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Riemannian manifold without putting any restriction on scalar curvature admitting a projective vector field

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Abstract

The purpose of present paper is to continue the work of present author[1,2] without putting any condition on the scalar curvature of Riemannian manifold M

Key words : Curvature tensor, Ricci tensor, Affine Vectorfield, Associated covariant Vector field, Globally isometric to a sphere.

1. Introduction

Let M be a connected Riemannian maniofold of dimension n covered by the system of coordinate neighbourhoods $\{U; x^h\}$ where the indices i,j,k,….. run over the range {1,2,3…,n}. Let $g_{ji}, \{^h_{ji}\}, \nabla_j, K_{kji}{}^h, K_{ji}$ and K, be the covariant components of the metric tensor g, the Christoffel symbols formed by g_{ji}, the operator of the covariant differentiation with respect to $\{^h_{ji}\}$, the components of curvature tensor and the components of Ricci tensor and the scalar curvature of M respectively.

The vector field v^h is called a projective vector field if it satisfies[4]

$$(1.1)\quad L_v\{^h_{ji}\} = \nabla_j\nabla_i v^h + v^k K_{kji}{}^h = \delta^h_j\rho_i + \delta^h_i\rho_j$$

for a certain covariant vector field ρ_i, called the associated vector field of v^h, where L_v denotes the operator of Lie derivation with respect to the vector field v^h. When we refer to a projective vector field v^h, we always mean ρ_i, the associated covariant vector field given in (1.1). In particular if ρ_i is zero, then a projective vector field is called an affine vector field[3].

In 1980, H. Hiramatu[4] has obtained series of integral formulas and integral inequalities in a compact orienable Riemannian manifold assuming that scalar curvature of M as constant. In this paper using projective and the conformal curvature tensor field of type (1,3), we have obtained the series of integral formulas and integral inequalities without putting condition on scalar curvature K of M .we get necessary and sufficient conditions for Riemannian manifold to be isometric to a sphere of radius $\sqrt{\frac{n(n-1)}{K}}$.

2. *Preliminaries :*

This section deals with preliminaries which are needed in the rest of the sections.

(2.1) $\nabla_j \nabla_t v^t = (n+1)\rho_j$

(2.2) $\nabla^j \nabla_j v^t + k_j{}^t v^j = 2\rho^t$

(2.3) $\rho = \frac{1}{n+1} \nabla_t v^t$

(2.4) $\nabla_j(\nabla_i v_h + \nabla_h v_i) = 2\rho_j g_{ih} + \rho_j g_{ih} + \rho_h g_{ji}$

(2.5) $\nabla_j L_v g^{ih} = -2\rho_j g^{ih} - \rho^i \delta_j^h - \rho^h \delta_j^i$

(2.6) $\nabla^j L_v g^{ih} = -2\rho^j g^{ih} - \rho^i g^{jh} - \rho^h g^{ji}$

(2.7) $L_v K_{kji}{}^h = -\delta_k^h \nabla_j \rho_i + \delta_j^h \nabla_k \rho_i$

(2.8) $L_v K_{ji} = -(n-1)\nabla_j \rho_i$

(2.9) $G_{ji} = K_{ji} - \frac{k}{n} g_{ji}$

(2.10) $P_{kji}{}^h = K_{kji}{}^h - \frac{1}{n-1}(\delta_k^h K_{ji} - \delta_j^h K_{hi})$

(2.11) $P_{kji}{}^h = -P_{jki}{}^h$

(2.12) $P_{kji}{}^h + P_{ikj}{}^h + P_{jik}{}^h = 0$

(2.13) $P_{tji}{}^t = 0$, $P_{kjt}{}^t = 0$

(2.14) $P_{kjih} g^{ji} = \frac{n}{n-1} G_{kh}$

(2.15) $\nabla^j K_{ji} = \frac{1}{2} \nabla_i K$

(2.16) $\nabla_t K_{kji}{}^t = \nabla_t K_{kji} - \nabla_j K_{ki}$

(2.17) $\nabla^k P_{kji}{}^h = \frac{n-2}{n-1} \nabla^h G_{ji} - \nabla_i G_j{}^h$

(2.18) $L_v G_{ji} + \frac{1}{n}(L_v K) g_{ji} = -\nabla_j w_i - \nabla_i w_j$

(2.19) $w^h = \frac{n-1}{2} \rho^h + \frac{k}{n} v^h$

(2.20) $L_v P_{kji}{}^h = 0$

(2.21) $\nabla^j \nabla_j \rho_i - K_{ji} \rho^j - \nabla_i \Delta\rho = 0$

(2.22) $\nabla^j(\nabla_j v_i + \nabla_i v_j) = (n+3)\rho_i$

(2.23) $\nabla^j G_{ji} = \frac{n-2}{2n} \nabla_i K$

(2.24) Let M be a complete, connected and simply connected Riemannian manifold. In order for M to admit a non-trivial solution of a system φ of partial differential equations

$$\nabla_j \nabla_i \varphi_h + K(2\varphi_i g_{ih} + \varphi_i g_{jh} + \varphi_h g_{ji}) = 0$$

Where $\varphi_h = \nabla_h \varphi$, k being positive constant , it is necessary and sufficient that M is globally isometric to a sphere of radius $\frac{1}{\sqrt{k}}$ in the Euclidean (n+1) space.

