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Some Theorems On A Conformal Transformation of A Kaehlerian Space

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Abstract

Ishihara and Obata[2] have proved that if M is a differentiable and connected Riemannian manifold of dimension > 2, which is not locally Euclidean and if M admits a conformal transformation $\emptyset$ such that the associated function $\alpha_{\emptyset}$ satisfies $\alpha_{\emptyset}$ (x) < 1 ε, or $\alpha_{\emptyset}$ (x) > 1+ε, for each x M, being a positive number, then $\emptyset$ has no fixed point. Further, Hirmatu[1] has studied that a differentiable and connected Riemannian manifold admitting a conformal transformation group of sufficiently high dimension is locally conformal Euclidean.

In the present paper, we have obtained results concerning the fixed point of a conformal transformation of a Kaehlerian space and concerning the locally conformally flatness of the Kaehlerian space.

Key words & phrases: Kaehler and Riemannian geometry, conformal transformation.

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Introduction

An n (= 2m) dimensional Kaehlerian space $\mathbf{K}_{\mathbf{n}}^{\mathbf{c}}$ is an even dimensional Riemannian space, which admits a structure tensor $\boldsymbol{F_i^h}$ satisfying the following condition:

$$F_i^h F_h^j = -\delta_i^j , \qquad (1.1)$$

$$F_{ij} = -F_{ji} , \ (F_{ij} = F_i^h g_{hj}) \qquad (1.2)$$

And

$$F^h_{i,j}=0, \qquad (1.3)$$

Where the comma (,) followed by an index denotes the operates of covariant differentiation w.r.t. the metric tensor g_{ij} of the Riemannian space.

2. Conformal transformation of a kaehlerian space:

Let $\boldsymbol{K_n^c}$ be a differentiable and connected Kaehlerian space with a fundamental metric tensor field g. A diffeomorphism on K_n^c is called a conformal transformation on K_n^c if there exists a positive valued function α_ϕ on K_n^c such that g = g holds, and a homothetic transformation on K_n^c if is constant on K_n^c. The function connected with is called the associated function of . The is necessarily differentiable. If is identically equal to unity, then is nothing else than on isometry on K_n^c.

Let be a conformal transformation on K_n^c and and A denote **inf** $\{\alpha_\phi(x), x \in K_n^c\}$ $(\geqq 0)$ and **sup**$\{\alpha_\phi(x), x \in K_n^c\}$ $(\leqq \infty)$ respectively. Then $< A$ iff is not a homothetic transformation, $=A$ if and only if is a homothetic transformation and $=A$ $=1$ if and only if is an isometry[1-3].

We shall denote by (A) the following property: there exists a real number such that 0 < < 1 and such that for each point x K_n^c either $\alpha_\phi(x) < 1-\in$, or $\alpha_\phi(x) > 1+\in$ holds. Since K_n^c is assumed to be connected and is continuous on K_n^c, $\{\alpha_\phi(x), x \in K_n^c\}$ is a connected subset in real number space. Therefore the property (A) is equivalent to a property that only one of the following (1) and (2) occurs: (1) $\alpha_\phi(x) < 1 - \epsilon$ for all x K_n^c and (2) $\alpha_\phi(x) > 1 + \epsilon$ for all x K_n^c. We remark that if (A) is assumed then is not an isometry[3].

Lemma (2.1): If (A) is assumed, then >1 or $\alpha_\phi - 1 > 1, \phi^{-1}$ being the inverse of and $\alpha_\phi - 1 = inf\{\alpha_\phi - 1(x), x \epsilon K_n^c\}$.

Proof: If the case (2) occurs, the result is clear. To prove our result, it suffices to consider the case in which (1) occurs. Considering the inverse ϕ^{-1} of , we have

$$(\phi^{-1}\phi)\mathrm{g} = \phi^{-1}(\alpha_\phi \mathrm{g}) = (\phi^{-1}\alpha_\phi . \alpha_{\phi-1})\ \mathrm{g}$$

From which $1/\alpha_\phi \circ \phi = \alpha_{\phi^{-1}}$ because $\alpha_{\emptyset^{-1}\emptyset} = 1$. It follows that $\alpha_{\phi^{-1}} > 1$.

Under the condition (A), by considering the inverse ϕ^{-1} of if necessary, we can assume without loss of generality that $\alpha_\phi > 1$. Hereafter we shall use this fact[4-6].

Lemma (2.2): If (A) is assumed, then for any given points p and q of K_n^c and for any given positive integer in the relation.

$$d(\phi^m p, \phi^m q) \leqq (a_\phi)^{-m/2}\, d(p,q)$$

holds, where ***d*** denotes the metric function of K_n^c connected with g.

