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Generalized Bernoulli polynomials and their basic properties

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Abstract

The Bernoulli polynomials have important applications in number theory and classical analysis. They appear in the integral representation of differential periodic functions. Since they are employed for approximating such function in terms of polynomials[3]. The object of the present paper is to prove some basic properties of generalized Bernoulli polynomials.

Keywords: Bernoulli polynomial, Bernoulli number, generalized Bernoulli polynomials, generating function.

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1. Introduction

For a real or complex parameter, the generalized Bernoulli polynomials $B_n^{\alpha}(x)$ of degree n in x as well as in α, are defined by [Norlund 4,chap.6] the generating function

$$\frac{t^{\alpha} e^{xt}}{(e^t-1)^{\alpha}} = \sum_{n=0}^{\infty} B_n^{\alpha}(x)\frac{t^n}{n!} \quad ,|t| \le 2\Pi \tag{1.1}$$

2. A recurrence relation of $B_n^{\alpha}(x)$:

In this section, we derive the following recurrence relation for $B_n^{\alpha}(x)$

$$B_n^{\alpha}(x) = \sum_{k=0}^{n} \binom{n}{k} B_n^{\alpha} x^{n-k} \tag{2.1}$$

Proof : By (1.1) we get

$$\frac{t^{\alpha}\left[1+xt+\frac{x^2t^2}{2!}+\frac{x^3t^3}{3!}+\ldots\ldots\ldots\ldots+\frac{x^nt^n}{n!}\right]}{\left[1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\ldots\ldots\ldots\ldots\ldots\ldots+\frac{t^n}{n!}-1\right]^{\alpha}}=\sum_{n=0}^{\infty}B_n^{\alpha}(x)\frac{t^n}{n!}$$

$$1+xt+\frac{x^2t^2}{2!}+\frac{x^3t^3}{3!}+\ldots\ldots\ldots\ldots+\frac{x^nt^n}{n!}=\left[1+\frac{t}{2!}+\frac{t^2}{3!}+\ldots\ldots\ldots\ldots\ldots\ldots+\frac{t^n}{(n+1)!}\right]^{\alpha}\cdot\sum_{n=0}^{\infty}B_n^{\alpha}(x)\frac{t^n}{n!}$$

By the Binomial expansion, we get

$$B_0^{\alpha}(x)=1$$

$$B_1^{\alpha}(x)=x-\frac{\alpha}{2}$$

$$B_2^{\alpha}(x)=x^2-\alpha\left(x-\frac{\alpha}{2}\right)-\frac{\alpha}{3}-\frac{\alpha(\alpha-1)}{4}$$

And so on.
Clearly, we have

$B_n^1(x)=B_n(x)$ And these can be deduced by the recurrence relation.
We know that the generating function of generalized Bernoulli polynomial

$$\frac{t^{\alpha}e^{xt}}{(e^t-1)^{\alpha}}=\sum_{n=0}^{\infty}B_n^{\alpha}(x)\frac{t^n}{n!}$$

$$\sum_{n=0}^{\infty}B_n^{\alpha}\frac{t^n}{n!}\cdot\sum_{n=0}^{\infty}\frac{x^nt^n}{n!}=\sum_{n=0}^{\infty}B_n^{\alpha}(x)\frac{t^n}{n!}$$

By the Cauchy product rule for multiplying power series

$$B_n^{\alpha}(x)=\sum_{k=0}^{n}\binom{n}{k}B_n^{\alpha}x^{n-k}$$

Particular case: It is interesting to note that (2.1) reduces to the well-known recurrence relation of Bernoulli polynomial for α = 1.

$$B_n(x)=\sum_{k=0}^{n}\binom{n}{k}B_k\,x^{n-k} \qquad (2.2)$$

3. Properties of generalized Bernoulli polynomial:

In this section some of well-known properties of generalized Bernoulli polynomials are derived from the generating function (1.1)

Property-1: (Differentiation) for the differentiation of generalized Bernoulli polynomial can be used the relations

$$\frac{d}{dx}B_n^{\alpha}(x)=nB_{n-1}^{\alpha}(x) \qquad (3.1)$$

Proof: By the (1.1) we get

$$\frac{t^{\alpha}e^{xt}}{(e^{t}-1)^{\alpha}}=\sum_{n=0}^{\infty}B_{n}^{\alpha}(x)\frac{t^{n}}{n!}$$

Differentiating above equation with respect to x

$$\frac{t.t^{\alpha}e^{xt}}{(e^{t}-1)^{\alpha}}=\sum_{n=0}^{\infty}\frac{d}{dx}B_{n}^{\alpha}(x)\frac{t^{n}}{n!}$$

$$t\sum_{n=0}^{\infty}B_{n}^{\alpha}(x)\frac{t^{n}}{n!}=\sum_{n=0}^{\infty}\frac{d}{dx}B_{n}^{\alpha}(x)\frac{t^{n}}{n!}$$