Known lemmas :

Lemma A: In a compact and orientable Riemannian manifold M, we have

$$\int_M (\nabla_i f)(\nabla^i h)dv = -\int_M f\nabla h\, dv = -\int_M h\nabla f\, dv$$

For any function f and h on M, where $\Delta = g^{ji}\nabla_j\nabla_i$ and dv denotes the volume element of M

Lemma B : If, in a compact and orientable Riemannian manifold M, a non-constant function φ satisfies a system of partial differential equations[5-6]

$$\nabla_j \nabla_i \varphi_h + K(\varphi_i g_{ih} + \varphi_i g_{jh} + \varphi_h g_{ji}) = 0$$

where $\varphi_h = \nabla_h \varphi$, k being a constant, then the constant k is a necessarily positive.

Lemma C: If a complete and simply connected Riemannian manifold M with scalar curvature k of dimension n>1 admits a non-affine projective vector field v^h and if the vector field w^h defined by (2.19) is a killing vector field, then M is globally isometric to a sphere of radius $\sqrt{\frac{n(n-1)}{K}}$ in the Euclidean (n+1) space.

3 Lemmas :

In this section we prove to series of Lemmas without putting any restriction on scalar curvature K of M which are needed to establish main theorem in section 4.

Lemma 3.1 : for a projective vector field v^h on a compact and orientable Riemannian manifold M of dimension n>1, we have

$$(3.1)\quad \int_M G_{ji}\rho^j w^i dv = \frac{2}{n-1}\int_M (\nabla_t w^t)^2 dv - \frac{1}{2(n-1)}\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j)dv$$
$$+\frac{2(n+1)}{n(n-1)}\int_M (\nabla_i K)\rho w^i dv - \frac{2}{n(n-1)}\int_M (\nabla_t k)v^t(\nabla_i w^i)dv$$

proof: we have, $G_{ji}=k_{ji}-\frac{k}{n}g_{ji}$

$$\therefore G_{ji}\rho^j w^i = k_{ji}\rho^j w^i - \frac{k}{n}g_{ji}\rho^j w^i$$
$$= k_{ji}\rho^j w^i - \frac{k}{n}\rho_i w^i$$

From (2.21) we have, $\nabla^j\nabla_j\rho_i - k_{ji}\rho^j - \nabla_i\Delta\rho = 0$

$$\therefore k_{ji}\rho^j = \nabla^j\nabla_j\rho_i - \nabla_i\Delta\rho$$

$$(3.2)\quad \therefore G_{ji}\rho^j w^i = [\nabla^j\nabla_j\rho_i - \nabla_i\Delta\rho]w^i - \frac{k}{n}\rho_i w^i$$
$$= (\nabla^j\nabla_j\rho_i)w^i - (\nabla_i\Delta\rho)w^i - \frac{k}{n}\rho_i w^i$$

Integrating (3.2) over M, we get

$$\int_M G_{ji}\rho^j w^i dv = -\int_M (\nabla_i \Delta\rho) w^i dv + \int_M \left(\nabla^j \Delta_j \rho_i\right) w^i dv - \frac{1}{n}\int_M k\rho_i w^i dv$$

$$= -\int_M (\Delta\rho)\left(\nabla_i w^i\right) dv + \int_M \left(\nabla^j \Delta_j \rho_i\right) w^i dv - \frac{1}{n}\int_M k\rho_i w^i dv$$

from Green's theorem

Now (3.3) $\int_M (\Delta\rho)\left(\nabla_i w^i\right) dv = \int_M \left(\Delta_t \nabla^t \rho\right)\left(\nabla_i w^i\right) dv$

$$= \int_M (\Delta_t \rho^t)(\nabla_t w^t) dv$$

From (2.19) we have, $w^t = \frac{n-1}{2}\rho^t + \frac{k}{n} v^t$

(3.4) $\therefore \rho^t = \frac{2}{n-1}[w^t - \frac{k}{n} v^t]$

From (3.4) equation (3.3) reduces to

(3.5) $$\int (\Delta\rho)\left(\nabla_i w^i\right) dv = \int_M \left[\frac{2}{n-1}(\nabla_t w^t) - \frac{2}{n(n-1)}\nabla_t (kv^t)\right]\left(\nabla_i w^i\right) dv$$

$$= \int_M \left[\frac{2}{n-1}(\nabla_t w^t) - \frac{2}{n(n-1)} k(\nabla_t v^t) - \frac{2}{n(n-1)}(\nabla_t k) v^t\right] \nabla_i w^i dv$$

$$= \frac{2}{n-1}\int_M (\nabla_t w^t)^2 dv - \frac{2}{n(n-1)}\int_M k(\nabla_t v^t)\left(\nabla_i w^i\right) dv - \frac{2}{n(n-1)}\int_M (\nabla_t k) v^t \left(\nabla_i w^i\right) dv$$

Consider , (3.6)

$$\nabla_i [k\left(\nabla_t v^t\right) w^i] = (\nabla_i k)\left(\nabla_t v^t\right) w^i + k(\nabla_i (\nabla_t v^t)) w^i + k(\nabla_t v^t)(\nabla_i w^i)$$

