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One modulo N gracefulness of acyclic graphs

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Abstract

A function f is called a graceful labelling of a graph G with q edges if f is an injection from the vertices of G to the set $(0, 1, 2, ..., q)$ such that, when each edge xy is assigned the label $|f(x) — f(y)|$, the resulting edge labels are distinct. A graph G is said to be one modulo N graceful (where N is a positive integer) if there is a function ϕ from the vertex set of G to $\{0, 1. N, (N+1), 2N, (2N+1), ..., N(q-1). N(q-1)+1\}$ in such a way that (i) ϕ is 1-1 (ii) ϕ induces a bijection ϕ^* from the edge set of G to $\{1, N+1, 2N+1, ..., N(q-1)+1\}$ where $\phi^*(uv) = |\phi(u) - \phi(v)|$. In this paper we prove that the acyclic graphs *viz.* Paths, Caterpillars, Stars and $S_{2,n} \underline{\ominus} S_{2,n}$ are one modulo N graceful for all positive integer N; Lobsters, Banana trees and Rooted tree of height two are one modulo N graceful for $N > 1$. where $S_{m,n} \underline{\ominus} S_{m,n}$ is a graph obtained by identifying one pendant vertex of each $S_{m,n}$. This is a fire cracker of subdivisioned stars.

Key words : Graceful, modulo N graceful, Path, $S_{m,n}$, $S_{m,n} \underline{\ominus} S_{m,n}$, Caterpillar, Star, Lob-ster, Banana tree and Rooted tree of height two.

AMS Subject Classification (2010):05C78

1. Introduction

S.W. Golomb[1] introduced graceful labelling. Odd gracefulness is introduced by R.B. Gnanajothi[2]. C. Sekar[6] intoduced one modulo three graceful labelling. In this paper we introduce the concept of one modulo N graceful where N is a positive integer. In the case $N = 2$, the labelling is odd graceful and in the case $N = 1$ the labelling is graceful. We prove that the acyclic graphs like Paths, Caterpillars, Stars and $S_{2,n} \underline{\ominus} S_{2,n}$, are one modulo N graceful where N is a positive integer, whereas Lobsters, Banana trees and Rooted tree of height two are one modulo N graceful for $N > 1$.

2 Main Results

Definition 2.1. A graph G is said to be one modulo N graceful (where N is a positive integer) if there is a function ϕ from the vertex set of G *to* $\{0, 1, N, (N+1), 2N, (2N+1), \ldots, N(q-1), N(q-1)+1\}$ in such a way that (i) ϕ *is* $1 - 1$ (ii) ϕ induces a bijection ϕ^* from the edge set of G to $\{1, N+1, 2N+1, \ldots, N(q-1)+1\}$ where $\phi^*(uv) = |\phi(u) - \phi(v)|$.

Definition 2.2. Consider $S_{m,n}$ (a star with n spokes in which each spoke is a path of length m) $S_{m,n} \ominus S_{m,n}$ is a graph obtained by identifying one pendant vertex of each $S_{m,n}$. This is a fire cracker of subdivisioned stars[3].

Definition 2.3. Caterpillar is a tree with the property that the removal of its end points or pendant vertices (vertices with degree. 1) leaves a path.

Definition 2.4. A banana tree is a graph obtained by connecting a vertex v to one leaf of each of any number of stars (v is not in any of the stars).

Definition 2.5. A lobster is a tree with the property that the removal of the end points (pendant vertices) leaves a caterpillar[4].

Theorem 2.6. Every path P_n is one modulo N graceful[5] (where N is a positive integer).

Proof: *Case (i)* n is odd.

Let $n = 2k + 1$, $k \geq 1$. Let $u_1, u_2, \ldots, u_n$ be the vertices P_n

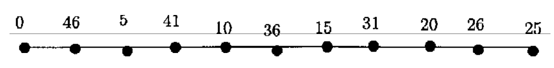
Define

$\phi(u_{2i-1}) = N(i-1)$ for $i = 1, 2, 3, \ldots, k+1$

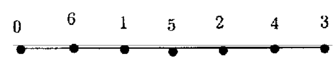
$\phi(u_{2i}) = 2Nk - (N-1) + N(i-1)$ for $i = 1, 2, 3, \ldots, k$

Clearly ϕ is 1 - 1 and ϕ defines a one modulo N graceful labelling of P_n.

