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Setting up a Public Utility Service-Application of graph theory

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Abstract

The concepts like distance, eccentricity, radius etc. are very common in graph theory. Applying these concepts, Prajapati[3] has designed a model to set up a fire station in a town. This paper is an extension of this idea and it deals with how can a public utility service in a certain region be made available to the various places in another region in the most democratic way. It also discusses at which place a public utility service shall be set up when two regions are connected by a bridge.

Key words : Graph, distance, walk, path, eccentricity, radius, bridge, connectedness.

Introduction

1. Preliminaries

A ***graph*** G is a pair G=(V, E) consisting of a finite set V and a set E of two element subsets of V. The elements of V are called ***vertices*** and the elements of E are called ***edges***. Two vertices u and v of G are said to be ***adjacent*** if there is an edge $e = (u, v) \in E$. Two edges are said to be ***adjacent*** if they have a common vertex. A ***walk*** in G is a finite sequence $v_0\ e_1\ v_1\ e_2\ v_2 \ldots\ldots v_{k-1}\ e_k\ v_k$ whose terms are alternatively vertices and edges. The number of edges in the walk is called ***length*** of the walk. If $v_0, v_1, \ldots\ldots, v_k$ are distinct, the walk is called a ***path***. A vertex u is said to be ***connected*** to a vertex v if there is a path in G from u to v. A graph G is called ***connected*** if every two of its vertices are connected.

1.1. Definition[1,2,4]. For any two vertices u and v connected by a path in a graph G, the ***distance*** between u and v denoted by d(u,v) is defined as the length of the shortest u – v path.

In a connected graph the distance function d is a metric.

1.2. Definition[1,2,4]. Let G be a connected graph with vertex set V. For any $v \in V$, the ***eccentricity*** of v denoted by e(v) is defined by

$$e(v) = \max \{d(u, v) : u \in V, u \neq v\}.$$

The ***radius*** of G denoted by r(G) is defined by

$$r(G) = \min \{e(v) : v \in G\}.$$

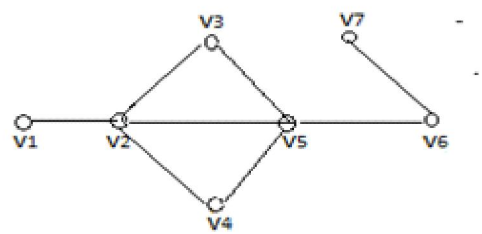
A vertex v is said to be a ***central vertex*** of G if $e(v) = r(G)$.

2. Pre-requisites of a public utility service:

Whenever it is planned to set up a public utility service like a hospital or a fire station etc., it should be easily accessible to people living in nearby localities and is in the most democratic way without any locality being discriminated. In other words it should be set up in the most convenient place in the most democratic way.

To do so imagine the various nearby places together with the link roads as a graph. Find that vertex which is maximum nearer to all other vertices. The eccentricity is a measure of nearness. We define ***accessibility*** of a vertex v as $a(v) = -e(v)$.

2.1. Example: Consider the following graph



In this example the following is the matrix of distances and corresponding accessibilities

v	v_1	v_2	v_3	v_4	v_5	v_6	v_7	e(v)	a(v)
v_1	0	1	2	2	3	4	5	5	-5
v_2	1	0	1	1	2	3	4	4	-4
v_3	2	1	0	2	1	2	3	3	-3
v_4	2	1	2	0	1	2	3	3	-3
v_5	3	2	1	1	0	1	2	3	-3
v_6	4	3	2	2	1	0	1	4	-4
v_7	5	4	3	3	2	1	0	5	-5

It is evident from the table of distances that v_3 is a vertex which is maximum nearer to other vertices where as v_1 is not, because, there is a place whose distance from v_1 is 5 units. Like v_3, the vertices v_4 and v_5 are also having maximum accessibility to other vertices. Note that these are central vertices. Hence in order to achieve maximum accessibility to other vertices, we have to find the central vertices and establish the public utility service at a central vertex.

