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Characterizations of Topological Properties using Closed and Open Set Subspaces

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Abstract

Within this paper, subspace questions are further investigated for compact spaces, hereditarily compact spaces, and T_0 spaces using closed set subspaces and open set subspaces.

Key words: subspaces, proper subspace inherited properties, T_0, and compactness.

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1. Introduction

Given property P in a topological space, there are questions that logically and naturally arise. As examples: (1) What about images of spaces with property P? (2) What about product spaces of spaces with property P? and (3) What about subspaces of spaces with property P? Within this paper the subspace question is further examined and the properties T_0, compact, and hereditarily compact are further investigated and characterized using closed set and open set subspaces.

Until recently, the question concerning subspaces for a property P has been "Does a space have property P iff every subspace has property P? Within this paper, properties satisfying the above statement are called subspace properties. In the past and in all cases of subspace properties, the proofs of the converse statement for the subspace theorem cited above are all the same and having nothing to do with the property: "Since the space is a subspace of itself and every subspace has the property, then the space has the property." As a result proper subspace inherited properties were introduced and investigated [2] giving the property itself a new and important role in the subspace problem.

Definition 1.1. Let (X,T) be a space and let P be a property of topological spaces. If (X,T) has property P when every proper subspace of (X,T) has property P, then P is said to be a proper subspace inherited property[2].

Included in the previous investigation[2] was a proof establishing the subspace T_0 property as a proper subset inherited property, which was used to give the following new characterization of T_0 spaces.

Theorem 1.1. A space is T_0 iff every proper subspace of (X,T) is T_0.

Compactness is not a subspace property and in the paper cited above[2], the question of whether or not compactness could be strengthened to obtain a new property that is a subspace and/or proper subspace inherited property led to a new property; the hereditarily compact property[2].

Definition 1.2. A space is hereditarily compact iff every subspace of the space is compact[2].

The investigation of the hereditarily compact property[2] included the results below.

Theorem 1.2. Let (X,T) be a space. Then the following are equivalent: (a) every proper subspace of (X,T) is compact, (b) (X,T) is compact and, in (X,T), compact and hereditarily compact are equivalent, (c) every subspace of (X,T) is hereditarily compact, and (d) every proper subspace of (X,T) is hereditarily compact.

Thus, as in the case of the T_0 separation axioms cited above, hereditarily compact proved to be both a subspace and a proper subspace inherited property. As long known[3], a closed subset of a compact space is compact, which, together with the work given above, raised the question of whether or not closed sets and/or open sets could be used to further investigate the subspace question for the T_0, compact, and hereditarily compact properties and led to the work below.

2. New Chararacterizations of Compactness and Hereditarily Compactness Using Open and Closed Set Subspaces.

Theorem 2.1. Let (X,T) be a space. Then the following are equivalent: (a) (X,T) is compact, (b) every closed set subspace of (X,T) is compact, and (c) every proper closed set subspace of (X,T) is compact.

Proof: (a) implies (b): Since a closed subset of a compact space is compact, then for each closed set C, (C,T_C) is compact.

Clearly (b) implies (c).

(c) implies (a): Let $\{O_a$: a in A$\}$ be an open cover of X. If $O_b = X$ for some b in A, then the open cover has a finite subcover. Thus consider the case that O_a is not X for all a in A. Let b be in A such that O_b is not the empty set. Then $\{O_a$: a in A and a not b$\}$ is an open cover of $C = X \setminus O_b$, which is a proper closed subset of X, and there exists a finite subcover $\{O_{(a)i} : i = 1, \ldots, n\}$ that covers C. Then $\{O_a$: a $= b$ or a $= (a)_i$; $i = 1,\ldots, n\}$ is an open cover of X. Hence (X,T) is compact.

Theorem 2.2. Let (X,T) be a space. Then the following are equivalent: (a) (X,T) is hereditarily compact, (b) for each nonempty subset Y of X, for each T_Y-closed set C, (C,T_C) is compact, (c) for each nonempty subset Y of X, for each proper T_Y-closed set C, (C,T_C) is compact, (d) for each nonempty proper subset Y of X, for each T_Y-closed set C, (C,T_C) is compact, and (e) for each nonempty proper subset Y of X, for each proper T_Y-closed set C, (C,T_C) is compact.

The proof is straightforward using the results above and that a subspace of a subspace is the subspace of the initial space[3] and is omitted.

