Ultra Scientist Vol. 25(3)A, 373-380 (2013).

L(d,1) – labeling of join of path and cycle and join of complete graph with path and cycle

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(Acceptance Date 6th September, 2013)

Abstract

Given a graph G and a positive integer d, an L(d, 1)-labeling of G is a function f that assigns to each vertex of G a non-negative integer such that if two vertices u and v are adjacent, then $|f(u)-f(v)| \geq d$; if u and v are not adjacent but there is a two-edge path between them, then $|f(u)-f(v)| \geq 1$. The L(d, 1)-number of G, $\lambda_d(G)$, is defined as the minimum m such that there is an L(d, 1)-labeling f of G with f(V) $\subseteq$ {0, 1, 2,…, m}.

Motivated by the channel assignment problem introduced by Hale, the L(2, 1)-labeling and the L(1, 1)-labeling (as d=2 and 1, respectively) have been studied extensively in the past decade.

This article extends the study to all positive integers d.The aim of this paper is to determine the λ_d-number of the join of path and cycle and two graphs, $K_m + P_n$ and $K_m + C_n$.

Key words: L(d,1)-labeling, λ_d-number, join of two graphs.

Introduction

The assignment of frequencies to television and radio transmitters subject to restrictions imposed by the distance between transmitters is known as the Channel assignment problem.

The L(2,1)-labeling problem proposed by Griggs and Roberts is a variation of the frequency assignment problem (or the T-coloring problem) introduced by Hale. Suppose we are given a number of transmitters or stations. The L(2,1)-labeling problem is to assign frequencies (nonnegative integers) to the transmitters so that "close" transmitters must receive different frequencies and "very close" transmitters must receive frequencies that are at least two frequencies apart. Instead of the condition of two frequencies apart, Chang *et al.*[2] considered the condition of d frequencies apart.

In 1988 Roberts proposed a variation of the channel assignment problem in which distinction is made between close and very close transmitters. This variation led Griggs and Yeh[10] to the problem of labeling the vertices of a graph with a condition at distance two. They investigated the λ-number of paths, cycles, multipartite graphs, hypercubes and established a bound on the λ-number of trees. They also investigated the relation between $\lambda(G)$ and other graph invariants such as $\chi(G)$, $\Delta(G)$ and $|V(G)|$.

A recent survey on labeling graphs[15] by Yeh is very useful.

In[5], Georges and Mauro considered L(j,k)-labelings, a generalization of L(2,1)-labelings, where j and k are positive integers $j \geq k$ and gave the value of $\lambda_{p,q}(G)$ where G is a cycle or a cartesian product of paths where at most one of the paths is P_2. They examined the sizes of graphs in G(n,k) in[6], where G(n,k) denote the set of all graphs with order n and λ-number k. They explored the relationship between the λ-number of a graph G and the path-covering number of G^c [8]. They determined $\lambda_k^j(K_n \times K_m)$ for all j,k,m,n in[9]. They later used a coding theory approach in[14] to improve the upper bound on the λ-number of the hypercube obtained by Griggs and Yeh. Several properties of $\lambda_{j,k}$ have been investigated[2,5,7].

Sakai[12] considered the λ-number of chordal graphs and intersection graphs. Chang and Kuo[1] investigated classes of interval graphs, established a polynomial-time algorithm that λ-labels trees, and obtained an upper bound for $\Delta^2+\Delta$ for $\lambda(G)$.

Chang and Liaw[4] considered the case when the transmitters have direction constraints. In this case, they studied the L(2,1)-labeling on digraphs, and gave results on L(2,1)-labelings for ditrees. Chang *et al.*[3] gave the exact values of the L(d,1)-labeling numbers for ditrees T. Lin and Lam[11] determined $\lambda(K_m x K_n)$ for m, $n \geq 2$, $\lambda(K_m x P_n)$ for $m \geq 3$, $n \geq 1$, and $\lambda(K_m x C_n)$ for $m \geq 3$ and some special values of n.

