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Degree of Approximation of Functions Belonging to the Generalized Lipschitz Class By Product Means of its Fourier series

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Abstract

In this paper,a new theorem on the degree of approximation of functions belonging to the class Lip(ξ(t),r) class by Euler and Matrix (E,q)*A*-product means of the Fourier series has been established.

Key words: Degree of approximation, Lip(ξ(t),r) class of function, (E,q) mean, *A* -mean, (E,q) *A* -Product means, Fourier series, Lebesgue integral.

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1.Introduction

The degree approximation of functions belonging to Lip α, Lip (α,p), Lip(ξ(t),p) and W(L_p,ξ(t)) classes by using Cesáro, Nörlund and generalized Nörlund summability methods has been discussed number of researchers like Chandra[1], Holland[3], Holland and Sahney[4], Khan[5], Lal and Tripathi[7], Qureshi[9,10]. Thereafter Lal and Kushwaha[6] discussed the degree of approximation of Lipschitz Function by (C,1)(E,q) product Summability methods, Padhy *et al.*[8] discussed the degree of approximation of Fourier series by the product means (E,q)A. Therefore, In the present paper, a new theorem on the degree of approximation of a function belonging to the Lip(ξ(t),r) class by (E,q)*A* -product means of Fourier series. Our theorem extands the result of Padhy *et al.*[8] on degree of approximation of Fourier by the product means (E,q)*A*.

2. Definition and Notation:

Let f (x) be a 2π- periodic function and integrable in the Lebesgue sense. The Fourier series f (x) is given by

$$f(x) \sim \frac{1}{2}a_o + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x) \tag{2.1}$$

with n^{th} partial sum s_n (f ; x).

A function $f \in$ Lip α, if

$$f(x+t)-f(x) = O\left(|t|^{\alpha}\right) \quad ,\text{for } 0<\alpha\le 1 \tag{2.2}$$

A function $f(x) \in$Lip (α, r), for $0 \le x \le 2\pi$ if

$$\left(\int_0^{2\pi} |f(x+t)-f(x)|^r\, dx\right)^{\frac{1}{r}} = O\left(|t|^{\alpha}\right),$$

$$0<\alpha\le 1, r\ge 1 \tag{2.3}$$

Given a positive increasing function $\xi(t)$ and integer $r \ge 1$, $f \in$ Lip $(\xi(t), r)$, if

$$\left(\int_0^{2\pi} |f(x+t)-f(x)|^r\, dx\right)^{\frac{1}{r}} = O\left(\xi(t)\right) \tag{2.4}$$

L_r- norm is defined by

$$\|f\|_r = \left(\int_0^{2\pi} |f(x)|^r\, dx\right)^{1/r}, \quad r\ge 1 \tag{2.5}$$

L_∞-norm of a function f : R→R is defined by

$$\|f\|_\infty = \sup\left\{|f(x): x\in R|\right\} \tag{2.6}$$

The degree of approximation of a function f: R→R by trigonometric polynomials t_n of order n is defined by zygmund[13]

$$\| t_n - f \|_\infty = Sup \left\{ | t_n(x) - f(x) | : x \in R \right\} \tag{2.7}$$

Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series with the sequence of its n^{th} partial sums $\{s_n\}$ and $A = (a_{mn})$ be infinite matrix satisfying Töeplitz[12] conditions of regularity, *i.e.*

$$\lim_{m\to\infty} a_{mn} = 0 \tag{2.8}$$

$$\lim_{m\to\infty} \sum_{n=0}^{\infty} a_{mn} = 1 \tag{2.9}$$

$$\sup_m \sum_{n=0}^{\infty} |a_{mn}| < M \quad , \text{ where M is constant.} \tag{2.10}$$

Then the sequence-to-sequence transformation

$$t_n = \sum_{\upsilon=0}^{n} a_{m\upsilon} s_\upsilon \tag{2.11}$$

defines the sequence $\{t_n\}$ of matrix *A*- mean of a sequence $\{s_n\}$ generated by the sequence of coefficients (a_{mn}). The series $\sum_{n=0}^{\infty} u_n$ is said to be *A*-summable to the sum s by matrix method if limit of t_n is exists and is equal to s (Zygmund[13]) and we write

$t_n \to s$, as $n\to\infty$.