Integrating above equation over M, we get

$$\int_M \nabla_i [k\left(\nabla_t v^t\right) w^i] dv = \int_M (\nabla_i k)\left(\nabla_t v^t\right) w^i dv + \int_M k(\nabla_i (\nabla_t v^t)) w^i dv + \int_M k(\nabla_t v^t)(\nabla_i w^i) dv$$

Hence by Green's theorem we obtain

(3.7) $$\int_M (\nabla_i k)\left(\nabla_t v^t\right) w^i dv + \int_M k(\nabla_i (\nabla_t v^t)) w^i dv + \int_M k(\nabla_t v^t)(\nabla_i w^i) dv = 0$$

$$\int_M k(\nabla_t v^t)(\nabla_i w^i)dv = -\int_M (\nabla_i k)\left(\nabla_t v^t\right)w^i dv - \int_M k(\nabla_i(\nabla_t v^t))w^i dv$$

$$\text{from(2.1)} = -\int_M (\nabla_i k)(n+1)\rho w^i dv - \int_M k(\nabla_i(n+1)\rho)w^i dv$$

$$(3.8) \qquad = -(n+1)\int_M (\nabla_i k)\rho w^i dv - (n+1)\int_M k\rho_i w^i dv$$

From (3.8) equation (3.7) reduces to

$$(3.9) \quad \int_M G_{ji}\rho^j w^i dv = \frac{2}{(n-1)}\int_M \left(\nabla_t w^t\right)^2 dv + \frac{2(n+1)}{n(n-1)}\int_M \left(\nabla_i k\right)\rho w^i dv + \frac{2(n+1)}{n(n-1)}\int_M k\rho_i w^i dv - \frac{2}{(n-1)}\int_M \left(\nabla_t k\right)v^t(\nabla_i w^i)dv + \int_M \left(\nabla^j\nabla_j \rho_i\right)w^i dv - \frac{1}{n}\int_M k\rho_i w^i dv$$

$$= \frac{2}{(n-1)}\int_M \left(\nabla_t w^t\right)^2 dv + \frac{2(n+1)}{n(n-1)}\int_M \left(\nabla_i k\right)\rho w^i dv + \int_M \left(\nabla^j\nabla_j \rho_i\right)w^i dv + \frac{(n+3)}{n(n-1)}\int_M k\rho_i w^i dv - \frac{2}{(n-1)}\int_M \left(\nabla_t k\right)v^t(\nabla_i w^i)dv$$

$$= \frac{2}{(n-1)}\int_M \left(\nabla_t w^t\right)^2 dv + \frac{2(n+1)}{n(n-1)}\int \rho L_w k dv + \frac{1}{2(n-1)}\int_M \left(\nabla_j w_i + \nabla_i w_j\right)(\nabla^j w^i + \nabla^i w^j)dv - \frac{2}{n(n-1)}\int_M \left(\nabla_t k\right)v^t(\nabla_i w^i)dv$$

This completes the proof of Lemma 3.1

Lemma 3.2 : for a projective vector field v^h on a compact and orientable Riemannian manifold M we have ,

$$(3.10) \quad \int_M \left(\nabla^j\left(L_v G_{ji}\right)w^i\right)dv + \frac{1}{n}\int_M \left(\nabla^j\left(L_v k\right)\left(w_j\right)\right)dv = \frac{1}{2}\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j)dv$$

Proof : consider, (3.11)

$$\nabla^j\left\{\left(\nabla_j w_i + \nabla_i w_j\right)w^i\right\} = \left(\nabla^j\left(\nabla_j w_i + \nabla_i w_j\right)\right)w^i + \left(\nabla_j w_i + \nabla_i w_j\right)(\nabla^j w^i)$$

Integrating (3.11) over M, we get

$$\int_M \nabla^j\{(\nabla_j w_i + \nabla_i w_j)w^i\}dv = \int_M (\nabla^j(\nabla_j w_i + \nabla_i w_j))w^i dv + \frac{1}{2}\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j)dv$$

If k is not constant then from (2.18) we have,

$$L_v G_{ji} + \frac{1}{n}(L_v k)g_{ji} = -\nabla_j w_i - \nabla_i w_j$$

$$\therefore \nabla_j w_i + \nabla_i w_j = -L_v G_{ji} - \frac{1}{n}(L_v k)g_{ji}$$

Now by green’s theorem, we get

$$\int_M \{\nabla^j(-L_vG_{ji} - \frac{1}{n}(L_vk)g_{ji})w^i\}dv + \frac{1}{2}\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j)dv = 0$$

$$\therefore \int_M \{\nabla^j(L_vG_{ji} + \frac{1}{n}(L_vk)g_{ji})w^i\}dv = \frac{1}{2}\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j)dv$$

$$\therefore \int_M \{\nabla^j(L_vG_{ji}) + \frac{1}{n}(\nabla^j(L_vk))g_{ji})w^i\}dv = \frac{1}{2}\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j)dv$$

(3.12)

$$\int_M \{(\nabla^j(L_vG_{ji}))w^i dv + \frac{1}{n}\int_M (\nabla^j(L_vk))w_j\}dv = \frac{1}{2}\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j)dv$$