The second pre-requisite of our planning is that it should be maximum democratic. But this is attained only if no vertex is too near to the point of establishment or too far from the point of establishment. In other words the difference between the highest and lowest distances, that is, the range of distances should be least. In the following theorem, we prove that the range of distances is least if and only if the point of establishment is a central vertex ■

2.2. Theorem. Let G be a connected graph and v be a vertex of G. Then the range of distances of the remaining vertices from v is least if and only if v is a central vertex.

Proof. Let v be a vertex of G. Let $v_1, v_2, \ldots\ldots, v_k$ be the remaining vertices of G. Let their distances from v be $d_1, d_2, \ldots\ldots., d_k$. Let $d = \max \{d_1, d_2, \ldots\ldots., d_k\}$. Then $d = e(v)$. Since the graph is connected, c has an immediate neighbour whose distance from v is 1. Hence $\min \{d_1, d_2, \ldots\ldots., d_k\} = 1$. Therefore, the range of distances from $v = d-1$.

Consider a vertex different from v say v_i. Suppose the distances of the remaining vertices $v_1, v_2, \ldots\ldots, v_{i-1}, v, v_{i+1}, \ldots\ldots.., v_k$ be

$e_{i1}, e_{i2}, \ldots\ldots, e_{ik}$. Let $e_i = \max\{e_{i1}, e_{i2}, \ldots\ldots, e_{ik}\}$. Then $e_i = e(v_i)$. Also $\min\{e_{i1}, e_{i2}, \ldots\ldots, e_{ik}\} = 1$. Therefore range of distances from $v_i = e_i - 1$.
Now v is a central vertex $\Leftrightarrow e(v) \le e(v_i)\ \forall\ i \Leftrightarrow d \le e_i\ \forall\ i \Leftrightarrow d - 1 \le e_i - 1\ \forall\ i \Leftrightarrow$ range of distances from v ≤ range of distances from $v_i\ \forall$ i. This proves the theorem ■

From the above theorem, it is clear that the choice of a central vertex for establishing a public utility service ensures high democracy in addition to maximum accessibility.

If there are more than one central vertex for a graph, again there is an uncertainty about which central vertex be chosen for setting up the public utility service. If we are determined to avoid unnecessary delay and unnecessary expenses by travelling more, we choose that central vertex to which the sum of the distances of the remaining vertices is lowest.

In example 2.1, the vertices v_3, v_4 and v_5 are central vertices. The respective sum of distances towards the remaining vertices are $S_{v_3} = 11, S_{v_4} = 11, S_{v_5} = 10$. Clearly the sum of distances is lowest if we choose v_5. Hence v_5 is the best choice for setting up the utility service.

3. Accessibility of a public utility service in a certain regain to the various places in another region:

Here we examine how can we made accessible the public utility service in a certain region in the most democratic way to the various places in another region by making a bridge between the two. The following theorem gives the answer to this question.

3.1. Theorem. Let G be a connected graph and w be a disconnected vertex. If a bridging is made between G and w, then the accessibility of G to w is maximum if the bridging is made with a central vertex of G.

Proof. Let c be a central vertex of G and v be another vertex not a central vertex. Let G_1 be the graph obtained by making a bridge between w and v and G_2 be the graph obtained by making a bridge between w and c. Let e, e_1 and e_2 be the eccentricity functions and a, a_1 and a_2 be the accessibility functions of G, G_1 and G_2 respectively. Then

$$e(v) = \max\{d(u, v) : u \in G\}$$

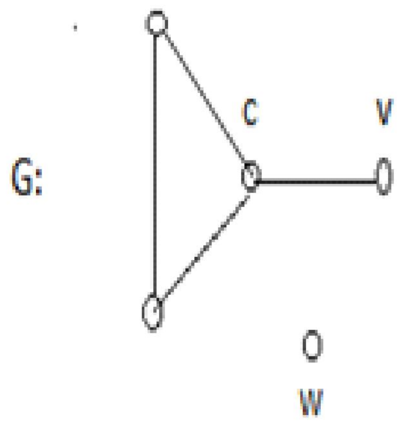
$$\begin{aligned} e_1(w) &= \max\{d(u, w) : u \in G_1\} \\ &= \max\{d(u, v) : u \in G\} \\ &= \max\{d(u, v) : u \in G\} + 1 \quad (\because d(v, w) = 1 \\ &\quad \text{and any path from w to u in } G_1 \text{ first} \\ &\quad \text{enters v}) \\ &= e(v) + 1 \end{aligned}$$