Equating the coefficient of $\frac{t^{n}}{n!}$

Hence

$$\frac{d}{dx}B_{n}^{\alpha}(x)=nB_{n-1}^{\alpha}(x)$$

Particular case: When α=1, we get the ordinary Bernoulli polynomial

$$\frac{d}{dx}B_{n}(x)=nB_{n-1}(x) \tag{3.2}$$

Property-2 : (Differences) for $n \geq 1$

$$nB_{n-1}^{\alpha-1}(x)=B_{n}^{\alpha}(x+1)-B_{n}^{\alpha}(x) \tag{3.3}$$

Proof: We know that

$$\frac{t^{\alpha}e^{xt}}{(e^{t}-1)^{\alpha}}=\sum_{n=0}^{\infty}B_{n}^{\alpha}(x)\frac{t^{n}}{n!} \tag{3.4}$$

Replace $x\to x+1$ in the above equation.

$$\frac{t^{\alpha}e^{(x+1)t}}{(e^{t}-1)^{\alpha}}=\sum_{n=0}^{\infty}B_{n}^{\alpha}(x+1)\frac{t^{n}}{n!} \tag{3.5}$$

Subtracting equation (3.5) from equation (3.4)

$$\frac{t^{\alpha}e^{(x+1)t}}{(e^{t}-1)^{\alpha}}-\frac{t^{\alpha}e^{xt}}{(e^{t}-1)^{\alpha}}=\sum_{n=0}^{\infty}\left[B_{n}^{\alpha}(x+1)-B_{n}^{\alpha}(x)\right]\frac{t^{n}}{n!}$$

$$\frac{t^{\alpha-1+1}e^{xt}(e^{t}-1)}{(e^{t}-1)^{\alpha-1+1}}=\sum_{n=0}^{\infty}\left[B_{n}^{\alpha}(x+1)-B_{n}^{\alpha}(x)\right]\frac{t^{n}}{n!}$$

$$\frac{t.t^{\alpha-1}e^{xt}}{(e^{t}-1)^{\alpha-1}}=\sum_{n=0}^{\infty}\left[B_{n}^{\alpha}(x+1)-B_{n}^{\alpha}(x)\right]\frac{t^{n}}{n!}$$

$$t\sum_{n=0}^{\infty}B_{n}^{\alpha-1}(x)\frac{t^{n}}{n!}=\sum_{n=0}^{\infty}\left[B_{n}^{\alpha}(x+1)-B_{n}^{\alpha}(x)\right]\frac{t^{n}}{n!}$$

$$\sum_{n=0}^{\infty}B_{n}^{\alpha-1}(x)\frac{t^{n+1}}{n!}=\sum_{n=0}^{\infty}\left[B_{n}^{\alpha}(x+1)-B_{n}^{\alpha}(x)\right]\frac{t^{n}}{n!}$$

Equating the coefficient of $\frac{t^{n}}{n!}$

$$nB_{n-1}^{\alpha-1}(x)=B_{n}^{\alpha}(x+1)-B_{n}^{\alpha}(x)$$

Particular case: α = 1, furthermore,

Since $B_{n}^{0}(x)=x^{n}$, therefore we get the difference property of ordinary Bernoulli polynomial[5].

$$B_{n}(x+1)-B_{n}(x)=nx^{n-1} \tag{3.6}$$

Property-3:

$$B_{n}^{\alpha}(\alpha-x)=(-1)^{n}B_{n}^{\alpha}(x) \tag{3.7}$$

Proof: By the generating function (1.1)

$$\frac{t^{\alpha}e^{xt}}{(e^{t}-1)^{\alpha}}=\sum_{n=0}^{\infty}B_{n}^{\alpha}(x)\frac{t^{n}}{n!}$$

Now put $x\to\alpha-x$ in above equation

$$\frac{t^{\alpha} e^{(\alpha-x)t}}{(e^{t}-1)^{\alpha}} = \sum_{n=0}^{\infty} B_n^{\alpha}(\alpha-x)\frac{t^n}{n!}$$

$$\frac{t^{\alpha} e^{\alpha t}.e^{-xt}}{(e^{t})^{\alpha}\left[1-e^{-t}\right]^{\alpha}} = \sum_{n=0}^{\infty} B_n^{\alpha}(\alpha-x)\frac{t^n}{n!}$$

$$\frac{t^{\alpha} e^{-xt}}{-\left[e^{-t}-1\right]} = \sum_{n=0}^{\infty} B_n^{\alpha}(\alpha-x)\frac{t^n}{n!}$$

$$\sum_{n=0}^{\infty} B_n^{\alpha}(x)\frac{(-t)^n}{n!} = \sum_{n=0}^{\infty} B_n^{\alpha}(\alpha-x)\frac{t^n}{n!}$$