Example 2.7. One modulo 5 graceful labelling of P_{11}



Example 2.8. Graceful labelling of P_7



Case (ii) n is even.

Let $n = 2k$

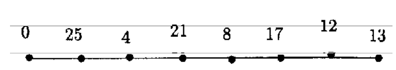
Define

$\phi(u_{2i-1}) = N(i-1)$ for $i = 1, 2, 3, \ldots, k$

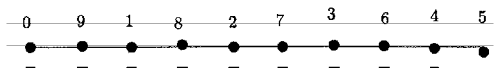
$\phi(u_{2i}) = 2Nk - (2N-1) - N(i-1)$ for $i = 1, 2, 3, \ldots, k$

Clearly ϕ is 1-1 and ϕ defines a one modulo N graceful labelling of P_n.

Example 2.9. One modulo 4 graceful labelling of P_8



Example 2.10. Graceful labelling of P_{10}



Theorem 2.11. Caterpillars are one modulo N graceful (where N is a positive integer).

Proof: Let G be a caterpillar.

Let $u_1, u_2, u_3, \ldots, u_m$ be the vertices of the caterpillar having degree at least 2 such that $u_i, u_{i+1} \in E(G)$.

Let $v_1, v_2, v_3, \ldots, v_n$ be the pendant vertices of the caterpillar. This caterpillar has $m + n$ vertices and $m + n - 1$ edges. Caterpillar is a bipartite graph in which the two independent vertex sets are

$V_1(G) = \{u_1, u_2, u_3, \ldots$ and all the pendant vertices $v_j's$ adjacent to $u_2, u_4, u_6 \ldots\}$

$V_2(G) = \{u_2, u_4, u_6, \ldots$ and all the pendant vertices $v_j's$ adjacent to $u_1, u_3, u_5, \ldots\}$

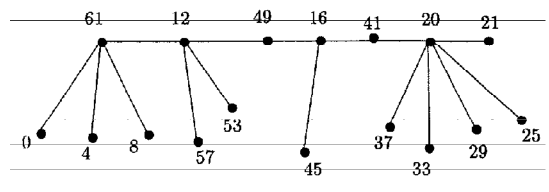
Reorder the elements of $V_1(G)$ as follows:

u_1, pendant vertices $v_j's$ adjacent to u_2, u_3, pendant vertices $v_j's$ adjacent u_4, u_5 and so on. Similarly reorder the elements of $V_2(G)$ as follows: pendant vertices adjacent to u_1, u_2, pendant vertices $v_j's$ adjacent u_3, u_4 and so on.

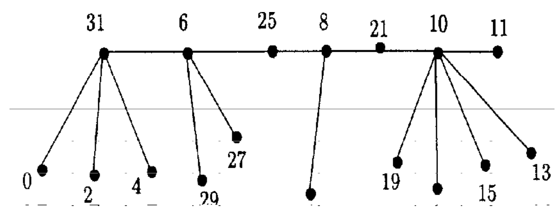
Label the elements of $V_1(G)$ as $N(m + n) - (2N - 1)$, $N(m + n) - (3N - 1)$, $N(m + n) - (4N - 1) \ldots$ in order and the elements of $V_2(G)$ as $0, N, 2N, 3N, \ldots$ in order. Clearly this labeling is one modulo N graceful.

Clearly Caterpillars are one modulo N graceful labelling.

Example 2.12. One modulo 4 graceful labelling Caterpillars.



Example 2.13. Odd graceful labelling Caterpillars.



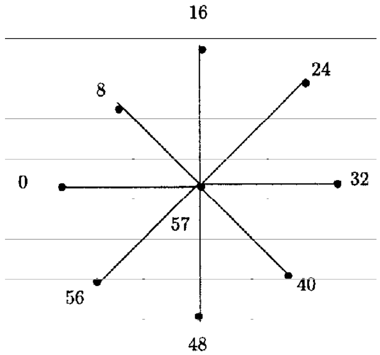
Theorem 2.14. Stars S_n are one modulo N graceful (where N is a positive integer).

