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Common Fixed Point Theorem for Weakly Compatible Mappings in Fuzzy Metric Spaces

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Abstract

We prove common fixed point theorem for weakly compatible mappings in fuzzy metric space. We extend results of Pathak, Khan and Tiwari to fuzzy metric space.

Introduction

In 1965 the concept of fuzzy sets was introduced by Zadeh[14]. It was developed extensively by many authors and used in various fields. Especially, Deng[3], Erceg[4], and Kramosil and Michalek[10] have introduced the concepts of fuzzy metric spaces in different ways.

Recently, George and Veeramani[7,8] modified the concept of fuzzy metric spaces introduced by kramosil and Michalek[10] and defined the Hausdoff topology of fuzzy metric spaces. They showed also that every metric induces a fuzzy metric.

Grabiec[6] extended the well known fixed point theorem of Banach[1] and Edelstein[5] to fuzzy metric spaces in the sense of Kramosil and Michalek[10].

Here we extend results of Pathak, Khan and Tiwari[12] to fuzzy metric space.

Preliminaries :

Definition[13] *1 :* A binary operation *: [0,1] [0,1] [0,1] is called a continuous t-norm if ([0,1], *) is an Abelian topological monoid with the unit 1 such that a * b c * d whenever a c and b d for all a, b, c, d are in [0,1].
Examples of t-norm are a * b = ab and a * b = min {a,b}.

Definition 2 :[10] The 3-tuple (X,M, *) is called a fuzzy metric space (shortly FM-space) if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set in X^2 [0,) satisfying the following conditions for all x, y, z in X and t, s > 0,

(FM-1) M(x, y, 0) = 0,

(FM-2) M(x, y, t) = 1 for all t > 0 if and only if x = y,

(FM-3) M(x, y, t) = M(y, x, t),

(FM-4) $M(x, y, t) * M(y, z, s) \quad M(x, z, t+s)$,
(FM-5) $M(x, y, .):[0,1] \quad [0,1]$ is left continuous.

In what follows, $(X, M, *)$ will denote a fuzzy metric space. Note that $M(x, y, t)$ can be thought as the degree of nearness between x and y with respect to t. We identify $x = y$ with $M(x, y, t) = 1$ for all $t > 0$ and $M(x, y, t) = 0$ with and we can find some topological properties and examples of fuzzy metric spaces in (George and Veeramani[7]).

Example 1 :[7] Let (X, d) be a metric space. Define $a * b = ab$ or $a * b = \min \{a, b\}$ and for all x, y in X and $t > 0$,

$$M(x,y,t) = \frac{t}{t + d(x,y)}$$

Then (X, M, *) is a fuzzy metric space. We call this fuzzy metric M induced by the metric d the standard fuzzy metric.

For workers of this line we are giving Lemmas.

Lemma 1:[6] For all x,y X, $M(x,y,.)$ is non-decreasing.

Definition 3:[6] Let (X, M, *) be a fuzzy metric space :

(1) A sequence $\{x_n\}$ in X is said to be convergent to a point x X (denoted by $\lim_n x_n = x$), if

$$\lim_n \quad M(x_n, x, t) = 1,$$

for all $t > 0$.

(2) A sequence $\{x_n\}$ in X called a Cauchy sequence if

$$\lim_n \quad M(X_{n+p}, x_n, t) = 1,$$

for all $t > 0$ and $p > 0$.

(3) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Remark 1 : Since * is continuous, it follows from (FM-4) that the limit of the sequence in FM-space is uniquely determined. Let (X,M,*) be a fuzzy metric space with the following condition:

(FM-6) $\lim_t \quad M(x,y,t) = 1$ for all x,y X.

Lemma 2 :[11] If for all x,y X, $t > 0$ and for a number k (0,1),

$$M(x,y,kt) \quad M(x,y,t)$$

then $x = y$.

Lemma[2,11] *3:* Let $\{y_n\}$ be a sequence in a fuzzy metric space (X, M, *) with the condition (FM-6). If there exists a number k (0,1) such that

$$M(y_{n+2}, y_{n+1}, kt) \quad M(y_{n+1}, y_n, t)$$

for all $t > 0$ and $n = 1, 2, \ldots\ldots..$ then $\{y_n\}$ is a Cauchy sequence in X.

