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Additional Characterizations of Separation Axioms Using Proper Subspaces

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Abstract

Within this paper recent characterizations of separation axioms obtained by using proper subspaces and proper subspace inherited properties are used to further characterize the separation axioms.

Key words: subspaces, separation axioms, proper subspace inherited properties.

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Introduction

When studying a property of topological spaces, the following question often arises: "Does the space have the property iff each subspace has the property?, *i.e.*, is the property a subspace property?

In 2013[1] onto functional open equal closed spaces were introduced and characterized and in a follow up paper[2], the onto functionally open equal closed property was proven to be a subspace property.

Definition 1.1. A space (X,T) is onto functionally open equal closed (ofoec) iff for each space (Y,S) and each function f from (X,T) onto (Y,S), f is open iff it is closed[1].

Theorem 1.1. A space (X,T) is ofoec iff (1) T is the power set of X or (2) T is the indiscrete topology on X, or (3) X has exactly two elements a and b and T = {ϕ ,X,{a}} or T= {ϕ ,X,{b}}[1].

Theorem 1.2. A space is ofoec iff each subspace is ofoec[2].

As is characteristic in the study of subspace properties, the proof of the converse statement in Theorem 1.2 was quick and easy using the space is a subspace of itself. After submitting the paper [ ], the long overlooked, long overdue observation that the properties of subspace properties have no role in the proof of the converse statement for subspace property theorems was recognized and proper subspaces were used giving the properties themselves a new, central role in the study of subspaces[3].

Definition 1.2. Let P be a property of topological spaces. If for a space (X,T), each proper subspace of (X,T) has property P implies (X,T) has property P, then property P is called a proper subspace inherited property (psip)[3].

Surprisingly, the meaningful role played by proper subspaces in the study of subspaces and properties of spaces did not surface until the paper above[3]. Below the role of proper subspaces is expanded in additional characterizations of separation axioms.

Since one element spaces satisfy many properties, in this paper and the earlier cited paper concerning proper subspace inherited properties[3], only spaces with three or more elements are considered.

New Characterizations of T_i; i = 0,1,2, and Urysohn Separation Axioms :

Within the introductory paper[3], it was proven that each separation axiom T_i; i = 0,1,2, and Urysohn is a proper subspace inherited property. Combining this result with the fact that each of the cited separations is a subspace property gave the following results.

Theorem 2.1. A space (X,T) is T_i; i = 0,1,2, or Urysohn iff each proper subspace of (X,T) is T_i; i =0,1,2,3, or Urysohn, respectively[3].

The results above are used below to further characterize separation axioms using proper subspaces.

Theorem 2.2. Let (X,T) be a space. Then the following are equivalent: (a) (X,T) is T_0, (b) for each x in X and $Y = X \setminus \{x\}$, (Y,T_Y) is T_0, (c) for each proper subset $Y = \{x_i: i = 1, —, n\}$ of X, (Y,T_Y) is T_0, and (d) for $Y = \{a,b\}$, a not b, (Y,T_Y) is T_0.

Proof: Clearly, by Theorem 2.1, (a) implies (b).

(b) implies (c): Let $Y = \{x_i: i = 1,—,n\}$ be a proper subset of X. Let x be in $X \setminus Y$. Let $Z=X\setminus\{x\}$. Since (Z,T_Z) is T_0 and T_0 is a subspace property, then $(Y,(T_Z)_Y)=(Y, T_Y)$ is T_0.

Clearly, (c) implies (d).

(d) implies (a): Suppose (X,T) is not T_0. Let x and y be distinct elements of X such that every one set containing one of x and y contains both of x and y and let $Y = \{x,y\}$. Then T_Y is the indiscrete topology on Y and (T,T_Y) is not T_0. Hence (X,T) is T_0.

Theorem 2.3. Let (X,T) be a space with $X = \{a,b\}$; a not b. Then the following are equivalent: (a) (X,T) is T_0, (b) (X,T) is onto functional open equal closed and T is not the indiscrete topology, and (c) T is not the indiscrete topology on X.

Proof: (a) implies (b): Since (X,T) is $T_{0,}$ {a} or {b} is in T, say {a} is in T. If {b} is not in T, then (X,T) is ofoec and if {b} is in T, then T is the discrete topology on X and (X,T) is ofoec. Hence (X,T) is ofoec and T is not the indiscrete topology on X.

Clearly (b) implies (c) and (c) implies (a).

Corollary 2.1. Let (X,T) be a space. Then the following are equivalent: (a) (X,T) is T_0, (b) for each distinct pair x and y in X and Y = {x,y}, (Y,T_Y) is ofoec and T_Y is not the indiscrete topology on Y, and (c) for each distinct pair x and y in X and Y = {x,y}, T_Y is not the indiscrete topology on Y.