Examples are well-known showing that closed set in Theorem 2.1 can not be replaced by open. However, as established below, open sets can be used to further characterize hereditarily compact spaces.

Theorem 2.3. Let (X,T) be a space. Then the following are equivalent: (a) (X,T) is hereditarily compact, (b) for each nonempty open set O in X, (O,T_O) is hereditarily compact, and (c) for each nonempty open set O in X, (O,T_O) is compact.

Proof: From the results above, (a) implies (b) and (b) implies (c).

(c) implies (a): Let Y be a nonempty subset of X. Let $\{O_a$: a in A$\}$ be a T_Y-open cover of Y. For each a in A, let U_a be open in T such that O_a is the intersection of Y and U_a. Then U; the union of all the U_a; a in A, is an open in X and there exit finitely many $(U_a)_i$; i = 1,.., n, that cover U. Then $\{O_{(a)i}$: i = 1,…, n$\}$ is a finite cover of Y. Hence (Y,T_Y) is compact. Thus every subspace of (X,T) is compact, which implies (X,T) is hereditarily compact.

Theorem 2.4. Let (X,T) be a space. Then (a) for each nonempty proper open set O in X, (O,T_O) is hereditarily compact iff (b) for each nonempty proper open set O in X, (O,T_O) is compact.

Proof : Clearly (a) implies (b).

(b) implies (a): Let O be a nonempty proper open set in X. Let U be a nonempty proper open set in O. Then U is a nonempty proper open set in X and (U,T_U) is compact. Since $(U,(T_O)_U) = (U,T_U)$, then $(U,(T_O)_U)$ is compact. Since (O,T_O) is compact, then every nonempty open subspace of (O,T_O) is compact, which implies (O,T_O) is hereditarily compact.

Theorem 2.5. Let (X,T) be a space such that T is the indiscrete topology on X or there exists a proper open set U and a x in U such that Cl({x}) is a subset of U. Then the following are equivalent: (a) (X,T) is hereditarily compact, (b) for each nonempty proper open set O in X, (O,T_O) is hereditarily compact, and (c) for each nonempty proper open set O in X, (O,T_O) is compact.

Proof: Clearly (a) implies (b) and (b) and (c) are equivalent.

(b) implies (a): If T is the indiscrete topology on X, then (X,T) is hereditarily compact. Thus consider the case that T is not the indiscrete topology on X. Let O be a nonempty proper open set in X such that there exists a x in O such that Cl({x}) is a subset of O. Then U = X\Cl({x}) is a nonempty proper open set in X and (U,T_U) is hereditarily compact, and thus compact. Since O is hereditarily compact, then every nonempty subspace of (O,T_O) is compact. Since Cl({x}) is a nonempty subset of O, then $(Cl(\{x\}),T_{Cl(\{x\})})$ is compact. Hence X is the union of two compact sets and is compact. Thus every nonempty open subspace of X is compact and (X,T) is hereditarily compact.

In 1961 [1] the R_0 axiom was characterized as follows: A space is R_0 iff for each open set O and each x in O, Cl({x}) is a subset of O. Combining this characterization of the R_0 property with a proof similar to that of Theorem 2.5 gives the following result whose proof is omitted.

Theorem 2.6. Let (X,T) be a R_0 space. Then the following are equivalent: (a) (X,T) is hereditarily compact, (b) for each nonempty proper open set O in X, (O,T_O) is hereditarily compact, and (c) for each nonempty proper open set O in X, (O,T_O) is compact.

Theorem 2.7. Let (X,T) be a space.

Then the following are equivalent: (a) (X,T) is hereditarily compact, (b) for each nonempty subset Y of X, for each nonempty open set O in Y, (O,T_O) is hereditarily compact, (c) for each nonempty subset Y of X, for each nonempty open set O in Y, (O,T_O) is compact, (d) for each nonempty proper subset Y of X, for each open set O in Y, (O,T_O) is hereditarily compact, (e) for each nonempty proper subset Y of X, for each open set O in Y, (O,T_O) is compact, (f) for each nonempty open set O in X, for each T_Y-closed set C, (C,T_C) is compact, and (g) for each nonempty open set O in X, for each proper T_Y-closed set C, $(C.T_C)$ is compact.

The proof is straightforward using the results above and is omitted.

Within a recent introductory graduate level topology class the question of whether or not the open image of a T_0 space is T_0 arose. This question has led to many topological characterizations of nonempty sets. Thus the question arises of whether or not hereditarily compactness could be used to further topological characterize a nonempty finite set.