Similarly,

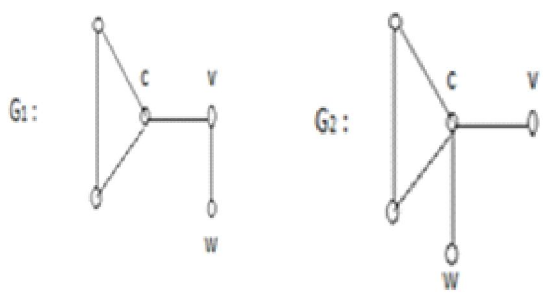
$$e_2(w) = e(c) + 1$$

Since c is a central vertex, $e(c) \le e(v)$. Hence $e(c) + 1 \le e(v) + 1$. i.e., $e_2(w) \le e_1(w)$, hence $a_1(w) \le a_2(w)$. In other words the accessibility of w to G is maximum when the bridge is made between w and a central vertex of G ■

3.2. Example. Consider the following graph G and a disconnected vertex w.



Clearly c is a central vertex of G and v is another vertex not a central vertex. The graph G_1 obtained by making a bridge between w and v and the graph G_2 obtained by making a bridge between w and c are shown below.

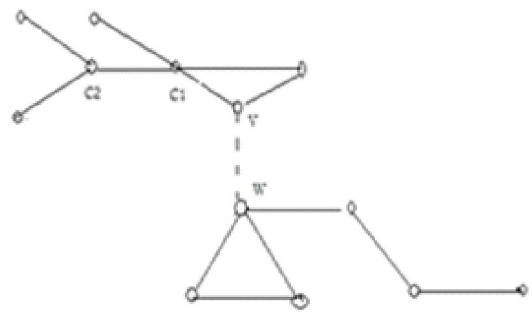


Not that $a_1(w) = -3$ and $a_2(w) = -2$. Thus accessibility of w to G is maximum when the bridge is made between w and c ■

The following result is an immediate consequence of the above theorem

3.3. Theorem. Let G and H be two connected graphs. Then the accessibility of G to a vertex w of H is maximum if G is connected to H by bridging w with a central vertex of G.

3.4. Example. Suppose G and H are two graphs. Let there be a public utility service at the vertex w of H. To make accessible the various places of G to w, we shall bridge G and H.



fig(i)

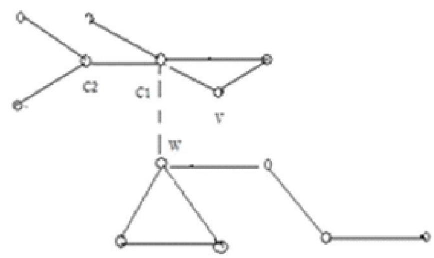


fig (ii)

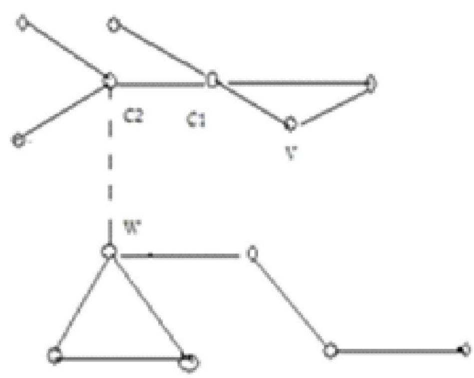
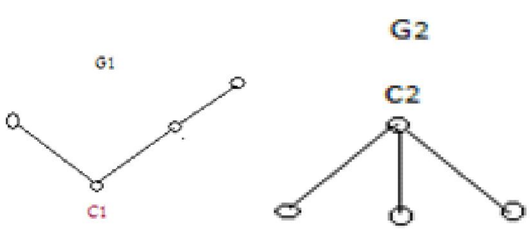


fig (iii)