L(2,1)-labeling of graphs form an important model for the channel assignment problem. It is more applicable and extensively studied over the past decade[13].

1. Definitions :

A $L(d,1)$-labeling of a graph *G* is a function *f* from the vertex set *V(G)* to the set of all nonnegative integers such that

$$|f(u)-f(v)| \geq 1 \quad \text{if} \quad d(u,v)=2$$
$$|f(u)-f(v)| \geq d \quad \text{if} \quad d(u,v)=1.$$

For a nonnegative integer *k*, a *k*-$L(2,1)$-labeling is an $L(d,1)$-labeling such that no label is greater than *k*. The $L(d,1)$-labeling number of G, denoted by $\lambda_d(G)$, is the smallest number k such that G has a k-$L(d,1)$-labeling.

The $L(d,1)$-labeling problem is to find the $L(d,1)$-labeling number $\lambda_d(G)$ of a graph G, which is the minimum cardinality k such that G has a k-$L(d,1)$-labeling.

2. Previous results :

Theorem 2.1.[5]

If $d \geq 2$, then $\lambda_1^d(G) \leq \frac{d}{2}\lambda_1^2(G)$.

Georges and Mauro[5] determined $L(j,k)$ for paths (when $j \geq k$) as follows:

Theorem 2.2.[5]

If $\frac{j}{k} \geq 2$, then

$$\lambda_{j,k}(P_n) = \begin{cases} j & \text{if } n = 2 \\ j+k & \text{if } n = 3 \text{ or } n = 4 \\ j+2k & \text{if } n \geq 5 \end{cases}$$

If $\frac{j}{k} \leq 2$, then

$$\lambda_{j,k}(P_n) = \begin{cases} j & \text{if } n = 2 \\ j+k & \text{if } n = 3 \text{ or } n = 4 \\ 2j & \text{if } n \geq 5 \end{cases}$$

As an immediate consequence we have:

Corollary 2.3 :

If $d = 1$, then

$$\lambda_{d,1}(P_n) = \begin{cases} 1 & \text{if } n = 2 \\ 2 & \text{if } n \geq 3 \end{cases}$$

If $d \geq 2$, then

$$\lambda_{d,1}(P_n) = \begin{cases} d & \text{if } n = 2 \\ d+1 & \text{if } n = 3 \text{ or } n = 4 \\ d+2 & \text{if } n \geq 5 \end{cases}$$

Georges and Mauro determined $L(j,k)$ for cycles (when $j \geq k$) as follows:

Theorem 2.4 [5]*:*

If $\frac{j}{k} \leq 2$, then

$$\lambda_{j,k}(C_n) = \begin{cases} 2j & \text{if } n \equiv 0 \pmod 3 \\ 4k & \text{if } n = 5 \\ j+2k & \text{otherwise} \end{cases}$$

If $\frac{j}{k} \geq 2$, then

$$\lambda_{j,k}(C_n) = \begin{cases} 2j & \text{if } n \text{ is odd and } n \geq 3 \\ j+2k & \text{if } n \equiv 0 \pmod 4 \\ 2j & \text{if } n \equiv 2 \pmod 4 \text{ and } \frac{j}{k} \leq 3 \\ j+3k & \text{if } n \equiv 2 \pmod 4 \text{ and } \frac{j}{k} \geq 3 \end{cases}$$

As an immediate consequence we have:

Corollary 2.5 :

If $d = 1$, then

$$\lambda_{d,1}(C_n) = \begin{cases} 2 & \text{if } n = 0 \pmod 3 \\ 4 & \text{if } n = 5 \\ 3 & \text{otherwise} \end{cases}$$

If $d = 2$, then

$$\lambda_{d,1}(C_n) = 4$$

If $d \geq 3$, then

$$\lambda_{d,1}(C_n) = \begin{cases} 2d & \text{if } n \text{ is odd} \\ d+2 & \text{if } n \equiv 0 \pmod 4 \\ d+3 & \text{if } n \equiv 2 \pmod 4 \end{cases}$$

3. Join of C_m and P_n

Theorem 3.1.