Theorem 2.8. Let X be a nonempty set. Then X is finite iff for each topology T on for which (X,T) is compact, (X,T) is hereditarily compact.

Proof: If X is finite, then each topology T on X is finite and (X,T) is both compact and hereditarily compact.

Conversely, suppose for each topology T on X for which (X,T) is compact, (X,T) is hereditarily compact. Then X is finite, for suppose not. Then X is infinite. Let y be an element in X. Then {O: O is a subset of X, y is in O, and X\O is finite} union with {{x}: x is in X and x is not y} is a base for a topology T on X, and (X,T) is compact but not hereditarily compact. Thus X is finite.

Theorem 2.9. Let X be a nonempty set. Then the following are equivalent: (a) X is finite, (b) for each topology T on X, (X,T) is hereditarily compact, and (c) for each topology T on X, (X,T) is compact.

Proof: (a) implies (b): If X is finite, then each topology T on X is finite and each T-subspace topology on X is finite, which implies (X,T) is hereditarily compact.

Clearly (b) implies (c).

(c) implies (a): Suppose X is infinite. Let T be the discrete topology on X. Then (X,T) is not compact, which is a contradiction.

3. Characterizations of T_0 Spaces Using Closed and Open Set Subspaces.

As in the initial investigation[2] all spaces will have three or more elements. Also, care must be taken for proper closed set subspaces and proper open set subspaces with more than one element to ensure the resulting subspace topology is not the indiscrete topology.

Theorem 3.1. Let (X,T) be a space. Then (X,T) is T_0 iff for x and y in X, x not y, y is not in Cl({x}) or x is not in Cl({y}).

The straightforward proof is omitted.

Theorem 3.1. Let (X,T) be a space. Then the following are equivalent: (a) (X,T) is T_0, (b) for each nonempty closed set C, (C,T_C) is T_0, and (c) for each nonempty proper closed set C, (C,T_C) is T_0 and for x and y in X, x not equal to y, Cl({x}) is not X or Cl({y}) is not X.

Proof: Since T_0 is a subspace property, (a) implies (b).

(b) implies (c): Since every nonempty closed set in X, the corresponding subspace is T_0, then the subspace for every nonempty proper closed set is T_0.

Let x and y be in X, x not y. Since X is a nonempty closed subset of itself, (X,T) is T_0 and x is not in Cl({y}) or y is not in Cl({x}) and Cl({x}) is not X and Cl({y}) is not X.

(c) implies (a): Let x and y be in X, x not y. Then Cl({x}) is not X or Cl({y}) is not X, say Cl({x}) is not X. If y is not in C = Cl({x}), then y is in O = X\Cl({x}), which is open and x is not in O. Thus consider the case that y is in Cl({x}). Then C = Cl({x}) is a proper closed subset of X and (C,T_C) is T_0. Let U be open in C containing only one of x and y. Let V be open in X such that U is V intersect C. Then V is open in X containing only one of x and y. Thus (X,T) is T_0.

Theorem 3.2. Let (X.T) be a space. Then the following are equivalent: (a) (X,T) is T_0, (b) for each nonempty closed set C in X, for each nonempty closed set Y in C, (Y,T_Y) is T_0, (c) for each nonempty closed set C in X, for each nonempty proper closed set Y in C, (Y,T_Y) is T_0, and for x and y in X, x not y, Cl({x}) is not X or Cl({y}) is not X, and (d) for each nonempty proper closed set C in X, for each nonempty proper closed set Y in C, (Y,T_Y) is T_0, and for each nonempty closed set W in X, for each x and y in W, x not y, there exits a proper closed set D in W such that x is in D or y is in D.

Proof: (a) implies (b): Let C be a nonempty closed set in X. By Theorem 3.1, (C,T_C) is T_0. Then, by Theorem 3.1, each nonempty closed set subspace of (C,T_C) is T_0.

(b) implies (c): Let C be a nonempty closed set in X. Since every nonempty closed set subspace in C is T_0, then every nonempty proper closed subspace in C is T_0. Since C is a nonempty closed subset of itself, then (C,T_C) is T_0. Hence every nonempty closed set subspace in (X,T) is T_0, which, by Theorem 3.1, implies (X,T) is T_0 and for x and y in X, x not y, Cl({x}) is not X or Cl({y}) is not X.

