I such that $X \subset \cup \{V_x : x\in I_0\}$. Thus $Y \subset \cup \{f(V_x) : x\in I_0\} \subset \cup \{U_{\alpha x} : x\in I_0\}$. Hence Y is mildly lindeloff.

Corollary 6.2: If f is sl.*vg*.c., surjection and X is locally *vg*-lindeloff then Y is locally mildly lindeloff.

Theorem 6.6: If f is al.c.*vg*.c.[resp: al.c.r.c] surjection and X is *vg*-lindeloff, then Y is nearly closed lindeloff.

Proof: Let $\{G_i : i\in I\}$ be any regular-closed cover for Y. Since f is al.c.*vg*.c., $\{f^{-1}(G_i)\}$ forms a *vg*-open cover for X and hence have a countable subcover, since X is *vg*-lindeloff. Since f is surjection, $Y = f(X) = \cup^{n}_{=1} G_i$. Therefore Y is nearly closed lindeloff.

Corollary 6.3: If f is al.c.*vg*.c., surjection, then the following statements hold:
(i) If X is locally *vg*-lindeloff, then Y is locally nearly closed lindeloff; [locally mildly lindeloff.]

(ii) If X is *vg*-closed[countably *vg*-closed], then Y is nearly lindeloff[nearly countably lindeloff].

Theorem 6.7: If f is al.c.*vg*.c., surjection and X is *vg*-lindeloff then Y is mildly closed lindeloff[mildly lindeloff].

Conclusion

In this paper author discussed *vg*-lindeloffness and studied its basic properties.

Acknowledgement

Author of the paper thank the referees for their suggestions and corrections for the development of the paper.

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