If $d \geq 3$, then

$$\lambda_1^d(C_m + P_n) = \begin{cases} 4d + 2 & \text{if } m \text{ is odd}, \\ 3d + 4 & \text{if } m = 0 \bmod 4 \\ 3d + 5 & \text{if } m = 2 \bmod 4 \end{cases}$$

Proof:

If $d = 1$, the result follows by inspection.

For $d = 2$ and $d = 3$ we have proved the join of the graphs earlier. Hence we consider $d \geq 4$.

In P_n for $n \leq 4$, the results follow by inspection.

Now we consider $n \geq 5$.

For the value of m in C_m, we consider the three cases in turn.

Case 1. m is odd.

Suppose there exists a labeling L of C_m with span $2d - 1$. Then each label of L is in precisely one of $X_1=\{0, 1, 2, ..., d - 1\}$ and $X_2=\{d, d + 1, d + 2, ..., 2d - 1\}$.

Since m is odd, there exist two adjacent vertices whose labels are in the same set, contradicting the distance one condition. Hence, $\lambda_1^d(C_m) \geq 2d$.

But by theorem 2.1, $\lambda_1^d(C_m) \leq \frac{d}{2}\lambda_1^2(C_m) = 2d$.

Therefore, $\lambda_1^d(C_m) = 2d$

The join of the two graphs C_m and P_n consists of $C_m \cup P_n$ and all edges between vertices of C_m and vertices of P_n.

We start labeling P_n with $2d+d$, by the condition of $L(d,1)$ labeling. Then we may label the vertices of P_n thus:

$3d, 3d+d+2, 3d+1, 3d+d+1, 3d, 3d+d+2, 3d+1, 3d+d+1, 3d, 3d+d+2, 3d+1, 3d+d+1,....$

Hence $\lambda_1^d(C_m + P_n) = 4d + 2$.

Case 2. $m = 0$ (mod 4).

If $m > 5$, then

$$\lambda_1^d(C_m) \geq \lambda_1^d(P_m) = d + 2.$$

To demonstrate a labeling of C_m with span $d+2$,

Let $C_m=W_1, W_2, W_3,....,W_m$, and assign the respective labels $0, d, 1, d+2, 0, d, 1, d+2,...$

Therefore, $\lambda_1^d(C_m) = d + 2$

The join of the two graphs C_m and P_n consists of $C_m \cup P_n$ and all edges between vertices of C_m and vertices of P_n.

We start labeling P_n with $d+2+d$, by the condition of $L(d,1)$ labeling. Then we may label the vertices of P_n thus:

$2d+2, 2d+2+d+2, 2d+2+1, 2d+2+d+1, 2d+2, 2d+2+d+2, 2d+2+1, 2d+2+d+1,....$

Hence $\lambda_1^d(C_m + P_n) = 3d + 4$.

Case 3. m = 2 (mod 4).

Let v_1, v_2, and v_3 denote vertices in C_m such that v_2 is adjacent to each of v_1 and v_3.

Suppose that L is an $L(d,1)$– labeling of C_m with span $d + 3$. Then each label of L is in precisely one of $X_1=\{0\}$, $X_2=\{1\}$, $X_3=\{2, 3,...., d\}$, $X_4=\{d + 1\}$, $X_5=\{d + 2\}$.

If $L(v_2) \in \{2, 3, ..., d-1\} \subseteq X_3$, then $L(v_1)$ and $L(v_3)$ must be greater than $L(v_2)$. Hence, the larger of $L(v_1)$ and $L(v_3)$ is at least $d + 3$, a contradiction of the span of L.

