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Fixed Point Theorems for Generalized Metric Space

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Abstract

In this paper a unique fixed point theorem is proved for generalized metric space satisfying a generalized contractive condition, using asymptotic regularity. Our result unifies and generalizes various known results.

Key wards: Generalized metric space, asymptotic regularity.

Introduction

Browder and Petryshyn[4] introduced the notion of asymptotic regularity for a Banach space. Its equivalent definition in metric space is given by Edelstein and Oberien. Dass and Gupta proved a fixed point theorem for metric space under asymptotic regularity. In the present paper a unique fixed point theorem is proved for generalized metric space under asymptotic regularity. Study of fixed point theorems in generalized metric spaces was indicated by Abbas and Rhoades[3]. Abbas *et al.*[2] obtained some periodic point results in generalized metric space. While, Chugh *et al.* obtained some fixed point results for maps satisfying p in G-metric spaces. Shadati *et al.* studied some fixed point results for contractive mappings in partially ordered G-metric spaces. Abbas *et al.*[1], Chaudhary[5] gave some new results on common fixed point theorem in two generalized metric spaces. The purpose of this paper is to prove fixed point theorems satisfying Hardy and Rogers type condition in generalized metric space under the asymptotic regularity.

Consistent with Mustafa and Sims, the following definitions are needed.

Definition 1.1. Let X is a nonempty set. Suppose that a mapping $G : X \times X \times X \to R^+$ satisfies

$G_1 : G(x, y, z) = 0$ if $x = y = z$

$G_2 : 0 < G(x, y, z)$ for all $x, y, z \in X$

$G_3 : G(x, x, y) < G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$

$G_4 : G(x, y, z) = G(x, z, y) = G(y, z, x)$, symmetry in all three variables

$G_5 : G(x, y, z) < G(\mathrm{x,a,a}) + \mathrm{G(a,y,z)}$, for all $x, y, z, a \in X$

Then G is called a G-metric on X and (X, G) is called a G metric space.

Definition 1.2. A sequence $\{x_n\}$ in a G-metric space X is G-Cauchy sequence if, for any $\in > 0$, there is an $n_0 \in N$(the set of natural numbers) such that for all $n, m, l \geq n_0, G(x_n, x_m, x_l) < \in$

Definition 1.3. A G-metric space on X is said to be symmetric if

$G(x, y, y) = G(y, x, x)$ for all $\mathrm{x, y} \in \mathrm{X}$.

Proposition: Every G metric on X is define a metric d_G on X by

$d_G(x, y) = G(x, y, y) + G(y, x, x) \forall x, y \in X$

For a symmetric G-metric

$d_G(x, y) = 2G(x, y, y) \; \forall x, y \in X$

Definition 1.4. Let X is a G-metric space. Then $f : X \to X$ is said to be asymptotic regular at $x \in X$ if $\lim G(f^n(x), f^{n-1}(x), f^{n-1}(x)$ approaches to zero as $n \to \infty$, where $f^n(x)$ is n^th^ iterate of f at $x \in X$.

Theorem 1: Let X be a complete G-metric space. Suppose f is a self mapping on X satisfying Hardy and Rogers type condition

1.1. $G(f(x), f(y), f(y)) \leq a_1 G(x, f(x), f(x)) + a_2 G(y, f(y), f(y)) + a_3 G(x, f(y), f(y)) + a_4 G(y, f(x), f(x)) + a_5 G(x, y, y)$

1.2. $G(f(y), f(x), f(x)) \leq a_1 G(fx, x, x) + a_2 G(fy, y, y) + a_3 G(fy, x, x) + a_4 G(fx, y, y) + a_5 G(y, x, x)$

for all $x,y \in X$, where $a_i > 0$ and $(a_3 + a_4 + a_5) \leq 1$, for i=1,2,3,4,5

Then f has a unique fixed point in X, if f is asymptotic regular at some point in X.

Proof: Consider the sequence { $f^n(x_0)$ } and assume that f is asymptotic regular at some point $x_0 \in X$. Then, for $n,m \geq 1$, we have by above conditions 1.1 and 1.2

1.3. $G(f^n(x_0), f^m(x_0), f^m(x_0)) \leq a_1 G(f^{n-1}(x_0), f^n(x_0), f^n(x_0)) + a_2 G(f^{m-1}(x_0), f^m(x_0), f^m(x_0))$
$a_3 G(f^{n-1}(x_0), f^m(x_0), f^m(x_0)) + a_4 G(f^{m-1}(x_0), f^n(x_0), f^n(x_0))$
$a_5 G(f^{n-1}(x_0), f^{m-1}(x_0), f^{m-1}(x_0))$

1.4. $G(f^m(x_0), f^n(x_0), f^n(x_0)) \leq a_1 G(f^n(x_0), f^{n-1}(x_0), f^{n-1}(x_0)) + a_2 G(f^m(x_0), f^{m-1}(x_0), f^{m-1}(x_0))$
$a_3 G(f^m(x_0), f^{n-1}(x_0), f^{n-1}(x_0)) + a_4 G(f^n(x_0), f^{m-1}(x_0), f^{m-1}(x_0))$
$a_5 G(f^{m-1}(x_0), f^{n-1}(x_0), f^{n-1}(x_0))$