This completes the proof of Lemma 3.2

Lemma 3.3 : for a projective vector field v^h on a compact and orientable Riemannian manifold M of

dimension $n > 1$ we have(3.13)

$$\int_M \left(\nabla^j L_v G_j^{\ i}\right) w_i dv$$

$$= -\frac{6}{n-1}\int_M (\nabla_t w^t)^2 dv - \frac{1}{n}\int_M \left(\nabla^j\left(L_v k\right)\right)\left(w_j\right)dv$$

$$+ \frac{(n+2)}{2(n-1)}\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j)dv$$

$$- \frac{6(n+1)}{n(n-1)}\int_M \left(\nabla_i k\right)\rho w^i dv + \frac{(n-2)}{2n}\int_M \left(\nabla_t k\right)(L_v g^{it})w_i dv$$

proof : consider, $\left\{\nabla^j L_v G_j^{\ i}\right\} w_i = \left\{\nabla^j L_v\left(G_{jt} g^{it}\right)\right\} w_i$

$$= \left\{\nabla^j[(L_vG_{it})g^{it} + G_{it}(L_v g^{it})]w_i\right\}$$

$$= (\nabla^j(L_vG_{it})g^{it} w_i + (\nabla^j G_{jt})(L_v g^{it})w_i + G_{jt}\nabla^j(L_v g^{it})w_i$$

from (2.23) ,

$$= (\nabla^j L_v G_{it})w^t + \frac{(n-2)}{2n}\nabla_t k(L_v g^{it})w_i + G_{jt}(\nabla^j L_v g^{it})w_i$$

$$\text{from}(2.7) \quad = (\nabla^j L_v G_{it})w^t + \frac{(n-2)}{2n}\nabla_t k(L_v g^{it})w_i$$

$$-[2G_{jt}\rho^j g^{ti} w_i - G_{jt}\rho^t g^{ji} w_i - G_{jt}\rho^i g^{jt} w_i]$$

$$= (\nabla^j L_v G_{jt})w^t + \frac{(n-2)}{2n}\nabla_t k(L_v g^{it})w_i - [2G_{jt}\rho^j w^t + G_{jt}\rho^t w^j + 0]$$

$$(3.14) \qquad = (\nabla^j L_v G_{jt})w^t + \frac{(n-2)}{2n}\nabla_t k(L_v g^{it})w_i - 3G_{jt}\rho^j w^t$$

Integrating (3.14) over M , we get

$$(3.15) \quad \int_M (\nabla^j L_v G_j{}^i) w_i dv$$

$$= \int_M (\nabla^j L_v G_{ji}) w^i dv - 3\int G_{ji}\rho^j w^i dv + \frac{(n-2)}{2n}\int_M (\nabla_t k)(L_v g^{it}) w_i dv$$

From Lemma(3.1) and Lemma(3.2) equation (3.15) reduces to

$$(3.16) \quad \int_M (\nabla^j L_v G_j{}^i) w_i dv = \frac{-1}{n}\int_M (\nabla^j (L_v k) w_j) dv + \frac{1}{2}\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j) dv$$

$$-\frac{6}{n-1}\int_M (\nabla_t w^t)^2 dv - \frac{6(n+1)}{n(n-1)}\int_M (\nabla_i k)\rho w^i dv$$

$$+\frac{3}{2(n-1)}\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j) dv$$

$$+\frac{(n-2)}{2n}\int_M (\nabla_t k)(L_v g^{it}) w_i dv$$

$$= -\frac{6}{n-1}\int_M (\nabla_t w^t)^2 dv + \frac{(n+2)}{2(n-1)}\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j) dv$$

$$-\frac{1}{n}\int_M (\nabla^j (L_v k) w_j) dv - \frac{6(n+1)}{n(n-1)}\int_M (\nabla_i k)\rho w^i dv$$

$$+\frac{(n-2)}{2n}\int_M (\nabla_t k)(L_v g^{it}) w_i dv$$

This completes the proof of Lemma 3.3

Lemma 3.4 : for a projective vector field v^h on a compact and orientable Riemannian manifold M of dimension n>1 we have,

$$(3.17)\ \int_M g^{kj}(L_v\nabla_k G_{ji})w^i dv$$
$$= -\frac{6}{n-1}\int_M (\nabla_t w^t)^2 dv + \frac{(n+2)}{2(n-1)}\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j)dv$$
$$-\frac{1}{n}\int_M (\nabla^j (L_v k)w_j)dv - \frac{6(n+1)}{n(n-1)}\int_M (\nabla_i k)\rho w^i dv + \frac{6}{n(n-1)}\int_M (\nabla_t k)v^t(\nabla_i w^i)dv$$

Proof: we have,

$$g^{kj}(L_v\nabla_k G_{ji})w^i = (\nabla^j L_v G_{ji})w^i - g^{kj}(\delta_k{}^t\rho_j + \delta_j{}^t\rho_k)G_{ti}w^i$$
$$- g^{kj}(\delta_k{}^t\rho_i + \delta_i{}^t\rho_k)G_{jt}w^i$$

$$= (\nabla^j L_v G_{ji})w^i - (g^{kj}\delta_k{}^t\rho_j + g^{kj}\delta_j{}^t\rho_k)G_{ti}w^i - (g^{kj}\delta_k{}^t\rho_i + g^{kj}\delta_i{}^t\rho_k)G_{jt}w^i$$

$$= (\nabla^j L_v G_{ji})w^i - (\rho^t + \rho^t)G_{ti}w^i - (g^{jt}\rho_i + g^j\delta_i{}^t)w^i G_{jt}$$