If $L(v_2) \in \{d\} \subseteq X_3$, then the condition $d \geq 3$ implies that $L(v_1)$ and $L(v_3)$ must be less than $L(v_2)$. The smaller of $L(v_1)$ and $L(v_3)$ is therefore at most -1, a contradiction.

Therefore, no vertex in $V(C_m)$ has label in X_3.

Furthermore, due to the distance one condition, neither $X_1 \cup X_2$ nor $X_4 \cup X_5$ contains labels of adjacent vertices. Thus, if we denote C_m by $W_1, W_2, W_3, ..., W_m$, then label membership alternates between $X_1 \cup X_2$ and $X_4 \cup X_5$.

Without loss of generality, suppose that $X_1 \cup X_2$ contains the labels of $W_{1i+1,\ 0 \leq i \leq \frac{n}{2}-1}$.

Then for $0 \leq i \leq \frac{n}{2} - 2$, the distance two condition requires that the labels of W_{2i+1} and W_{2i+3} alternate between X_1 and X_2.

Without no loss of generality, suppose $L(W_1) \in X_1$.

Then $L(W_{n-1}) \in X_1$ due to the condition $n = 2 \bmod 4$, which violates the distance two condition. Hence, $\lambda_1^d(C_m) \geq d + 3$.

To demonstrate a labeling of C_m with span $d + 3$, assign respective labels

$0, d + 1, 1, d + 2, 0, d + 1, 1, d + 2, ..., 0, d + 1, 1, d + 2, 2, d + 3$.

Hence, $\lambda_1^d(C_m) = d + 3$.

The join of the two graphs C_m and P_n consists of $C_m \cup P_n$ and all edges between vertices of C_m and vertices of P_n.

We start labeling P_n with $d+3+d$, by the condition of $L(d,1)$ labeling. Then we may label the vertices of P_n thus:

$2d + 3, 2d+3+d+2, 2d + 3 + 1, 2d + 3 + d + 1, 2d+3, 2d+3+d+2, 2d+3+1, 2d+3+d+1, ...$

Hence $\lambda_1^d(C_m + P_n) = 3d + 5$.

We illustrate λ_1^d - labelings of C_m for each of the three cases given in theorem 3.1.

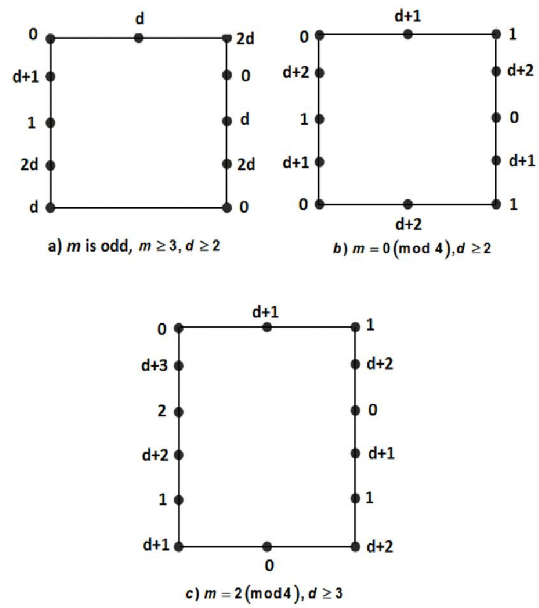

Figure 3.1

l_1^d-labeling of C_m

4. L(d,1)-labeling of join of K_m with C_m and P_n

In this section, we find the $L(d,1)$-labeling of join of K_m with C_m and P_n.

Theorem 4.1

If K_m is a complete graph and P_n is a path on n vertices, then

$$\lambda_d(K_m+P_n)=\begin{cases} d(m+1), & if\ n=2, \\ d(m+1)+1, & if\ n=3,4, \\ d(m+1)+2, & if\ n\geq 5. \end{cases}$$

Where $m(\geq 3)$ is an integer.