Proof: Let v_0 be the centre of the star and $v_1, v_2, v_3, \ldots, v_n$ be the pendant vertices of S_n. Define $\phi(u_0) = N(n - 1) + 1$

$\phi(v_i) = N(i - l)$ for $i = 1, 2, 3, \ldots, n$

Clearly ϕ is 1-1 and ϕ defines a one modulo N graceful labelling of S_n.

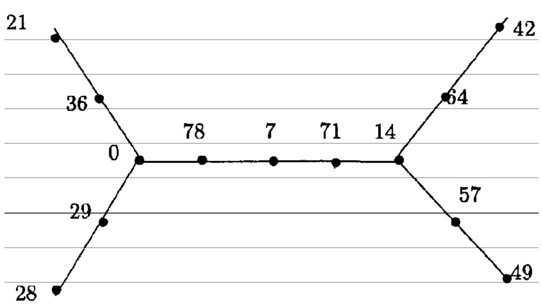
Example 2.15. One modulo 8 graceful labelling of S_8



Theorem 2.16. $S_{2,n}\ \underline{\ominus}\ S_{2,n}$ is one modulo N graceful for $n \geq 3$ (where N is a positive integer).

Proof: *Case (i)* $n = 3$

One modulo 7 gracefulness of $S_{2,3}\ \underline{\ominus}\ S_{2,3}$ is given below



Case (ii) $n \geq 4$

Let u_0 be the centre of first sub divisioned star $S_{2,n}$ and v_0 be the centre of second sub divisioned star $S_{2,n}$. Let $u_1,u_2,u_3,\ldots,u_n,u'_1.u'_2,u'_3,\ldots .u'_n$ be the vertices of first $S_{2,n}$ where u_i and u'_i are adjacent. Let $v_1,v_2,v_3,\ldots,v_n$, $v'_1.v'_2,v'_3,\ldots .v'_n$ be the vertices of second $S_{2,n}$ where v_i and v'_i are adjacent. Identify u'_n and v'_n.

Define $\phi(u_0) = 0$

$\phi(u_n)=4Nn - (N - 1)$

$\phi(u'_n)=\phi(v'_n) = N$

$\phi(v_n)=4Nn-(2N - 1)$

$\phi(v_0)=2N$

$\phi(v_i)=4Nn-(3N-1)-N(i-1)$ for $i=1,2,3,\ldots,n-1$

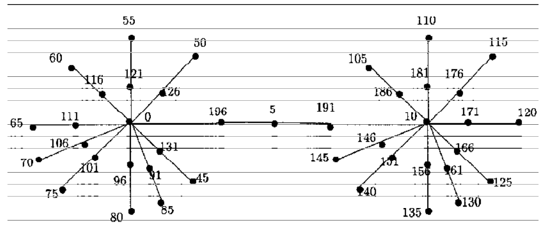
$\phi(u_i)= 3Nn-(4N-1)-N(i-1)$for $i=1,2,3,\ldots,n-1$

$\phi(u'_i)=N(n -1)+N(i - 1)$ for $i= 1,2,3,\ldots ,n-1$

$\phi(v'_i)= 2Nn +N + N(i - 1)$ for $i = 1,2,3,\ldots, n - 1$

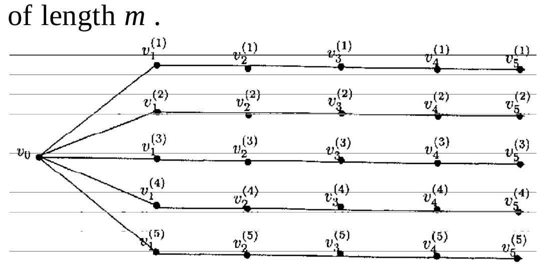
Clearly ϕ is 1-1 and ϕ defines a one modulo N graceful labelling of $S_{2,n}\ \underline{\ominus}\ S_{2,n}$.

One modulo 5 graceful labelling of $S_{2,10}\ \underline{\ominus}\ S_{2,10}$ is given below



Theorem 2.17. Let $S_{m,n}$ stand for a star with n spokes in which each spoke is a path of length m. Then $S_{m,n}$ is one modulo N graceful for all m and n (where N is a positive integer).