Definition 4 :[9] A pair of mappings S and T is called weakly compatible pair in fuzzy metric space if they commute at coincidence points; *i.e.* , if $Tu = Su$ for some u X, then $TSu = STu$.

It is easy to see that if S and T are compatible , then they are weakly compatible and the converse is not true in general.

Example 2 : Let $X = R_+$. Define S and T by:
$Sx = x$ and $Tx = 2x-1$; $Sx = Tx$ iff $x = 1$,
As $ST(1) = S(1) = 1$, $TS(1) = T(1) = 1$
Therefore {S,T} are weakly compatible.

Let be the set of all continuous and increasing functions $_i$:[0,1] [0,1] in any coordinate and $_i$(t) > t for all t [0,1) and i = 1, 2, 3, 4, 5.

Main Results

We extend results of Pathak, Khan and Tiwari[12] to fuzzy metric spaces.

Theorem 1: Let (X, M, *) be a complete fuzzy metric space with t*t t for all t [0,1]. Let A, B, S and T be mappings of X into itself such that

(1.1) A(X) T(X) and B(X) S(X),

(1.2) there exists a constant k (0,1) such that

$M^{2p}(Ax, By, kt)$ min{ $_1(M^{2p}(Sx, Ty, t))$,
$_2(Mq(Sx, Ax, t).M^{q'}(Ty, By, t))$,
$_3(M^r(Sx, By, (2-)t).M^{r'}(Ty, Ax, t))$,
$_4(M^s(Sx, Ax, t).M^{s'}(Ty, Ax, t))$,
$_5(M^l(Sx, By, (2-)t). M^{l'}(Ty, By, t))$},

for all x, y X, 0, (0,2), t > 0, $_i$,
i = 1, 2, 3, 4, 5, 0 < p, q, q', r, r', s, s' l, l' 1, such that 2p = q + q' = r + r' = s + s' = l + l'.

(1.3) If the pairs {A, S} and {B, T} are weakly compatible, then
A, B, S and T have a unique common fixed point in X.

Proof : Since A(X) T(X), for arbitrary point x_0 in X, there exists a point x_1 X such that T x_1 = Ax_0. Since B(X) S(X), for this point x_1 we can choose a point x_2 X such that $Sx_2 = Bx_1$ and so on. Continuing in this manner, we can define a sequence $\{y_n\}$ in X such that

(1.4) $y_{2n} = T x_{2n+1} = Ax_{2n}$ and $y_{2n+1} = Sx_{2n+2} = Bx_{2n+1}$ for n = 1, 2, 3, . . .,

We need following Lemma for the proof of our mail Theorem.

Lemma 4 : Let A,B, S and T be self-mappings of a fuzzy metric space
(X, M, *) satisfying the conditions (1.1) and (1.2). Then the sequence $\{y_n\}$ denoted by (1.4) is a Cauchy sequence in X.

Proof. For t > 0, By putting x = x_{2n} and y = x_{2n+1} in (1.2), = 1–q, with q (0,1) we have