Proof:

If $n\leq 4$, the results follow by inspection. So consider $n \geq 5$.

Let $G = K_m + P_n$.

We know that $\lambda_d(K_m)=d(m-1)$.

The join of the two graphs K_m and P_n consists of $K_m \cup P_n$ and all edges between vertices of K_m and vertices of P_n.

By the definition of $L(d,1)$-labeling, we start labeling the vertex of the path by the next label $d(m-1)+d$.

Case (i): Assume $n = 2$. We shall show that $\lambda_d(G)=d(m-1)$.

By corollary 2.3, $\lambda_d(P_2) = d$,

It follows that in any $L(d,1)$ labeling of $K_m + P_n$,

$$\lambda_d(G)=d(m-1)+d+d$$
$$\Rightarrow \quad \lambda_d(G)=d(m+1)$$

Case (ii) : Assume n = 3 or 4.

We shall show that $\lambda_d(G)=d(m+1)+1$.

By corollary 2.3, $\lambda_d(P_3)=\lambda_d(P_4)=d+1$,

It follows that in any $L(d,1)$ labeling of $K_m + P_n$,

$$\lambda_d(G)=d(m-1)+d+d+1$$
$$\Rightarrow \quad \lambda_d(G)=d(m+1)+1.$$

Case (iii): Assume $n \geq 5$.

We shall show that $\lambda_d(G)=d(m+1)+2$.

By corollary 2.3, $\lambda_d(P_n)=d+2$, for $n \geq 5$,

It follows that in any $L(d,1)$ labeling of K_m+P_n,

$$\lambda_d(G)=d(m-1)+d+d+2$$
$$\Rightarrow \quad \lambda_d(G)=d(m+1)+2.$$

Therefore the theorem is proved.

Theorem 4.2 :

If K_m is a complete graph and C_n is a cycle on n vertices, then, for $m,n\geq 3$,

$$\lambda_d\left(K_m+C_n\right)=\begin{cases} d(m+2), & if\ n\ is\ odd \\ d(m+1)+2, & if\ n\equiv 0 \pmod 4 \\ d(m+1)+3, & if\ n\equiv 2 \pmod 4 \end{cases}$$

Proof:

Let $G = K_m + C_n$.

We know that $\lambda_d(K_m)=d(m-1)$.

The join of the two graphs K_m and C_n consists of $K_m \cup C_n$ and all edges between vertices of K_m and vertices of C_n.

By the definition of $L(d,1)$-labeling, we start labeling the vertex of the cycle by the next label $d(m-1)+d$.

Case(i): n is odd

By corollary 2.5, in this case, $\lambda_d(C_n)=2d$.

So it follows that in any $L(d,1)$ labeling of K_m+C_n,

$$\lambda_d(G)=d(m-1)+d+2d$$
$$\Rightarrow \quad \lambda_d(G)=d(m+2).$$

Case(ii): $n \equiv 0(\text{mod}\,4)$

By corollary 2.5, in this case, $\lambda_d(C_n)=d+2$.

So it follows that in any $L(d,1)$ labeling of K_m+C_n,

$$\lambda_d(G)=d(m-1)+d+d+2$$
$$\Rightarrow \quad \lambda_d(G)=d(m+1)+2.$$

Case(iii): $n \equiv 0(\text{mod}\,4)$

By corollary 2.5, in this case, $\lambda_d(C_n)=d+3$.

So it follows that in any $L(d,1)$ labeling of K_m+C_n,

$$\lambda_d(G)=d(m-1)+d+d+3$$
$$\Rightarrow \quad \lambda_d(G)=d(m+1)+3.$$

Therefore the theorem is proved.

Conclusion

We have made an attempt to establish ë-number for some well known graphs such as join of paths and cycles, join of complete graphs with paths and cycles. We have also extended our result for L(3,1)-labeling and L(d, 1)-labeling. We hope that it holds good for L(j,k)-labeling.

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