$$= (\nabla^j L_v G_{ji})w^i - 2G_{ti}\rho^t w^i - (g^{jt}G_{jt}\rho_i w^i + G_{jt}g^t w^i)$$

$$= (\nabla^j L_v G_{ji})w^i - 2G_{ti}\rho^t w^i - (0 + G_{jt}g^t w^i)$$

$$(3.18)\qquad = (\nabla^j L_v G_{ji})w^i - 3G_{ti}\rho^t w^i$$

Integrating (3.18) over M ,we get

$$(3.19)\qquad \int_M g^{kj}(L_v\nabla_k G_{ji})w^i dv = \int_M (\nabla^j L_v G_{ji})w^i dv - 3\int_M G_{ji}\rho^j w^i dv$$

From Lemma (3.1) and Lemma (3.3) equation (3.19) reduces to

$$(3.20)\ \int_M g^{kj}(L_v\nabla_k G_{ji})w^i dv = \frac{-1}{n}\int_M \nabla^j (L_v k)w_j dv$$
$$+\frac{1}{2}\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j)dv$$

$$-\frac{6}{n-1}\int_M(\nabla_t w^t)^2dv$$
$$+\frac{3}{2(n-1)}\int_M(\nabla_j w_i+\nabla_i w_j)(\nabla^j w^i+\nabla^i w^j)dv$$
$$-\frac{6(n+1)}{n(n-1)}\int_M(\nabla_i k)\rho w^i dv+\frac{6}{n(n-1)}\int_M(\nabla_t k)v^t(\nabla_i w^i)dv$$

$$=-\frac{6}{n-1}\int_M(\nabla_t w^t)^2dv$$
$$+\frac{(n+2)}{2(n-1)}\int_M(\nabla_j w_i+\nabla_i w_j)(\nabla^j w^i+\nabla^i w^j)dv$$
$$\frac{-1}{n}\int_M\nabla^j(L_v k)w_j dv$$
$$-\frac{6(n+1)}{n(n-1)}\int_M(\nabla_i k)\rho w^i dv$$
$$+\frac{6}{n(n-1)}\int_M(\nabla_t k)v^t(\nabla_i w^i)dv$$

4. *Theorems :*

Theorem *A :* If a connected, compact ,orientable and simply connected Riemannian manifold M without any restricted on scalar curvature k of dimension n>1 admits a non-affine projective vector field v^h, then M is globally isometric to a sphere of radius $\sqrt{\frac{n(n-1)}{K}}$ in the Euclidean (n+1) space.

Proof :- from (2.24) we have, $(L_v\rho_{kji}{}^h)g^{ji}$

$$=\frac{n}{n-1}L_vG_k{}^h-\rho_{kji}{}^hL_vg^{ji}$$

Taking co-varient derivative of both sides we get,

$$(4.1)\quad (\nabla^kL_v\rho_{kji}{}^h)g^{ji}=\frac{n}{n-1}(\nabla^kL_vG_k{}^h)-\nabla^k[\rho_{kji}{}^hL_vg^{ji}]$$
$$=\frac{n}{n-1}(\nabla^kL_vG_k{}^h)-(\nabla^k\rho_{kji}{}^h)(L_vg^{ji})-\rho_{kji}{}^h(\nabla^kL_vg^{ji})$$

Now multiply both sides of (4.1) by w_h we get

$$(\nabla^kL_v\rho_{kji}{}^h)g^{ji}w_h=\frac{n}{n-1}(\nabla^kL_vG_k{}^h)w_h-(\nabla^k\rho_{kji}{}^h)(L_vg^{ji})w_h-\rho_{kji}{}^h(\nabla^kL_vg^{ji})w_h$$

From (2.7) and (2.17)

$$=\frac{n}{n-1}(\nabla^kL_vG_k{}^h)w_h-\left[\frac{n-2}{n-1}\nabla^hG_{ji}-\nabla_iG_j{}^h\right](L_vg^{ji})w_h$$
$$-\rho_{kji}{}^h[-2\rho^kg^{ji}-\rho^jg^{ki}-\rho^ig^{kj}]w_h$$

$$=\frac{n}{n-1}(\nabla^kL_vG_k{}^h)w_h-\frac{(n-2)}{n-1}(\nabla^hG_{ji})(L_vg^{ji})w_h+\nabla_iG_j{}^h(L_vg^{ji})w_h$$
$$+[2\rho_{kji}{}^h\rho^kg^{ji}+\rho_{kji}{}^h\rho^jg^{ki}+\rho_{kji}{}^h\rho^ig^{kj}]w_h$$

$$=\frac{n}{n-1}(\nabla^k L_v G_k{}^h)w_h-\frac{(n-2)}{n-1}(\nabla^h G_{ji})(L_v g^{ji})w_h+\nabla_i G_j{}^h(L_v g^{ji})w_h$$
$$+[2\rho_{kjit}g^{ht}g^{ji}w_h-\rho_{jki}{}^h\rho^j g^{ki}w_h+\rho_{kji}{}^h g^{kj}\rho^i w_h]$$

$$=\frac{n}{n-1}(\nabla^k L_v G_k{}^h)w_h-\frac{(n-2)}{n-1}(\nabla^h G_{ji})(L_v g^{ji})w_h+\nabla_i G_j{}^h(L_v g^{ji})w_h$$
$$+[2\rho_{kjit}g^{ji}w^t\rho^k-\rho_{jkit}g^{ki}g^{ht}\rho^j w_h+\rho_{kji}{}^h g^{kj}\rho^i w_h]$$