Equating the coefficient of $\frac{t^n}{n!}$

$$B_n^{\alpha}(\alpha-x) = (-1)^n B_n^{\alpha}(x)$$

Particular case: (I) When α = 1 we get the ordinary property of Bernoulli polynomial

$$B_n(1-x) = (-1)^n B_n(x)$$

(II) When $x = 0$ in 3.3, then

$$B_n^{\alpha}(\alpha) = (-1)^n B_n^{\alpha}(0) = (-1)^n B_n^{\alpha}$$

In terms of generalized Bernoulli number [1, page 227, exercise 18]

Property -4 : (Addition theorem) from the generating function

$$B_n^{(\alpha+\beta)}(x+y) = \sum_{k=0}^{n}\binom{n}{k} B_k^{\alpha}(x) B_{n-k}^{\beta}(y) \quad (3.8)$$

Proof: By the (1.1), we get

$$\frac{t^{\alpha} e^{xt}}{(e^{t}-1)^{\alpha}} = \sum_{n=0}^{\infty} B_n^{\alpha}(x)\frac{t^n}{n!}$$

Now put $\alpha = \alpha + \beta$ in above generating function

$$\frac{t^{\alpha+\beta} e^{xt}}{(e^{t}-1)^{\alpha+\beta}} = \sum_{n=0}^{\infty} B_n^{\alpha+\beta}(x)\frac{t^n}{n!}$$

Put $x \to x+y$

$$\frac{t^{\alpha+\beta} e^{(x+y)t}}{(e^{t}-1)^{\alpha+\beta}} = \sum_{n=0}^{\infty} B_n^{\alpha+\beta}(x+y)\frac{t^n}{n!}$$

$$\frac{t^{\alpha} e^{xt}}{(e^{t}-1)^{\alpha}} \cdot \frac{t^{\beta} e^{yt}}{(e^{t}-1)^{\beta}} = \sum_{n=0}^{\infty} B_n^{\alpha+\beta}(x+y)\frac{t^n}{n!}$$

$$\sum_{m=0}^{\infty} B_m^{\alpha}(x)\frac{t^m}{m!} \sum_{n=0}^{\infty} B_n^{\beta}(y)\frac{t^n}{n!} = \sum_{n=0}^{\infty} B_n^{\alpha+\beta}(x+y)\frac{t^n}{n!}$$

By the Cauchy product rule for multiplying power series

$$B_n^{\alpha+\beta}(x+y) = \sum_{k=0}^{n}\binom{n}{k} B_k^{\alpha}(x) B_{n-k}^{\beta}(y)$$

Particular case: Which for $x = \beta = 0$, corresponds to the elegant representation

$$B_n^{\alpha}(x) = \sum_{k=0}^{n}\binom{n}{k} B_k^{\alpha} x^{n-k\backslash} \quad (3.9)$$

For the generalized Bernoulli polynomials as a finite sum of the generalized Bernoulli numbers[6]

Property-5: $B_n^{\alpha}(x+y)=\sum_{k=0}^{n} B_k^{\alpha}(x)y^{n-k}$ (3.10)

Proof: By the (1.1) we get

$$\frac{t^{\alpha}e^{xt}}{(e^t-1)^{\alpha}}=\sum_{n=0}^{\infty}B_n^{\alpha}(x)\frac{t^n}{n!}$$

Now put $x\to x+y$, we get

$$\frac{t^{\alpha}e^{(x+y)t}}{(e^t-1)^{\alpha}}=\sum_{n=0}^{\infty}B_n^{\alpha}(x+y)\frac{t^n}{n!}$$

$$\frac{t^{\alpha}e^{xt}\cdot e^{yt}}{(e^t-1)^{\alpha}}=\sum_{n=0}^{\infty}B_n^{\alpha}(x+y)\frac{t^n}{n!}$$

$$\sum_{n=0}^{\infty}B_n^{\alpha}(x)\frac{t^n}{n!}.\sum_{k=0}^{\infty}\frac{y^n t^n}{n!}=\sum_{n=0}^{\infty}B_n^{\alpha}(x+y)\frac{t^n}{n!}$$