$M^{2p}(y_{2n}, y_{2n+1}, kt) = M^{2p}(Ax_{2n},Bx_{2n+1}, kt)$
$M^{2p}(Ax_{2n},Bx_{2n+1}, kt)$ min{ $_1(M^{2p}(Sx_{2n}, Tx_{2n+1}, t))$,
$_2(M^q(Sx_{2n},Ax_{2n},t).M^{q'}(Tx_{2n+1}, Bx_{2n+1}, t))$,
$_3(M^r(Sx_{2n}, Bx_{2n+1},(2-)t).M^{r'}(Tx_{2n+1}, Ax_{2n}, t))$
$_4(M^s(Sx_{2n}, Ax_{2n},t).M^{s'}(Tx_{2n+1}, Ax_{2n}, t))$,
$_5(M^l(Sx_{2n}, Bx_{2n+1}, (2-)t). M^{l'}(Tx_{2n+1}, Bx_{2n+1}, t))$},
$M^{2p}(y_{2n}, y_{2n+1}, kt)$ min{ $_1(M^{2p}(y_{2n-1}, y_{2n}, t))$,
$_2(M^q(y_{2n-1}, y_{2n},t).M^{q'}(y_{2n}, y_{2n+1},t))$,
$_3(M^r(y_{2n-1}, y_{2n+1}, (1+q)t).M^{r'}(y_{2n}, y_{2n}, t))$,
$_4(M^s(y_{2n-1}, y_{2n}, t).M^{s'}(y_{2n}, y_{2n}, t))$,
$_5(M^l(y_{2n-1}, y_{2n+1}, (1+q)t). M^{l'}(y_{2n}, y_{2n+1}, t))$ },
$M^{2p}(y_{2n}, y_{2n+1}, kt)$ min { $_1(M^{2p}(y_{2n-1}, y_{2n}, t))$,
$_2(M^q(y_{2n-1}, y_{2n}, t).M^{q'}(y_{2n}, y_{2n+1}, t))$,
$_3(M^r(y_{2n-1}, y_{2n}, t)* M^r(y_{2n}, y_{2n+1}, qt))1$
$_4(M^s(y_{2n-1}, y_{2n}, t)1)$,
$_5$ $(M^l(y_{2n-1}, y_{2n}, t) * M^l(y_{2n}, y_{2n+1}, qt)$.
$M^{l'}(y_{2n}, y_{2n+1}, t))$},

Since the t-norm * is continuous and M(x, y, .)

is continuous , letting q 1, we have

$M^{2p}(y_{2n}, y_{2n+1}, kt)$ min { $_1(M^{2p}(y_{2n-1}, y_{2n}, t))$, $_2(M^{q}(y_{2n-1}, y_{2n}, t).M^{q'}(y_{2n}, y_{2n+1}, t))$, $_3(M^{r}(y_{2n-1}, y_{2n}, t)* M^{r}(y_{2n}, y_{2n+1}, t))$ $_4$ $(M^{s}(y_{2n-1}, y_{2n}, t))$, $_5$ $(M^{l}(y_{2n-1}, y_{2n}, t) * M^{l}(y_{2n}, y_{2n+1}, t). M^{l'}(y_{2n}, y_{2n+1}, t))$},

$M(y_{2n}, y_{2n+1}, kt)$

$$(1.5) > \begin{cases} M(y_{2n-1}, y_{2n}, t), \text{ if } M(y_{2n-1}, y_{2n}, t) < M(y_{2n}, y_{2n+1}, t) \\ M(y_{2n}, y_{2n+1}, t), \text{ if } M(y_{2n-1}, y_{2n}, t) \quad M(y_{2n}, y_{2n+1}, t), \end{cases}$$

as $_i(t) > t$ for $0 < t < 1$. Thus {$M(y_{2n}, y_{2n+1}, t)$, n 0} is an increasing sequence of positive real numbers in [0,1] and therefore tends to a limit l 1. We assert that $l = 1$. If not, $l < 1$ which on letting n in (1.5) one gets l (l) > l a contradiction yielding thereby $l = 1$. Therefore for every n N, using analogous arguments one can show that {$M(y_{2n+1}, y_{2n+2}, t)$, n 0} is a sequence of positive real numbers in [0, 1] which tends to a limit $l = 1$. Therefore for every n N, $M(y_n, y_{n+1}, t) > M(y_{n-1}, y_n, t)$ and

$\lim_n$ $M(y_n, y_{n+1}, t) = 1$.

Now for any positive integer p

$M(y_n, y_{n+p}, t)$ $M(y_n, y_{n+1}, t/p)*^{p\text{-times}} \ldots *M(y_{n+p-1}, y_{n+p}, t/p)$.

Since $\lim_n$ $M(y_n, y_{n+1}, t) = 1$ for $t > 0$, it follows that

$\lim_n$ $M(y_n, y_{n+p}, t)$ $1* 1 \ldots * 1 = 1$

which shows that {y_n} is a Cauchy sequence in X.