Proof: Let v_0 be the centre of the star. Let v_i^j, $1 \leq i \leq m, j = 1. 2,. .. ,n$ be the other vertices of vortices of the j th spoke of length m .



Case (i) $N = 1$.

It has been proved in[6] that $S_{m,n}$ is graceful for all m and n

Case (ii) $N > 1$.

Subcase (ii)(a) m is odd and n is odd. Let $m = 2r + 1$ and $n = 2s + 1$.

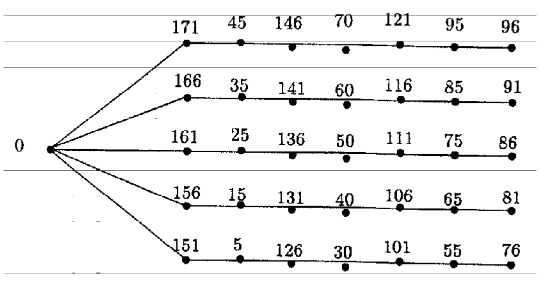
Define $\phi(v_0) = 0$

$\phi(v_{2i+1}^{(j)}) = N(2r+1)(2s+1)-(N-1)-Ni(2s+1)-N(j-1)$ for $i = 0, 1, 2, \ldots, r$ and $j = 1, 2, 3, \ldots, 2s+1$

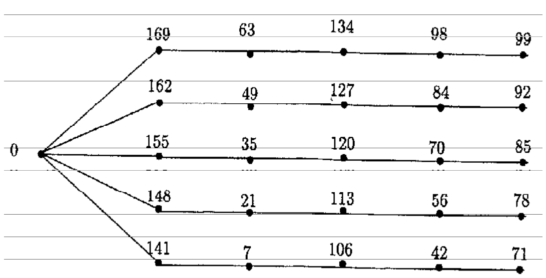
$\phi(v_{2i}^{(j)}) = 4Ns + N + N(i-1)(2s+1)-2N(j-1)$ for $i = 1, 2, \ldots, r$ and $j = 1, 1, 3, \ldots, 2s + 1$

Clearly ϕ is 1 - 1 and ϕ defines a one modulo N graceful labelling of $S_{m,n}$.

Example 2.18. One modulo 5 graceful labelling of $S_{7,5}$



Example 2.19. One modulo 7 graceful labelling of $S_{5,5}$



Subcase (ii)(b) m is even and n is odd. Let $m = 2r$ and $n = 2s + 1$.

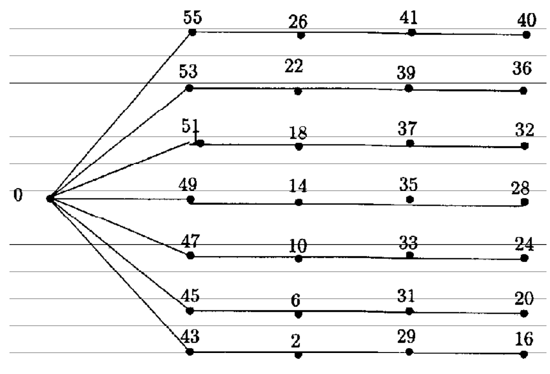
Define $\phi(v_0) = 0$

$\phi(v_{2i+1}^{(j)}) = 2Nr(2s+1)-(N-1)-Ni(2s+1)-N(j-1)$ for $i = 0, 1, 2, \ldots, r - 1$ and $j = 1, 2, 3, \ldots, 2s+1$

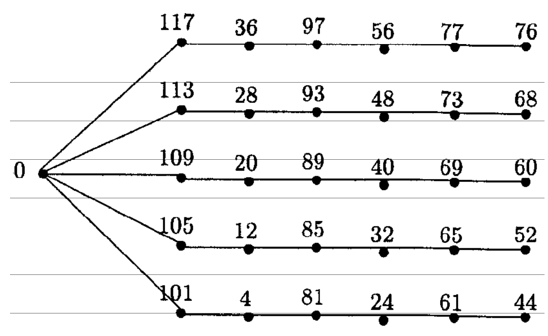
$\phi(v_{2i}^{(j)}) = 4Ns + N(i-1)(2s + 1) - 2N(j - 1)$ for $i = 1, 2, \ldots, r$ and $j = 1, 2, 3, \ldots, 2s + 1$

Clearly ϕ is 1 - 1 and ϕ defines a one modulo N graceful labelling of $S_{m,n}$.