Using the definition of asymptotic regularity the above two conditions reduces

1.5 $G(f^n(x_0), f^m(x_0), f^m(x_0)) \le a_3 G(f^{n-1}(x_0), f^m(x_0), f^m(x_0)) + a_4 G(f^{m-1}(x_0), f^n(x_0), f^n(x_0))$

$$+\ a_5 G(f^{n-1}(x_0), f^{m-1}(x_0), f^{m-1}(x_0))$$

And

1.6 $G(f^m(x_0), f^n(x_0), f^n(x_0)) \le a_3 G(f^m(x_0), f^{n-1}(x_0), f^{n-1}(x_0)) + a_4 G(f^n(x_0), f^{m-1}(x_0), f^{m-1}(x_0))$

$$a_5 G(f^{m-1}(x_0), f^{n-1}(x_0), f^{n-1}(x_0))$$

Adding 1.5 and 1.6

$$d_G(f^n(x_0), f^m(x_0)) \le a_3 d_G(f^{n-1}(x_0), f^m(x_0)) + a_4 d_G(f^{m-1}(x_0), f^n(x_0)) + a_5 d_G(f^{n-1}(x_0), f^{m-1}(x_0))$$

$$\le a_3\{d_G(f^{n-1}(x_0), f^{m-1}(x_0)) + d_G(f^{m-1}(x_0), f^m(x_0))\} +$$

$$a_4 d_G(f^{m-1}(x_0), f^n(x_0)) + a_5 d_G(f^{n-1}(x_0), f^{m-1}(x_0))$$

$$= (a_3 + a_5) d_G(f^{n-1}(x_0), f^{m-1}(x_0)) + a_4 d_G(f^{m-1}(x_0), f^n(x_0))$$

$$\le (a_3 + a_5) d_G(f^{n-1}(x_0), f^{m-1}(x_0)) +$$

$$a_4\{d_G(f^{m-1}(x_0), f^{n-1}(x_0)) + d_G(f^{n-1}(x_0), f^n(x_0))\}$$

$$\therefore d_G(f^m(x_0), f^n(x_0)) \le (a_3 + a_4 + a_5) d_G(f^{m-1}(x_0), f^{n-1}(x_0))$$

$\therefore d_G(f^m(x_0), f^n(x_0)) \le d_G(f^{m-1}(x_0), f^{n-1}(x_0))$, if $(a_3 + a_4 + a_5) < 1$

$$\le d_G(f^{m-2}(x_0), f^{n-2}(x_0))$$

$$\le d_G(f^{m-3}(x_0), f^{n-3}(x_0))$$

$$\le d_G(f^{m-4}(x_0), f^{n-4}(x_0))$$

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$\therefore 2G(f^m(x_0), f^n(x_0), f^n(x_0)) \to 0$, if G is symmetric

Hence sequence $f^n(x_0)$ is a Cauchy sequence. Further G-metric space on X is complete, thus sequence $\{f^n(x_0)\}$ is convergent to the point z in X, therefore $z = \lim_{n\to\infty} f^n(x_0)$.

Let $z \ne f(z)$

$$G(z, f(z), f(z)) \le G(z, f^n(x_0), f^n(x_0))$$

$$+ G(f^n(x_0), f(z), f(z))$$

$$\le G(z, f^n(x_0), f^n(x_0)) + a_1 G(f^{n-1}(x_0), f^n(x_0), f^n(x_0)) + a_2 G(z, f(z), f(z))$$

$$+ a_3 G(f^{n-1}(x_0), f(z), f(z))$$

$$a_4 G(z, f^n(x_0), f^n(x_0)) + a_5 G(f^{n-1}(x_0), z, z)$$

$$= G(z,z,z) + 0 + (a_2 + a_3) G(z, f(z), f(z))$$

$$+ a_4 G(z,z,z) + a_5 G(z,z,z)$$

$$\therefore (G(z, f(z), f(z)) \le (a_2 + a_3) G(z, f(z), f(z))$$

This is contradiction. Hence $f(z) = z$. Thus z is a fixed point of f.

To prove the uniqueness of fixed point, let w be another fixed point such that $f(w) = w$, $z \neq w$. Then

$$\begin{aligned} G(z, w, w) &= G(fz, fw, fw) \\ &\leq a_1 G(z, fz, fz) + a_2 G(w, fw, fw) \\ &\quad + a_3 G(z, fw, fw) + a_4 G(w, fz, fz) \\ &\quad + a_5 G(z, w, w) \\ &= a_1 G(z, z, z) + a_2 G(w, w, w) \\ &\quad + a_3 G(z, w, w) + a_4 G(w, z, z) \\ &\quad + a_5 G(z, w, w) \end{aligned}$$

$$\therefore G(z, w, w) \leq (a_3 + a_4 + a_5) G(z, w, w)$$

This implies z = w

Thus f has unique fixed point. This completes the proof of theorem 1.

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