Now we prove our main result as follows: Since X is complete, it follows by Lemma 4, that the sequence {y_n} converges to a point z in X. On the other hand, the sub sequences {Ax_{2n}}, {Bx_{2n+1}}, {Sx_{2n}} and {$T x_{2n+1}$} of {y_n} also converges to the point z.

Now suppose that the subsequence {y_{2n}} is contained in S(X) and has a limit in S(X) call it z. Let u $S^{-1}(z)$. Then Su = z.

By (1.2) with = 1, we have

$M^{2p}(Au, y_{2n+1}, kt) = M^{2p}(Au, Bx_{2n+1}, kt)$

min { $_1(M^{2p}(Su, Tx_{2n+1}, t))$, $_2(M^{q}(Su, Au, t). M^{q'}(Tx_{2n+1}, Bx_{2n+1}, t))$, $_3(M^{r}(Su, Bx_{2n+1},t).M^{r'}(Tx_{2n+1}, Au, t))$, $_4(M^{s}(Su, Au, t). M^{s'}(Tx_{2n+1}, Au, t))$, $_5(M^{l}(Su, Bx_{2n+1}, t). M^{l'}(Tx_{2n+1}, Bx_{2n+1}, t))$ },

which implies that as n , we have

$M^{2p}(Au, z, kt)$ min { $_2(M^{q}(z, Au, t))$, $_3(M^{r'}(z, Au, t))$, $_4(M^{s+s'}(z, Au, t)$}

$M^{2p}(Au, z, kt)$ $_4(M^{2p}(z, Au, t)) > M^{2p}(z, Au, t)$

a contradiction. Therefore Au = z = Su , i. e. u is a coincidence point of A and S.

Now suppose that the subsequence {y_{2n}} is contained in T(X) and has a limit in T(X) call it z . Let v $T^{-1}(z)$. Then Tv = z .

Again by (1.2) with = 1, we have

$M^{2p}(y_{2n}, Bv, kt) = M^{2p}(Ax_{2n}, Bv, kt)$

min { $_1(M^{2p}(Sx_{2n}, Tv, t))$, $_2(M^{q}(Sx_{2n}, Ax_{2n}, t). M^{q'}(Tv, Bv, t))$, $_3(M^{r}(Sx_{2n}, Bv, t).M^{r'}(Tv, Ax_{2n}, t))$, $_4(M^{s}(Sx_{2n}, Ax_{2n}, t).M^{s'}(Tv, Ax_{2n}, t))$, $_5(M^{l}(Sx_{2n}, Bv, t). M^{l'}(Tv, Bv, t))$ },

which implies that as n , we have

$M^{2p}(z, Bv, kt) \quad \min\{ {}_1(M^{2p}(z, z, t)),\ {}_2(M^q(z, z, t).M^{q'}(z, Bv, t)),$
$\quad {}_3(M^r(z, Bv, t).M^{r'}(z, z, t)),\ {}_4(M^s(z, z, t).M^{s'}(z, z, t)),$
$\quad {}_5(M^l(z, Bv, t).\ M^{l'}(z, Bv, t)) \},$

or

$M^{2p}(z, Bv, kt) \quad \min\{ {}_1(1),\ {}_2(M^{q'}(z, Bv, t)),\ {}_3(M^r(z, Bv, t)),$
$\quad {}_4(1),\ {}_5(M^{l+l'}(z, Bv, t)) \},$

or

$M^{2p}(z, Bv, kt) \quad {}_5(M^{l+l'}(z, Bv, t)) > M^{2p}(z, Bv, t)$

a contradiction Therefore $Bv = z$. Since $Tv = z$ thus $Tv = Bv = z$
i.e. v is a coincidence point of B and T.
Since the pair {A,S} is weakly compatible therefore, A and S commute at their coincidence point, i.e. if $ASw = SAw$ or $Az = Sz$.