Example 2.20. Odd graceful labelling of $S_{4,7}$



Example 2.21. One modulo 4 graceful labelling of $S_{6,5}$



Subcase (ii)(c) m is odd and n is even. Let $m = 2r + 1$ and $n = 2s$.

Define $\phi(v_0) = Nr$

$\phi(v_{2i+1}^{(j)}) = 2Ns(2r+1) - Nr + Ni - (N-1)$ for $i = 0,1,2,\ldots,r$

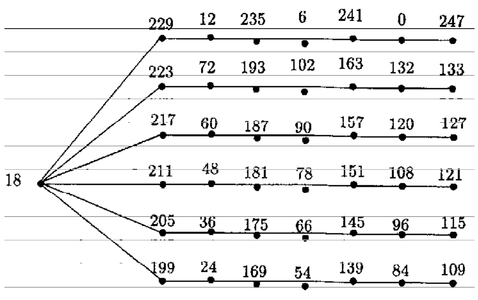
$\phi(v_{2i}^{(j)}) = Nr - N - N(i-1)$ for $i = 1,2,\ldots,r$

$\phi(v_{2i+1}^{(j)}) = 2Ns(2r+1) - Nr - (2N-1) - Ni(2s-1) - N(j-2)$ for $i = 1, 2,\ldots, r$ and $j = 2,3,\ldots,2s$

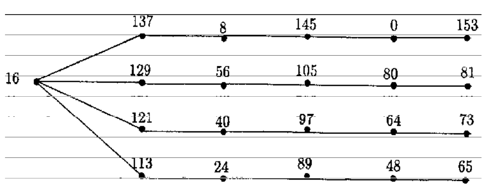
$\phi(v_{2i}^{(j)}) = Nr + N + 2N(2S-2) + N(i-1)(2s-1) - 2N(j-2)$ for $i = 1, 2,\ldots, r$ and $j = 2,3,\ldots,2s$

Clearly ϕ is 1 - 1 and ϕ defines a one modulo N graceful labelling of $S_{m,n}$.

Example 2.22. One modulo 6 graceful labelling of $S_{7,6}$



Example 2.23. One modulo 8 graceful labelling of $S_{5,4}$



Subcase (ii)(d) m is even and n *is* even. Let $m = 2r$ and $n = 2s$.

Define $\phi(v_0) = Nr$

$\phi(v_{2i+1}^{(j)}) = 4Nrs - Nr + 1 + Ni$ for $i = 0,1,2,\ldots,r-1$

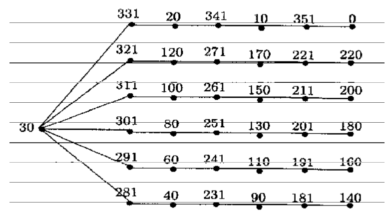
$\phi(v_{2i}^{(j)}) = Nr - N - N(i-1)$ for $i = 1,2,\ldots, r$

$\phi(v_{2i+1}^{(j)}) = 4Nrs - Nr - (N-1) - Ni(2s-1) - N(j-2)$ for $i = 0,1,2,\ldots,r-1$ and $j = 2,3,\ldots,2s$

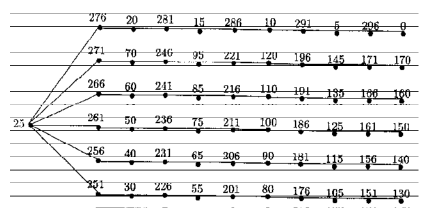
$\phi(v_{2i}^{(j)}) = Nr + N + 2N(2S-2) + N(i-1)(2s-1) - 2N(j-2)$ for $i = 1, 2,\ldots,r$ and $j = 2,3,\ldots, 2s$

Clearly ϕ is 1 - 1 and ϕ defines a one modulo N graceful labelling of $S_{m,n}$.