Theorem 2.4. Let (X,T) be a space. Then the following are equivalent: (a) (X,T) is T_1, (b) for each x in X and Z = X \ {x}, (Z,T_Z) is T_1, (c) for each finite proper subset Y = {x_i: i = 1,---,n} of X, T_Y is the discrete topology on Y, and (d) for distinct elements x and y in X and Y = {x,y}, T_Y is the discrete topology on Y.

Clearly, by the results above, (a) implies (b).

(b) implies (c): Let Y = {x_i: i = 1,—,n} be a proper subset of X. Let x be in X \ Y and let Z = X \ {x}. Then (Y,T_Z) is a subspace of the T_1 space (Z,T_Z) and $(Y,(T_Z)_Y)$ is a finite T_1 space, which implies $T_Y = (T_Z)_Y$ is the discrete topology on Y.

Clearly (c) implies (d).

(d) implies (a): Suppose (X,T) is not T_1. Let x and y be in X such that every open set containing x contains y or every open set containing y contains x. Let Y = {x,y}. Then T_Y is not the discrete topology on Y.

Theorem 2.5. Let (X,T) be a space. Then the following are equivalent: (a) (X,T) is T_2, (b) for each x in X and Y = X \ {x}, (Y,T_Y) is T_2, (c) for each proper subset Z of X and for each nonempty finite set of distinct elements Y = {x_i: i = 1,---,n} of Z there exist disjoint T_Z-open set O_i ; i = 1,---,n, such that x_i is in O_i for all i, and (d) for each proper subset Z of X and subset Y = {x,y}, x not y, of Z, there exist disjoint T_Z-open set U and V such that x is in U and y is in V.

Proof: From the results above (a) implies (b).

(b) implies (c). Let Z be a proper subset of X and let Y = {x_i: i=1,---,n} be a nonempty finite set of distinct elements of Z. Let x be in X \ Z and let W = X \ {x}. Then Z is a subset of the T_2 space (W,T_W) and (Z,T_Z) is T_2. Since Y = {x_i: i = i,---,n} is a nonempty finite set of distinct elements in the T_2 space (Z,T_Z), there exist disjoint T_Z- open sets O_i such that x_i is in O_i for each[4] i .

Clearly (c) implies (d).

(d) implies (a): Let x and y be distinct elements of X and let Z = Y = {x,y}. Then T_Y is the discrete topology on Y and by Theorem 2.4, (X,T) is T_1.

Since X has three or more elements, let z be in X different from both x and y. Then {w} is T-closed and W = X \ {w} is T-open. Let U and V be disjoint T_W-open sets such that x is in U and y is in V. Since W is T-open, then U and V are T-open. Hence (X,T) is T_2.

Theorem 2.6. Let (X,T) be a space. Then the following are equivalent: (a) (X,T) is

Urysohn, (b) for each x in X and $Y = X \setminus \{x\}$, (Y,T_Y) is Urysohn, (c) for each proper subset Z of X and for each nonempty finite subset Y = $\{x_i: i = 1,---,n\}$of distinct elements of Z there exist T_Z-open sets O_i ; i = 1,--- ,n}, such that x_i is in O_i and whose T_Z-closures are disjoint, and (d) for each proper subset Z of X and subset $Y = \{x,y\}$, x not y, of Z, there exist T_Z-open sets U and V such that x is in U, y is in V, and whose T_Z –closures are disjoint.

Proof: From the results above (a) implies (b).

(b) implies (c): Let Z be a proper subset of X and let $Y = \{x_i: i = 1,---,n\}$ be a nonempty finite set of distinct elements of Z. Let w be in $X \setminus Z = W$. Then (Z,T_Z) is a subspace of the Urysohn space (W,T_W) and is Urysohn. Since $Y = \{x_i: i = 1,---,n\}$ is a finite set of distinct elements of the Urysohn space (Z,T_Z), there exist T_Z-open sets O_i ; i = 1,---, n, such that x_i is in O_i whose T_Z-closures are disjoint[4].

Clearly (c) implies (d).

(d) implies (a): By Theorem 2.5 (X,T) is T_2. Let x and y be distinct elements of X and let w be in X different from both x and y. Let U, V, and W be disjoint open sets in the T_2 space (X,T) such that x is in U, y is in V, and w is in W [4]. Then $Z = X \setminus \{w\}$ is T-open, {x,y} is a subset of the proper subspace (Z,T_Z), and there exist T_Z-open sets A and B such that x is in A, y is in B, and the T_Z-closures of A and B are disjoint. Then x is in $C = U \cap A$ and y is in $D = V \cap B$, both of which are T-open. Since w is not in the T-closure of U, then the T-closure of C equals the T_Z-closure of C. Similarly the T-closure of D equals the T_Z-closure of D. Hence (X,T) is Urysohn.

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