Similarly, since the pair {B,T} is weakly compatible therefore, B and T commute at their coincidence point, i.e. if $BTw = TBw$ or $Bz = Tz$.
Now, we prove $Az = z$. By (1.2) with $= 1$, we have

$M^{2p}(Az, y_{2n+1}, kt) = M^{2p}(Az, Bx_{2n+1}, kt)$
$\min\{ {}_1(M^{2p}(Sz, Tx_{2n+1}, t)),\ {}_2(M^q(Sz, Az, t).M^{q'}(Tx_{2n+1}, Bx_{2n+1}, t)),$
$\quad {}_3(M^r(Sz, Bx_{2n+1}, t).M^{r'}(Tx_{2n+1}, Az, t)),$
$\quad {}_4(M^s(Sz, Az, t).M^{s'}(Tx_{2n+1}, Az, t)),$
$\quad {}_5(M^l(Sz, Bx_{2n+1}, t).\ M^{l'}(Tx_{2n+1}, Bx_{2n+1}, t))\},$

which implies that as n , we have

$M^{2p}(Az, z, kt) \quad \min\{ {}_1(M^{2p}(Az, z, t)),\ {}_2(1),\ {}_3(M^r(Az, z, t).M^{r'}(z, Az, t)),$
$\quad {}_4(M^{s'}(z, Az, t)),\ {}_5(M^l(Az, z, t)) \},$

$M^{2p}(Az, z, kt) \quad {}_1(M^{2p}(Az, z, t) > M^{2p}(Az, z, t)$

a contradiction. Therefore $Az = z$. Thus $Az = Sz = z$.
Now, we prove $Bz = z$. By (1.2) with $= 1$, we have

$M^{2p}(Ax_{2n}, Bz, kt) \quad \min\{ {}_1(M^{2p}(Sx_{2n}, Tz, t)),$
$\quad {}_2(M^q(Sx_{2n}, Ax_{2n}, t).M^{q'}(Tz, Bz, t)),$
$\quad {}_3(M^r(Sx_{2n}, Bz, t).M^{r'}(Tz, Ax_{2n}, t)),\ {}_4(M^s(Sx_{2n}, Ax_{2n}, t).M^{s'}(Tz, Ax_{2n}, t)),$
$\quad {}_5(M^l(Sx_{2n}, Bz, t).\ M^{l'}(Tz, Bz, t)) \},$

which implies that as n , we have

$M^{2p}(Ax_{2n}, Bz, kt) \quad \min\{ {}_1(M^{2p}(z, Bz, t)),\ {}_2(1),\ {}_3(M^{r+r'}(z, Bz, t)),$
$\quad {}_4(M^{s'}(Bz, z, t)),\ {}_5(M^l(z, Bz, t)) \},$

or

$M^{2p}(z, Bz, kt) \quad {}_1(M^{2p}(z, Bz, t) > M^{2p}(z, Bz, t).$

a contradiction. Therefore $Bz = z$. Since $Tz = z$ thus $Tz = Bz = z$.
Combining the above results, we have
$Az = Bz = Sz = Tz = z$, z is a common fixed point of A, B, S and T.
For the uniqueness of common fixed point let w (z w) be another common fixed point of A, B, S and T. Then by (1.2) with $= 1$, we have

$M^{2p}(z, w, kt) = M^{2p}(Az, Bw, kt)$
$\min\{ {}_1(M^{2p}(Sz, Tw, t)),\ {}_2(M^q(Sz, Az, t).M^{q'}(Tw, Bw, t)),$
$\quad {}_3(M^r(Sz, Bw, t).M^{r'}(Tw, Az, t)),\ {}_4(M^s(Sz, Az, t).M^{s'}(Tw, Az, t)),$
$\quad {}_5(M^l(Sz, Bw, t).\ M^{l'}(Tw, Bw, t))\},$

or

$M^{2p}(z, w, kt) \quad \min\{ {}_1(M^{2p}(z, w, t)),\ {}_2(M^q(z, z, t).\ M^{q'}(w, w, t)),$
$\quad {}_3(M^r(z, w, t).M^{r'}(w, z, t)), {}_4(M^s(z, z, t).M^{s'}(w, z, t)),$
$\quad {}_5(M^l(z, w, t).\ M^{l'}(w, w, t)) \},$

$$M^{2p}(z, w, kt) \quad {}_1(M^{2p}(z, w, t)) > M^{2p}(z, w, t) .$$

a contradiction. Therefore $z = w$.
This completes the proof of the Theorem.

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