From (2.16)

$$=\frac{n}{n-1}(\nabla^k L_v G_k{}^h)w_h-\frac{(n-2)}{n-1}(\nabla^h G_{ji})(L_v g^{ji})w_h+\nabla_i G_j{}^h(L_v g^{ji})w_h$$
$$+\left[\frac{2n}{n-1}G_{kt}w^t\rho^k-\frac{n}{n-1}G_{jt}w^t\rho^j+\rho_{kji}{}^h g^{kj}\rho^i w_h\right]$$

$$=\frac{n}{n-1}(\nabla^k L_v G_k{}^h)w_h-\frac{(n-2)}{n-1}(\nabla^h G_{ji})(L_v g^{ji})w_h$$
$$+\left[\frac{2n}{n-1}G_{ji}w^i\rho^j-\frac{n}{n-1}G_{ji}w^i\rho^j+\rho_{kji}{}^h g^{kj}\rho^i w_h\right]$$

From(2.10) consider , $\rho_{kji}{}^h=k_{kji}{}^h-\frac{1}{n-1}\left(\delta_k{}^h k_{ji}-\delta_j{}^h K_{ki}\right)$

(4.2) $\quad \therefore \rho_{kji}{}^h g^{kj}=k_{kji}{}^h g^{kj}-\frac{1}{n-1}\left(\delta_k{}^h k_{ji}-\delta_j{}^h K_{ki}\right)g^{kj}$

$$=k(\delta_k{}^h g_{ji}-\delta_j{}^h g_{ki})g^{kj}-\frac{1}{n-1}\left(\delta_k{}^h g^{kj}k_{ji}-\delta_j{}^h g^{kj}K_{ki}\right)$$

$$=k(\delta_k{}^h\delta_i{}^k-\delta_j{}^h\delta_i{}^j)-\frac{1}{n-1}\left(g^{hj}k_{ji}-g^{hk}K_{ki}\right)$$

$$=k(\delta_i{}^h-\delta_i{}^h)-\frac{1}{n-1}\left(k_i{}^h-K_i{}^h\right)$$

$=0$

From (4.2) equation (4.1) reduces to

$$(4.3)\qquad (\nabla^k L_v \rho_{kji}{}^h) g^{ji} w_h = \frac{n}{n-1}(\nabla^k L_v G_k{}^h) w_h - \frac{(n-2)}{n-1}(\nabla^h G_{ji})(L_v g^{ji}) w_h + \nabla_i G_j{}^h (L_v g^{ji}) w_h + \frac{n}{n-1} G_{ji} \rho^j w^i$$

Integrating (4.3) over M , we get

$$(4.4)\quad \int_M (\nabla^k L_v \rho_{kji}{}^h) g^{ji} w_h dv = \frac{n}{n-1}\int_M (\nabla^k L_v G_k{}^h) w_h dv - \frac{(n-2)}{n-1}\int_M (\nabla^h G_{ji})(L_v g^{ji}) w_h dv + \int_M (\nabla_i G_j{}^h)(L_v g^{ji}) w_h dv + \frac{n}{n-1}\int_M G_{ji}\rho^j w^i \; dv$$

$$= \frac{n}{n-1}\int_M (\nabla^j L_v G_j{}^i) w_i dv - \frac{(n-2)}{n-1}\int_M (\nabla_t G_{ji})(L_v g^{ji}) w^t dv + \int_M (\nabla_k G_{ji})(L_v g^{kj}) w^i dv + \frac{n}{n-1}\int_M G_{ji}\rho^j w^i \; dv$$

Consider (4.5)

$$\nabla_t (G_{ji} L_v g^{ji} w^t) = (\nabla_t G_{ji})(L_v g^{ji}) w^t + G_{ji}(\nabla_t L_v g^{ji}) w^t + G_{ji}(L_v g^{ji})(\nabla_t w^t)$$

Integrating (4.5) over M we get

$$\int_M \nabla_t (G_{ji} L_v g^{ji} w^t) dv = \int_M (\nabla_t G_{ji})(L_v g^{ji}) w^t dv + \int_M G_{ji}(\nabla_t L_v g^{ji}) w^t dv + \int_M G_{ji}(L_v g^{ji})(\nabla_t w^t) dv$$