Example 2.24. One modulo 10 graceful labelling of $S_{6,6}$



Example 2.25. One modulo 5 graceful labelling of $S_{10,6}$



Theorem 2.26. Lobsters are one modulo N graceful (where N is a positive integer with $N > 1$).

Proof: Let $u_1,u_2,u_3,\ldots, u_m$ be the first level vertices. Let $v_1,v_2,v_3,\ldots,v_n$ be the second level vertices. Let $w_1,w_2, w_3,\ldots, w_r$ be the third level vertices adjacent to v_i' s in order. Let $x_1,x_2, x_3,\ldots, x_s$ be the fourth level vertices adjacent to u_i' s in order. Since total number of vertices is $m + n + r + s$, this lobster has $m + n + r + s - 1$ edges.

Label the first level vertices $u_1, u_2, u_3,\ldots,u_m$ respectively by $N(m+n+r+s-1)-(N-1)$, $N(m+n+r+s-1)-(2N-1),\ldots\ldots, N(m+n+r+s-1)-Nm+1$.

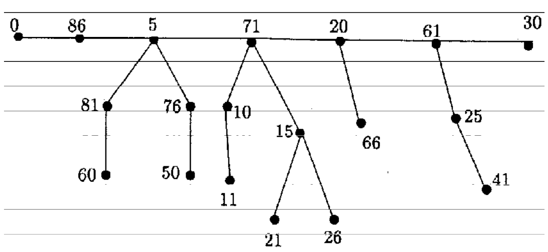
Label the second level vertices $v_1, v_2, v_3,\ldots, v_n$ respectively by $0, N, 2N, 3N,\ldots, N(n-1)$. Label the third level vertices $w_1, w_2,w_3,\ldots.,w_r$ as follows.

If w_i is adjacent to v_k put $f(w_i) = f(v_k) + 1 + N(i-1)$, k is arbitrary.

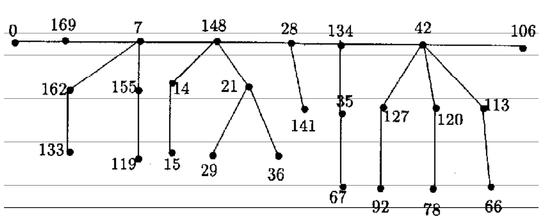
Label the fourth level vertices $x_1, x_2, x_3,\ldots, x_s$ as follows.

$f(x_i)=f(u_k) - N(r-1) - (N+1) - N(i-1)$ when u_k is adjacent to x_i. Clearly f is one-one and f gives a one modulo N graceful labelling.

Example 2.27. One modulo 5 graceful labelling of lobsters.



Example 2.28. One modulo 7 graceful labelling of lobsters.



Theorem 2.29. A banana tree is one modulo N graceful (where N is a positive integer with $N > 1$).

Proof: Let there be k stars $S_{n1}, S_{n2},S_{n3},\ldots, S_{nk}$. Connect a vertex v to one leaf of each star. Rearrange the stars in descending order of the number of leaves. Without loss of generality we assume that $n_1 \geq n_2 \geq \ldots \geq n_k$.

Let $u_1,u_2,u_3,\ldots,u_k$ be respectively the vertices of $S_{n1}, S_{n2},S_{n3},\ldots, S_{nk}$ connecting the vertex v. Let $u_1^{(0)}, u_2^{(0)}, u_3^{(0)},\ldots\ldots,u_k^{(0)}$ be respectively the centre of $S_{n1},S_{n2},S_{n3},\ldots, S_{nk}$. Let $u_i^{(j)}$, $j = 1,2,\ldots.,n_i-1$ $(1 \leq i \leq k)$ be the other vertices of the stars $S_{ni}(1 \leq i \leq k)$. This banana tree has $k + n_1 + n_2 +\ldots,+n_k$ edges and $k + 1 + n_1 + n_2 + \ldots, +n_k$ vertices.