Proof: Let $\sigma: [t_0, t_1] \longrightarrow K_n^c$ be a piecewisely C^1 — differentiable curve joining p to q. Then the length $L(\phi \circ \sigma)$ of the transformed curve $\phi \circ \sigma$ joining ϕp to ϕq

is given by the integral

$$L(\phi \circ\sigma) = \int_{t_0}^{t_1} \left[g_{(\phi\ \circ\ \sigma)(t)} \left(\phi \frac{d\sigma}{dt}, \phi \frac{d\sigma}{dt}\right)\right]^{\frac{1}{2}} dt$$

$$= \int_{t_0}^{t_1} \left[\frac{1}{\alpha_\phi((\phi\ \circ\ \sigma)(t))} g_{\sigma(t)} \left(\frac{d\sigma}{dt}, \frac{d\sigma}{dt}\right)\right]^{\frac{1}{2}} dt.$$

Since $a_\phi \le \alpha_\phi((\phi\ \circ\ \sigma)(t))$ for all $t\ \epsilon\ [t_o, t_1]$, we have

$$L(\phi\ \circ\ \sigma) \le (a_\phi)^{-\frac{1}{2}} \int_{to}^{t1} \left[g_{\sigma(t)} \left(\frac{d\sigma}{dt}, \frac{d\sigma}{dt}\right)\right]^{\frac{1}{2}} d\,t$$

$$= (\alpha_\phi)^{-\frac{1}{2}} L(\sigma),$$

Where $L(\sigma)$ denotes the length of σ,
It follows from the above relation that

$$d(\phi p, \phi q) \leqq (a_\phi)^{-\frac{1}{2}} d(p, q)$$

And consequently for any given positive integer **m**

$$d(\phi^m p, \phi^m q) \leqq (a_\phi)^{-m/2} d(p, q).$$

Now, we shall prove the following:

Theorem (2.1): Let K_n^c be differentiable, connected and complete Kaehlerian space and let ϕ be a conformal transformation on K_n^c. If (A) is assumed, then has only one fixed point[4,7].

Proof: From the assumption (A), by using ***Lemma (2.1),*** we can assume without loss of generality that $a_\phi > 1$, Take any point p of K_n^c. Then, for any given positive integer ***m*** and we have by using ***Lemma (2.2):***

$$d(\phi^m p, \phi^{m+1}) \leqq\ d(\phi^m p, \phi^{m+1} p) + d(\phi^{m+1} p, \phi^{m+2} p) + \cdots + \cdots + d(\phi^{m+l-1}\ p, \phi^{m+l}\ p)$$

$$\leqq (a_\phi)^{-m/2}\ d(p, \phi p) + \cdots + (a_\phi)^{-(m+l-1)/2}\ d(p, \phi p)$$

$$\leqq (a_\phi)^{-m/2} d(p, \phi p) \qquad (a_\phi)^{-s/2}$$

It follows from the above relation that a sequence of points $\{\emptyset^m\ p\}_{m=1}^{\infty}$ is a Cauchy sequence because the series in the right hand side of the above relation converges. Since K_n^c is assumed to be complete, the sequence of points has the limit point p_0. It is easily proved that $\emptyset$ leaves p_0 invariant. Next, if x_0 and y_0 are two fixed points of $\emptyset$, then from ***Lemma (2.2),*** we have

$$d(x_0, y_0) = d(\phi^m x_0, \phi^m y_0) \leqq (a_\phi)^{-m/2}\, d(x_0, y_0)$$

for any positive integer ***m*** from which $d(x_0, y_0) = 0$ and hence $x_0 = y_0.$

With help of **Theorem (2.1**), we have

Theorem (2.2): Let K_n^c be a differentiable, connected and complete Kaehlerian space of dimension > 2 and let be a conformal transformation on K_n^c. If (A) is assumed, then K_n^c is locally conformally Euclidean[7-9].

From **Theorem (2.2),** we have the following:

Corollary (2.1): Let K_n^c be a differentiable, connected and complete Kaehlerian space of dimension > 2, which is not locally conformally Euclidean. If K_n^c admits a conformal transformation , then can take value unity or an arbitrary value closed to unity[8].

Since K_n^c is assumed to be connected and is continuous, if K_n^c compact, then the set $\{\alpha_\phi(x), x\epsilon M\}$ is compact and connected subset in real number space and hence is a closed interval. Therefore, we have

Corollary (2.2): Let K_n^c be a differentiable, connected and compact Kaehlerian space of dimension > 2, which is not locally conformally Euclidean. If K_n^c admits a conformal transformation , then takes value unity.

References

1. Hiramatu, H., Riemannian manifolds and conformal transformation groups. *Tensor 8,* 123 (1958).
2. Ishihara, S. and Obata, M., On the group of conformal transformations of a Riemannian manifold. *Proc. Japan Acad. 31,* 426 (1955).
3. Otsuki, T. and Tashiro, Y., on curves in Kaehlerian spaces, *Math. Jour. Okayama Univ., 4,* 57 (1954).
4. Sumitomo, T., Projective and conformal transformations in compact Riemannian manifolds, *tensor, 9,* 113 (1959).
5. Tashiro, Y., On a holomorphically projective correspondence in an almost complex space, *Math, Jour. Okayama Univ., 6,* 147 (1957).
6. Yano, K., Lie derivatives and its applications, Amsterdam (1957).
7. Yano, K. and Nagano, T., Some theorems on projective and conformal transformations, *Indag. Math., 14,* 45 (1957).
8. Negi, U.S. and Gairola Kailash, On H-Projective transformations in almost Kaehlerian spaces, *Asian Journal of Current Engineering and Maths 1:3,* 162–165 (2012).
9. Negi, U.S. and Gairola Kailash, Admitting a conformal transformation group on Kaehlerian recurrent spaces, International *Journal of Mathematical Archive-3(4),* 1584-1589 (2012).