Now by Green's theorem we get

$$\int_M (\nabla_t G_{ji})(L_v g^{ji}) w^t dv + \int_M G_{ji}(\nabla_t L_v g^{ji}) w^t dv + \int_M G_{ji}(L_v g^{ji})(\nabla_t w^t) dv = 0$$

$$\therefore -\int_M (\nabla_t G_{ji})(L_v g^{ji}) w^t dv = \int_M G_{ji}(\nabla_t L_v g^{ji}) w^t dv + \int_M G_{ji}(L_v g^{ji})(\nabla_t w^t) dv$$

$$\therefore -\frac{(n-2)}{n-1}\int_M (\nabla_t G_{ji})(L_v g^{ji}) w^t dv = \frac{(n-2)}{n-1}\int_M G_{ji}(\nabla_t L_v g^{ji}) w^t dv$$

$$+\frac{(n-2)}{n-1}\int_M G_{ji}(L_v g^{ji})(\nabla_t w^t) dv$$

From (2.6) and (2.18)

$$=\frac{(n-2)}{n-1}\int_M G_{ji}(-2\rho_t g^{ji} - \rho^j \delta_t{}^i - \rho^i \delta_t{}^j) w^t dv - \frac{(n-2)}{n-1}\int_M (L_v G_{ji}) g^{ji}(\nabla_t w^t) dv$$

$$=\frac{(n-2)}{n-1}\int_M (-2\rho_t G_{ji} g^{ji} - G_{ji}\rho^j \delta_t{}^i - G_{ji}\rho^i \delta_t{}^j) w^t dv - \frac{(n-2)}{n-1}\int_M (L_v G_{ji}) g^{ji}(\nabla_t w^t) dv$$

$$=\frac{(n-2)}{n-1}\int_M (0 - \rho^j G_{jt} - \rho^i G_{it}) w^t dv - \frac{(n-2)}{n-1}\int_M \left(\frac{-1}{n}(L_v k) g_{ji} - \nabla_j w_i - \nabla_t w_j\right) g^{ji}(\nabla_t w^t) dv$$

$$=\frac{(n-2)}{n-1}\int_M (-2\rho^j w^i G_{ji}) dv - \frac{(n-2)}{n-1}\int_M \left(\frac{-1}{n}(L_v k) g^{ji} g_{ji} - g^{ji}\nabla_j w_i - g^{ji}\nabla_t w_j\right)(\nabla_t w^t) dv$$

$$=\frac{-2(n-2)}{n-1}\int_M G_{ji}\rho^j w^i dv + \frac{(n-2)}{n-1}\int_M [(L_v k) + (\nabla^i w_i + \nabla^j w_j)(\nabla_t w^t) dv$$

$$=\frac{-2(n-2)}{n-1}\int_M G_{ji}\rho^j w^i dv + \frac{(n-2)}{n-1}\int_M (L_v k) dv + \frac{(n-2)}{n-1}\int_M (\nabla^i w_i + \nabla^j w_j)(\nabla_t w^t) dv$$

$$=\frac{-2(n-2)}{n-1}\int_M G_{ji}\rho^j w^i dv + \frac{(n-2)}{n-1}\int_M (L_v k) dv + \frac{2(n-2)}{n-1}\int_M (\nabla_t w^t)^2 dv$$

Consider (4.6) $\nabla_k \{G_{ji}(L_v g^{kj})\} = (\nabla_k G_{ji})(L_v g^{kj}) + G_{ji}(\nabla_k (L_v g^{kj}))$

Integrating (4.6) over M, we get

$$\int_M \nabla_k \{G_{ji}(L_v g^{kj})\} dv = \int_M (\nabla_k G_{ji})(L_v g^{kj}) dv + \int_M G_{ji}(\nabla_k (L_v g^{kj})) dv \tag{4.7}$$

Now applying Green's theorem we get

$$\therefore \int_M (\nabla_k G_{ji})(L_v g^{kj})dv + \int_M G_{ji}\nabla_k (L_v g^{kj})dv = 0$$

$$\therefore \int_M (\nabla_k G_{ji})(L_v g^{kj})dv = -\int_M G_{ji}\nabla_k (L_v g^{kj})dv$$

$$= -\int_M g^{kj}(L_v \nabla_k G_{ji})dv$$

$$(4.8) \qquad \int_M (\nabla_k G_{ji})(L_v g^{kj})w^i dv = -\int_M g^{kj}(L_v \nabla_k G_{ji})w^i dv$$

From (4.7) and (4.8) equation (4.6) reduces to

$$(4.9) \quad \int_M (\nabla^k L_v {}_{kji}{}^h) g^{ji} w_h dv = \frac{n}{n-1}\int_M (\nabla^k L_v G_k{}^h) w_h dv - \frac{(n-4)}{n-1}\int_M G_{ji}\rho^j w^i dv$$
$$- \int_M g^{kj}(L_v \nabla_k G_{ji}) w^i dv + \frac{2(n-2)}{n-1}\int_M (\nabla_t w^t)^2 dv$$
$$+ \frac{(n-2)}{n-1}\int_M (L_v k)(\nabla_t w^t) dv$$

From Lemma 3.1 ,3.3,3.4 equation (4.9) reduces to

$$\int_M (\nabla^k L_v P_{kji}{}^h) g^{ji} w_h dv = \frac{-6n}{(n-1)^2}\int_M (\nabla_t w^t)^2 dv$$
$$+ \frac{n(n+2)}{2(n-1)^2}\int_M (\nabla_k w_h + \nabla_h w_k)(\nabla^k w^h + \nabla^h w^k) dv$$
$$- \frac{1}{n-1}\int_M \nabla^k (L_v k) w_k dv - \frac{6(n+1)}{(n-1)^2}\int_M (\nabla_h k)\rho w^h dv$$
$$+ \frac{n-2}{2(n-1)}\int_M (\nabla_t k)(L_v g^{ht}) w_h dv - \frac{2(n-4)}{(n-1)^4}\int_M (\nabla_t w^t)^2 dv$$
$$+ \frac{(n-4)}{2(n-1)^2}\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j) dv$$
$$- \frac{2(n^2-3n-4)}{n(n-1)^2}\int_M (\nabla_i k)\rho w^i dv + \frac{2(n-4)}{n(n-1)^2}\int_M (\nabla_t k) v^t (\nabla_i w^i) dv$$