(c) implies (d): Let C be a nonempty proper closed set in X. Let Y be a nonempty proper closed set in C. Then (Y,T_Y) is T_0. Let x and y be in C, x not y. Then C is a nonempty proper closed subset of X, which is closed, and (C,T_C) is T_0. Thus D = Cl({x}) is not C or D = Cl({y}) is not C.

(d) implies (a): Let C be a nonempty proper closed set in X. Then for each nonempty proper closed set Y in C, (Y,T_Y) is T_0. Let u and v be in C, u not v. Let Z be a proper closed set in C such that u is in Z or v is in Z. Then Cl({u}) is not C or Cl({v}) is not C. Thus, by Theorem 3.1, (C,T_C) is T_0.
Let x and y be in X, x not y. Then W = X is a closed set in X and there exists a proper closed set D in X such that x is in D or y is in D and Cl({x}) is not X or Cl({y}) is not X. Hence, by Theorem 3.1, (X,T) is T_0.

Theorem 3.3. Let (X,T) be a space. Then the following are equivalent: (a) (X,T) is T_0, (b) for each nonempty open set O in X, (O,T_O) is T_0, and (c) for each nonempty proper open set O in X, (O,T_O) is T_0 and for each x and y in X, x not y, there exists a proper open set containing x or a proper open set containing y.

Proof: Clearly (a) implies (b).

(b) implies (c): Clearly for every nonempty proper open set O in X, (O,T_O) is T_0. Let x and y be in X. Since X is a nonempty open set in itself, then (X,T) is T_0 and, as above, Cl({x}) is not X and Cl({y}) is not X.

(c) implies (a): Let x and y be in X, x not y. Then there exists a proper open set containing x or a proper subset containing y, say there exists a proper open set O containing x. If y is not in O, then O is open such that x is in O and y is not in O. Thus consider the case that y is in O. Then x and y are distinct elements in the proper open set subspace (O,T_O) and there exists an open set U in O containing only one of x and y. Since U is open in O, which is open in X, then U is open in X. Hence (X,T) is T_0.

Theorem 3.4. Let (X,T) be a space. Then the following are equivalent: (a) (X,T) is T_0, (b) for each nonempty open set O in X, for each open set U in O, (U,T_U) is T_0, (c) for each nonempty open set O in X, for each nonempty open proper subset U in O, (U,T_U) is T_0 and for each x and y in X, x not y, there exists an open proper set W in X such that x is in W or y is in W, and (d) for each nonempty proper open set O in X, for each nonempty proper open set U in O, (U,T_U) is T_0 and for each open set W in X and x and y in W, x not y, there exists a proper open set V in W such that x is in V or y is in V.

Proof: Clearly (a) implies (b).

(b) implies (c): Clearly, if U is a nonempty proper subset of the open set O, then (U,T_U) is T_0. Let x and y be in X, x not y. Since X is open in itself, then (X,T) is T_0 and there exists a proper open set containing x or a proper open set containing y.

(c) implies (d): Let u and v be in X, u not v. Let W be a proper open subset of X containing u or containing v, say u is in W. If v is not in W, then W is an open set containing only one of u and v. Thus consider the case that v is in W. Since X is open in itself and W is a nonempty proper open subset of X, then (W,T_W) is T_0. Let V be a T_W-open set containing only one of u and v. Then V is open in X containing only one of u and v. Hence (X,T) is T_0. Then by Theorem 3.3 for each open set W in X and x and y in W, x not y, (W,T_W) is T_0 and there exists a proper open set V in W such that x is in V or y is in V. Let O be a nonempty proper open set in X. By Theorem 3.3, (O,T_O) is T_0. Then by Theorem 3.3, for each nonempty proper open subset U of O, (U,T_U) is T_0.

(d) implies (a): Let x and y be in X, x not y. Since X is open and x and y are in X, x not y, let V be a proper open set in X such that x is in V or y is in V, say x is in V. If y is not in V, then V is an open set containing only one of x and y. Thus consider the case that both x and y are in V. Let U be a proper subset of V such that x is in U or y is in U, say y is in U. Then U is open in X and contains only one of x and y. Hence consider the case that both x and y are in U. Then (U,T_U) is T_0 and there exists an open set in W in U, which will also be open in X, containing only one of x and y. Therefore (X,T) is T_0.

As in the case of hereditarily compact spaces above, combining the previous results [2] and the results above for T_0 spaces will give additional characterizations of T_0 spaces.

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