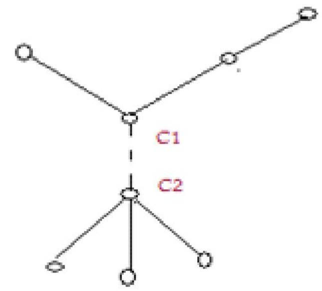
Consider the vertices v, c_1 and c_2 of G. The vertex v is non-central where as c_1 and c_2 are central vertices of G. In fig (i) the bridge is between w and v. The distances of various places of G to w are 4, 4, 3, 3, 2, 1, 2. In fig (ii), the bridge is between w and c_1. The distances of various places of G in this case are 3, 3, 2, 2, 1, 2, 2. In fig (iii), the bridge is between w and c_2. The distances of various places of G in this case are 2, 2, 1, 2, 3, 3, 3. Clearly, accessability to w is more if bridging is done between w and a central vertex than bridging is done between w and a non-central vertex of G. Again, in both fig (ii) and fig (iii),the accessibility to w is the same. But the respective sum of distances towards w are 15 and 16. Hence bridging of w to c_1 is the best choice ■

4. Setting up a public utility service in the join of two graphical regions.

4.1. Example. Consider the following two graphs G_1 and G_2.



Clearly c_1 is a central vertex of G_1 and c_2 is a central vertex of G_2. Let G be the graph obtained by joining G_1 and G_2 by making a bridge between c_1 and c_2.



The distances of various vertices in the joined organization from c_1 and c_2 are respectively 1, 1, 2, 1, 2, 2, 2 and 1, 1, 1, 1, 2, 2, 3. Therefore, $e(c_1) = 2$, $e(c_2) = 3$, hence $a(c_1) = -2$, $a(c_2) = -3$. It is therefore clear that setting a utility service at c_1 is more convenient than setting it at c_2. On further examination, it can be seen that c_2 is a central vertex of the joined organization ■

The following theorem substantiates this finding.

4.2. Theorem. Let G_1 and G_2 be two connected graphs. Let c_1 and c_2 be central vertices of G_1 and G_2 respectively. Then c_1 or c_2 is a central vertex of the graph G obtained by making a bridge between c_1 and c_2.

Proof. Let e_1 and e_2 be the eccentricity functions of G_1 and G_2 respectively. Let $e_1(c_1) = k_1$ and $e_2(c_2) = k_2$. Suppose $k_1 < k_2$.

Consider the joined organization G obtained by making a bridge between c_1 and c_2. Since the maximum span of c_2 to G_1 is $k_1 + 1$ and $k_1 + 1 \leq k_2$, we have

$$e(c_2) = k_2 \text{..........(1)}$$

Consider any vertex v of G_2 other than c_2.

Since c_2 is a central vertex of G_2, $e_2(c_2) \leq e_2(v)$. That is $k_2 \leq e_2(v)$. Also $e(v) \geq e_2(v)$. Therefore from (1),

$$e(c_2) \quad e(v) ....(2).$$

Again the maximum span of c_1 to G_2 is $k_2 + 1$ and $e_1(c_1) = k_1$ and that $k_1 < k_2$, we have

$$e(c_1) = k_2 + 1 ......(3)$$

Finally, consider any vertex u of G_1 other than c_1. Since $d(u, c_1) \geq 1$ and maximum span of c_1 to G_2 is $k_2 + 1$,

$$e(u) \geq k_2 + 2 ......(4).$$

Now from (1), (2), (3) and (4),

$r(G) = k_2 = e(c_2)$. Hence c_2 is a central vertex of G.

On the other hand if $k_2 < k_1$, then $r(G) = k_1 = e(c_2)$ and c_1 would be central vertex of G. Finally, if $k_1 = k_2$, then both c_1 and c_2 are central vertices of G and $r(G) = k_1 + 1 = k_2 + 1$. Thus c_1 or c_2 will always be a central vertex of G ■

4.3. Remark. If both c_1 andc_2 are central vertices of G, then that central vertex to which the sum of distances of the remaining vertices is lowest is the best choice for setting up the public utility service.

References

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