The sequence-to-sequence transformation (Hardy[2]),

$$T_n = \frac{1}{(1+q)^n} \sum_{\upsilon=0}^{n} \binom{n}{\upsilon} q^{n-\upsilon} s_\upsilon \tag{2.12}$$

defines the sequence $\{T_n\}$ of the (E,q) means of the sequence $\{s_n\}$.

If $T_n \to s$, as $n \to \infty$, then the series $\sum_{n=0}^{\infty} u_n$ is said to be (E,q) summable to the sum s. Cleary (E,q) method is regular.

The (E,q) transform of the *A*- transform of $\{s_n\}$ is defined by

$$\tau_n = \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}t_k$$

$$\tau_n = \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\sum_{\upsilon=0}^{k}a_{k\upsilon}\, s_\upsilon\right\} \quad (2.13)$$

If $\tau_n \to s$ as $n \to \infty$, then the series $\sum_{n=0}^{\infty}u_n$ is said to be (E,q)*A*- summable to the sum s. we use the following notations:

$$(i) \qquad \phi(t) = f(x+t)+f(x-t)-2f(x)$$

$$(ii) \qquad K_n(t) = \frac{1}{2\pi(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\sum_{\upsilon=0}^{k}a_{k\upsilon}\frac{\sin\left(\upsilon+\frac{1}{2}\right)t}{\sin t/2}\right\}$$

3.Known Theorem: Dealing with the degree of approximation by the product mean (E,q)*A* of Fourier series, Padhy *et al.*[8] proved the following theorem:

Theorem 3.1: Let $A = (a_{mn})_{\infty\times\infty}$ be a regular matrix . If *f* is a $2\pi-$ periodic function of class Lip α, then the degree of approximation by the product (E,q)*A*- summability means of its Fourier series is given by

$$\|\tau_n - f\|_\infty = O\left(\frac{1}{(n+1)^\alpha}\right), \quad 0<\alpha<1,$$

$$\text{where} \quad \tau_n = \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\sum_{\upsilon=0}^{k}a_{k\upsilon}\, s_\upsilon\right\}.$$

4. Main Theorem: In this paper our aim is to generalize the above result of Padhy *et al.*[8], In fact, we prove the following theorem:

Let $A = (a_{mn})$ be an infinite matrix. If *f* is a 2π- periodic, Lebesgue integrable function , belongs to Lip($\xi(t)$,r) class, then the degree of approximation by (E,q)*A*- summability means of its Fourier series (2.1) is given by[2]

$$\|\tau_n - f(x)\|_r = O\left[(n+1)^{\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right] \quad (4.1)$$

Provided $\xi(t)$ satisfies the following conditions:

$$\left\{\int_0^{\frac{1}{n+1}}\left(\frac{t\,|\phi(t)|}{\xi(t)}\right)^r dt\right\}^{1/r} = O\left(\frac{1}{n+1}\right) \quad (4.2)$$

and

$$\left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{t^{-\delta}\,|\phi(t)|}{\xi(t)}\right)^r dt\right\}^{1/r} = O\left((n+1)^\delta\right) \quad (4.3)$$

Where δ is an arbitrary number such that $s(1-\delta)-1>0$, $\frac{1}{r}+\frac{1}{s}=1$ Such that $1\le r<\infty$, Conditions (4.2) and (4.3) holds uniformly in *x*.

5. Lemmas: For the proof of our theorem, we required following lemmas:

Lemma 5.1: $|K_n(t)| = O(n+1)$; *for* $0 \le t \le \frac{1}{n+1}$.

Proof

For $0 \le t \le \frac{1}{n+1}$, we have $\sin nt \le n \sin t$

then

$$|K_n(t)| = \frac{1}{2\pi(1+q)^n}\left|\left[\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\sum_{\upsilon=0}^{k}a_{k\upsilon}\frac{\sin\left(\upsilon+\frac{1}{2}\right)t}{\sin t/2}\right\}\right]\right|$$

$$\leq \frac{1}{2\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\sum_{\upsilon=0}^{k}a_{k\upsilon}\frac{(2\upsilon+1)\sin t/2}{\sin t/2}\right\}\right|$$

$$\leq \frac{1}{2\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}(2k+1)\left\{\sum_{\upsilon=0}^{k}a_{k\upsilon}\right\}\right|$$

$$\leq \frac{M(2n+1)}{2\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\right|$$

$$= O(n+1)$$

This complete proof of the lemma (5.1).