$$+\frac{2(n-2)}{(n-1)}\int_M(\nabla_t w^t)^2 dv+\frac{(n-2)}{(n-1)}\int_M (L_v k)(\nabla_t w^t)dv$$

$$+\frac{6}{n-1}\int_M(\nabla_t w^t)^2 dv$$

$$-\frac{(n+2)}{2(n-1)}\int_M(\nabla_j w_i+\nabla_i w_j)(\nabla^j w^i+\nabla^i w^j)dv$$

$$+\frac{1}{n}\int_M \nabla^j(L_v k)w_j dv+\frac{6(n+1)}{n(n-1)}\int_M(\nabla_i k)\rho w^i dv$$

$$-\frac{6}{n(n-1)}\int_M(\nabla_t k)v^t(\nabla_i w^i)dv$$

$$=\left(-\frac{6n}{(n-1)^2}-\frac{2(n-4)}{(n-1)^2}+\frac{2(n-2)}{n-1}+\frac{6}{n-1}\right)\int_M(\nabla_t w^t)^2 dv$$

$$+\left(\frac{n(n+2)}{2(n-1)^2}+\frac{n-4}{2(n-1)^2}-\frac{(n+2)}{2(n-1)}\right)\int_M(\nabla_j w_i+\nabla_i w_j)(\nabla^j w^i+\nabla^i w^j)dv$$

$$+\left(\frac{-1}{n-1}+\frac{1}{n}\right)\int_M \nabla^j(L_v k)w_j dv$$

$$+\left(\frac{-6(n+1)}{(n-1)^2}-\frac{2(n^2-3n-4)}{n(n-1)^2}+\frac{6(n+1)}{n(n-1)}\right)\int_M(\nabla_i k)\rho w^i dv$$

$$+\frac{n-2}{2(n-1)}\int_M(\nabla_t k)(L_v g^{ht})w_h dv$$

$$+\frac{n-2}{n-1}\int_M(L_v k)(\nabla_t w^t)dv$$

$$+\left(\frac{2(n-4)}{n(n-1)^2}-\frac{6}{n(n-1)}\right)\int_M(\nabla_t k)v^t(\nabla_i w^i)dv$$

$$
\begin{aligned}
(4.10) = {} & \left(\frac{2(n-3)}{(n-1)}\right)\int_M (\nabla_t w^t)^2 dv \\
& + \left(\frac{1}{(n-1)}\right)\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j) dv \\
& - \left(\frac{2}{n(n-1)}\right)\int_M \nabla^j (L_v k) w_j dv - \left(\frac{2(n+1)}{n(n-1)}\right)\int_M \rho(L_w k) dv \\
& + \frac{n-2}{2(n-1)}\int_M (\nabla_t k)(L_v g^{ht}) w_h dv \\
& + \frac{n-2}{n-1}\int_M (L_v k)(\nabla_t w^t) dv \\
& - \left(\frac{2(2n+1)}{n(n-1)^2}\right)\int_M (\nabla_t k) v^t (\nabla_i w^i) dv
\end{aligned}
$$

From this and (2.20) we obtain

$$
\begin{aligned}
(4.11) \quad & \left(\frac{2(n-3)}{(n-1)}\right)\int_M (\nabla_t w^t)^2 dv \\
& + \left(\frac{1}{(n-1)}\right)\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j) dv \\
& - \left(\frac{2}{n(n-1)}\right)\int_M \nabla^j (L_v k) w_j dv \\
& + \left(\frac{2}{n(n-1)}\right)\int_M L_v L_w k \, dv + \frac{n-2}{2(n-1)}\int_M (\nabla_t k)(L_v g^{ht}) w_h dv \\
& + \frac{n-2}{n-1}\int_M (L_v k)(\nabla_t w^t) dv \\
& - \left(\frac{2(2n+1)}{n(n-1)^2}\right)\int_M (\nabla_t k) v^t (\nabla_i w^i) dv \quad = 0 \qquad \because \int_M \rho f dv = \frac{-1}{n+1}\int_M L_v f dv
\end{aligned}
$$

Remark A: If this scalar curvature k of M is constant then (4.11) of above theorem reduces to

$$\left(\frac{2(n-3)}{(n-1)}\right)\int_M (\nabla_t w^t)^2 dv + \left(\frac{1}{(n-1)}\right)\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j) dv = 0$$

If n>2 then we have, from the above relation

$$\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j) dv = 0$$

From which $\nabla_j w_i + \nabla_i w_j = 0$

i.e. vector field w^h defined by (2.19) is a killing vector field and theorem A follows from Lemma C

If n=2 then we have $\rho_{kji}{}^h = 0$

$\because$ from (2.16) $G_{ji} = 0$ consequently using lemma 3.2

We have $\int_M (\nabla_j w_i + \nabla_i w_j)(\nabla^j w^i + \nabla^i w^j) dv = 0$

from which $\nabla_j w_i + \nabla_i w_j = 0$

$\because$ theorem follow from Lemma C.

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