By the Cauchy product rule for multiplying power series

$$B_n^{\alpha}(x+y)=\sum_{k=0}^{n} B_k^{\alpha}(x)y^{n-k}$$

Now put y = 1 in (3.10), we get

$$B_n^{\alpha}(x+1)=\sum_{k=0}^{n}\binom{n}{k}B_k^{\alpha}(x) \quad (3.11)$$

Particular case:- (i) α = 1, we get the recurrence relation in terms of Bernoulli polynomials and numbers[4]

$$B_n(x+1)=\sum_{k=0}^{n}\binom{n}{k}B_k(x) \quad (3.12)$$

(II) x = 0 in (3.12) Then

$$B_n(1)=B_n(0)=B_n \quad (3.13)$$

Property-6: A recurrence relation

$$B_n^{\alpha+1}(x)=\left(1-\frac{n}{\alpha}\right)B_n^{\alpha}(x)+n\left(\frac{x}{\alpha}-1\right)B_{n-1}^{\alpha}(x)$$
(3.14)

Proof: We know that the generalized function

$$\frac{t^{\alpha}e^{xt}}{(e^t-1)^{\alpha}}=\sum_{n=0}^{\infty}B_n^{\alpha}(x)\frac{t^n}{n!}$$

Differentiating above equation with respect to t, we get

$$\sum_{n=0}^{\infty}\frac{nt^{n-1}}{n!}B_n^{\alpha}(x)=\left[\frac{\alpha t^{\alpha-1}e^{xt}}{(e^t-1)^{\alpha}}+\frac{\alpha t^{\alpha-1}xe^{xt}}{(e^t-1)^{\alpha}}-\frac{\alpha t^{\alpha}e^{xt}e^{t}}{(e^t-1)^{\alpha+1}}\right]$$

Multiplying t both side

$$\sum_{n=0}^{\infty}nB_n^{\alpha}(x)\frac{t^n}{n!}=\frac{\alpha t^{\alpha}e^{xt}}{(e^t-1)^{\alpha}}+\frac{xt^{\alpha+1}e^{xt}}{(e^t-1)^{\alpha}}-\frac{\alpha t^{\alpha+1}e^{(x+1)t}}{(e^t-1)^{\alpha+1}}$$

$$\sum_{n=0}^{\infty}nB_n^{\alpha}(x)\frac{t^n}{n!}=\alpha\sum_{n=0}^{\infty}B_n^{\alpha}(x)\frac{t^n}{n!}+xt\sum_{n=0}^{\infty}B_n^{\alpha}(x)\frac{t^n}{n!}-\alpha\sum_{n=0}^{\infty}B_n^{\alpha+1}(x+1)\frac{t^n}{n!}$$

Equating the coefficient of $\frac{t^n}{n!}$

$$nB_n^{\alpha}(x)=\alpha B_n^{\alpha}(x)+nxB_{n-1}^{\alpha}(x)-\alpha B_n^{\alpha+1}(x+1)$$
(3.15)

But we know that by the (3.3)

$$B_n^{\alpha+1}(x+1)=nB_{n-1}^{\alpha}(x)+B_n^{\alpha+1}(x) \quad (3.16)$$

Put the value of (3.16) in (3.15), we get

$$nB_n^{\alpha}(x) = \alpha B_n^{\alpha}(x) + nxB_{n-1}^{\alpha}(x) - \alpha\left[nB_{n-1}^{\alpha}(x) - B_n^{\alpha+1}(x)\right]$$

$$nB_n^{\alpha+1}(x) = (\alpha - n)B_n^{\alpha}(x) + (x-\alpha)nB_{n-1}^{\alpha}(x)$$

$$B_n^{\alpha+1}(x) = (1-\frac{n}{\alpha})B_n^{\alpha}(x) + (\frac{x}{\alpha}-1)nB_{n-1}^{\alpha}(x)$$

Particular case: (I) putting $x = 0$ in (3.14)

$$B_n^{\alpha+1} = \left(1-\frac{n}{\alpha}\right)B_n^{\alpha} + n\left(\frac{x}{\alpha}-1\right)B_{n-1}^{\alpha} \quad (3.17)$$

(II) Putting $x = 0$ in (3.15)

$$B_n^{\alpha+1}(1) = \left(1-\frac{n}{\alpha}\right)B_n^{\alpha} \quad (3.18)$$

On writing $\alpha \to \alpha + n$ in (3.18)

$$B_n^{\alpha+n+1}(1) = \left(\frac{\alpha}{\alpha+n}\right)B_n^{\alpha+n} \quad (3.19)$$

(III) Now putting $\alpha = 1$ in 3.14, we get ordinary Bernoulli polynomial[2]

$$B_n^{2}(x) = (1-n)B_n(x) + n(n-1)B_{n-1}(x) \quad (3.20)$$

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