Lemma 5.2 : $|K_n(t)| = O\left(\frac{1}{t}\right), for\ \frac{1}{n+1} \leq t \leq \pi.$

Proof:

For $\frac{1}{n+1} \leq t \leq \pi$, we have by Jordans Lemma,

$\sin\left(\frac{t}{2}\right) \geq \frac{t}{\pi}$, sin $nt \leq 1$, then

$$,|K_n(t)| = \frac{1}{2\pi(1+q)^n}\left|\left[\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\sum_{\upsilon=0}^{k}a_{k\upsilon}\frac{\sin\left(\upsilon+\frac{1}{2}\right)t}{\sin t/2}\right\}\right]\right|$$

$$\leq \frac{1}{2\pi(1+q)^n}\left|\left[\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\sum_{\upsilon=0}^{k}a_{k\upsilon}\frac{1}{t/\pi}\right\}\right]\right|$$

$$= \frac{\pi}{2\pi t(1+q)^n}\left|\left[\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\sum_{\upsilon=0}^{k}a_{k\upsilon}\right\}\right]\right|$$

$$\leq \frac{M}{2t(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\right|$$

$$= O\left(\frac{1}{t}\right)$$

This complete proof of the lemma (5.2).

6. Proof of the Main Theorem:

Following Titchmarsh[11] and using Riemann-Lebesgue theorem, that the nth partial sum s_n of Fourier series(2.1) at $t = x$ may be written as :

$$s_n(f;x) - f(x) = \frac{1}{2\pi}\int_0^{\pi}\phi(t)\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\left(t/2\right)}dt$$

Therefore, the *A*- transform of $s_n(f;x)$ is given by

$$t_n - f(x) = \frac{1}{2\pi}\int_0^{\pi}\phi(t)\sum_{k=0}^{n}a_{nk}\frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\left(t/2\right)}dt$$

Now denoting (E,q)*A*- transform of $s_n(f;x)$ by τ_n, we have

$$\tau_n - f(x) = \frac{1}{2\pi(1+q)^n}\int_0^{\pi}\phi(t)\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\sum_{\upsilon=0}^{k}a_{k\upsilon}\frac{\sin\left(\upsilon+\frac{1}{2}\right)t}{\sin\left(t/2\right)}\right\}dt$$

$$= \int_0^{\pi}\phi(t)K_n(t)\,dt$$

$$= \left[\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\right]\phi(t)K_n(t)\,dt$$

$$= I_1 + I_2 \quad (say) \tag{6.1}$$

We consider

$$|I_1| \le \int_0^{\frac{1}{n+1}} |\phi(t)| \, |K_n(t)| \, dt$$

Using Hölder's inequality and the fact that $\phi(t) \in Lip(\xi(t), r)$,

$$|I_1| \le \left[\int_0^{\frac{1}{n+1}} \left\{\frac{t|\phi(t)|}{\xi(t)}\right\}^r dt\right]^{1/r} \left[\int_0^{\frac{1}{n+1}} \left\{\frac{\xi(t)|K_n(t)|}{t}\right\}^s dt\right]^{1/s}$$

$$= O\left(\frac{1}{n+1}\right)\left[\int_0^{\frac{1}{n+1}} \left\{\frac{\xi(t)|K_n(t)|}{t}\right\}^s dt\right]^{1/s} \quad \text{(By (4.2))}$$

$$= O\left(\frac{1}{n+1}\right)\left[\int_0^{\frac{1}{n+1}} \left\{\frac{\xi(t)(n+1)}{t}\right\}^s dt\right]^{1/s} \quad \text{(By Lemma (5.1))}$$

Since $\xi(t)$ is a positive increasing function and using second mean value theorem for integrals,

$$I_1 = O\left\{\xi\left(\frac{1}{n+1}\right)\right\}\left[\int_{\in}^{\frac{1}{n+1}} \frac{dt}{t^s}\right]^{1/s}, \text{ for some } 0 \le \in < \frac{1}{n+1}$$

$$= O\left\{\xi\left(\frac{1}{n+1}\right)\right\}\left[\left\{\frac{t^{-s+1}}{-s+1}\right\}_{\in}^{1/n+1}\right]^{1/s}$$

$$= O\left\{\xi\left(\frac{1}{n+1}\right)\right\}\left[\left\{(n+1)^{1-\frac{1}{s}}\right\}\right]$$

$$= O\left\{(n+1)^{\frac{1}{r}} \xi\left(\frac{1}{n+1}\right)\right\} \quad , \frac{1}{r} + \frac{1}{s} = 1 \tag{6.2}$$