Define $\phi(v) = 0$

$\phi(u_i)=N(k+n_1+n_2+\ldots.+n_k)-(N-1) - N(k-i)$ for $i = 1,2,3,\ldots, k$

$\phi(u_i^{(0)}) = N+2N(i-1)$ for $i = 1, 2, 3,\ldots, k$

$\phi(u_i^{(1)}) = N(k+n_1+n_2+\ldots+n_k)-(N-1)-2Nk+Ni$ for $i = 1,2,3,\ldots, k$

$\phi(u_i^{(2)}) = \phi(u_i^{(1)})-Nk$ for $i = 1,2,3,\ldots, k$

$\phi(u_i^{(3)}) = \phi(u_i^{(2)})-Nr$ if $u_i^{(2)}$ exist for $i = 1,2,3,\ldots,r$.

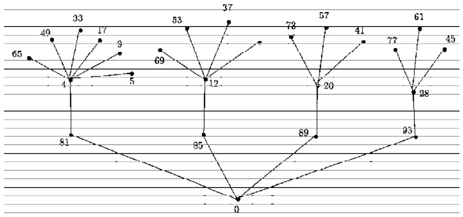
In general

$\phi(u_i^{(j+1)}) = \phi(u_i^{(j)})-Ns$ if $u_i^{(j)}$ exist for $i = 1,2,3,\ldots,s$.

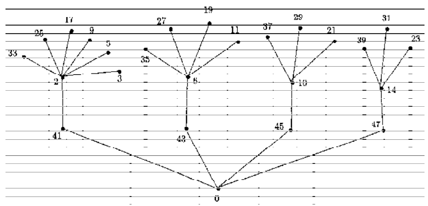
This process will stop when the edges have the labels $1,N+1,2N+1,\ldots,N(k+n_1+n_2+\ldots.+n_k) - (N-1)$

Clearly ϕ is 1- 1 and ϕ defines a one modulo N graceful labelling of banana tree.

Example 2.30. One modulo 4 graceful labelling of banana tree.



Example 2.31. Odd graceful labelling of banana tree.



Theorem 2.32. Rooted trees of height two are one modulo N graceful (where N is a positive integer with $N > 1$).

Proof: Let u_0 be the root. Let $u_1, u_2, u_3, \ldots, u_n$ be the vertices adjacent to u_0. Let $u_i^{(j)}$, $j = 1,2,\ldots,k_i$ be the pendant vertices adjacent to u_i, $i = 1,2,\ldots, n$. This rooted tree has $n + k_1 + k_2 + \cdots + k_n$ edges and $n + 1 + k_1 + k_2 + \cdots + k_n$ vertices

Define $\phi(u_0) = 0$

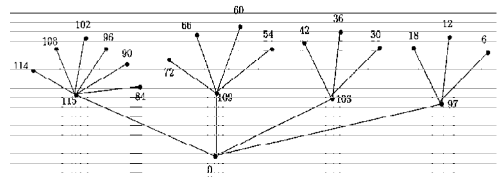
$\phi(u_i) = N(n+k_1+k_2 + \cdots + k_n)-(N-1)-N(i-1)$ for $i = 1,2,3,\ldots ,n$

$\phi(u_1^{(j)}) = N(n + k_1+k_2 + \cdots + k_n) -N- N(j - 1)$ for $j = 1,2,3,\ldots,k_1$

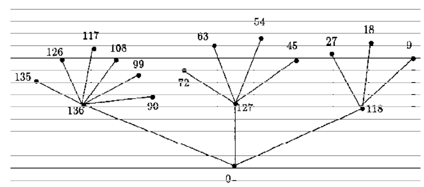
$\phi(u_{i+1}^{(j)}) = \phi(u_i^{(ki)})-2N- N(j - 1)$ for $i = 1$, $j = 1, 2, 3,\ldots, k_2$ and $i = 2$, $j = 1, 2, 3,\ldots,k_3,\ldots,$ $i=n-l$, $j=l,2,3,\ldots,k_n$

Clearly ϕ is 1-1 and ϕ defines a one modulo N graceful labelling of Rooted trees of height two.

Example 2.33. One modulo 6 graceful labelling of Rooted trees of height two.



Example 2.34. One modulo 9 graceful labelling of Rooted trees of height two.



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