Now we consider

$$|I_2| \le \int_{\frac{1}{n+1}}^{\pi} |\phi(t)| \, |K_n(t)| \, dt$$

Now using Hölder's inequality, equation (4.2) (4.3), Lemma (5.2) and second mean value theorem for integrals

$$|I_2| \le \left[\int_{\frac{1}{n+1}}^{\pi} \left(\frac{t^{-\delta}|\phi(t)|}{\xi(t)}\right)^r dt\right]^{1/r} \left[\int_{\frac{1}{n+1}}^{\pi} \left(\frac{\xi(t)|K_n(t)|}{t^{-\delta}}\right)^s dt\right]^{1/s}$$

$$= O\left\{(n+1)^{\delta}\right\}\left[\int_{\frac{1}{n+1}}^{\pi} \left(\frac{\xi(t)|K_n(t)|}{t^{-\delta}}\right)^s dt\right]^{1/s}$$

$$= O\left\{(n+1)^{\delta}\right\}\left[\int_{\frac{1}{n+1}}^{\pi} \left(\frac{\xi(t)}{t^{1-\delta}}\right)^s dt\right]^{1/s}$$

$$I_2 = O\left\{(n+1)^{\delta}\right\}\left[\int_{\frac{1}{\pi}}^{n+1} \left\{\frac{\xi\left(\frac{1}{y}\right)}{y^{\delta-1}}\right\}^s \frac{dy}{y^2}\right]^{1/s} \quad \text{(Putting } t = 1/y\text{)}$$

Since $\xi(t)$ is a positive increasing function and using second mean value theorem for integrals,

$$I_2 = O\left\{(n+1)^{\delta}\,\xi\left(\frac{1}{n+1}\right)\right\}\left[\int_{\eta}^{n+1} \frac{1}{y^{s(\delta-1)+2}} dy\right]^{1/s},$$

for some $\frac{1}{\pi} \le \eta \le (n+1)$

$$= O\left\{(n+1)^{\delta}\,\xi\left(\frac{1}{n+1}\right)\right\}\left[\int_{1}^{n+1} \frac{1}{y^{s(\delta-1)+2}} dy\right]^{1/s},$$

for some $\frac{1}{\pi} \le 1 \le (n+1)$

$$= O\left\{ (n+1)^{\delta}\, \xi\left(\frac{1}{n+1}\right) \right\} \left[\left\{ \frac{y^{s(1-\delta)-1}}{s(1-\delta)-1} \right\}_{1}^{n+1} \right]^{1/s}$$

$$= O\left\{ (n+1)^{\delta}\, \xi\left(\frac{1}{n+1}\right) \right\} \left[(n+1)^{(1-\delta)-\frac{1}{s}} \right]$$

$$= O\left\{ \xi\left(\frac{1}{n+1}\right) \right\} \left[(n+1)^{1-\frac{1}{s}} \right]$$

$$I_2 = O\left\{ (n+1)^{\frac{1}{r}}\, \xi\left(\frac{1}{n+1}\right) \right\}, \frac{1}{r} + \frac{1}{s} = 1 \quad (6.3)$$

Combining (6.1), (6.2) and (6.3), we have

$$|\tau_n - f(x)| = O\left\{ (n+1)^{\frac{1}{r}}\, \xi\left(\frac{1}{n+1}\right) \right\}$$

Now using L_r-norm, we get

$$\|\tau_n - f(x)\|_r = \left[\int_0^{2\pi} |\tau_n - f(x)|^r \, dx \right]^{1/r}$$

$$= O\left[\int_0^{2\pi} \left\{ (n+1)^{\frac{1}{r}}\, \xi\left(\frac{1}{n+1}\right) \right\}^r dx \right]^{1/r}$$

$$= O\left[(n+1)^{\frac{1}{r}}\, \xi\left(\frac{1}{n+1}\right) \right] \left\{ \int_0^{2\pi} dx \right\}^{1/r}$$

$$\|\tau_n - f(x)\|_r = O\left[(n+1)^{\frac{1}{r}}\, \xi\left(\frac{1}{n+1}\right) \right].$$

This completes the proof of the theorem.

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