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Theorems on special, union and hyper-asymptotic curves of a Tachibana recurrent hypersurface

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Abstract

Springer[5], has been studied and defined Union curves of a Riemannian hypersurface Mishra[1], has investigated the properties of these curves in a subspace of a Riemannian space. Further, Saxena and Behari[2], studied Hypersurfaces of Kaehler manifold. Singh[3], studied and defined hypernormal curves of a Finsler subspace.

In the present paper, we have studied on special, Union and hyper-asymptotic curves of a Tachibana Recurrent Hypersurface also several theorems have established and proved therein.

Key words : Union curves, Special curves, Hyper-asymptotic curves, Tachibana space, Recurrent space.

1. Introduction

An almost Tachibana space is an almost Hermite space *i.e* a 2n-dimensional space with an almost Complex structure F_i^h satisfying the relation

$$F_j^i \ F_i^h \ = \ -A_j^h \tag{1.1}$$

and with a Riemannian metric g_{ji} satisfying

$$F_j^t \ F_i^s \ g_{ts} = \ g_{ji} \tag{1.2}$$

from which, we find

$$F_{ji} \ = \ -F_{ij} \tag{1.3}$$

where
$$F_{ji} \ = \ F_j^t \ g_{ti} \tag{1.4}$$

and finally has the property that the skew-symmetric tensor F_{ih} is a killing tensor

$$F_{ih,j} + F_{jh,i} = 0 \tag{1.5}$$

from which

$$F_{i,j}^h + F_{j,i}^h = 0 \tag{1.6}$$

and

$$F_i \ = \ - F_{i,j}^j \tag{1.7}$$

where the comma (,) followed by an index denotes the operation of covariant differentiation with respect to the metric tensor g_{ij} of the Riemannian space.

If the space satisfies the condition ([4])

$$F^h_{i,j} = 0\,, \tag{1.8}$$

then the space is said to be a Tachibana space and is denoted by T^c_n .

The curvature tensor of the T^c_n is defined as

$$R^h_{ijk} = \partial_i \left\{ {h \atop j\ k} \right\} - \partial_j \left\{ {h \atop i\ k} \right\} + \left\{ {h \atop i\ a} \right\} \left\{ {a \atop j\ k} \right\} - \left\{ {h \atop j\ a} \right\} \left\{ {a \atop i\ k} \right\},$$

The Ricci tensor and scalar curvature are respectively given by

$$R_{ij} = R^a_{aij} \quad \text{and}\ R = g^{ij} R_{ij}\,.$$

A Tachibana hypersurface is said to be Tachibana recurrent hypersurface if its curvature tensor satisfies the following condition

$$R^h_{ijk,a} - \lambda_a R^h_{ijk} = 0\,, \tag{1.9}$$

It is said to be Ricci-recurrent hypersurface, if it satisfies

$$R_{ij,a} - \lambda_a R_{ij} = 0\,, \tag{1.10}$$

Multiplying (1.10) by g^{ij}, we have

$$R_{,a} - \lambda_a R = 0\,. \tag{1.11}$$

Consider a hypersurface C_n

$$z^i = z^i(u^\alpha)\,, z^{\bar{i}} = z^{\bar{i}}(u^{\bar{\alpha}})$$

of an (n+1)-dimensional Complex manifold C_{n+1} . whose metric tensor satisfies the Kaehlerian condition

$$\frac{\partial g_{i\bar{j}}}{\partial z^{\bar{k}}} = \frac{\partial g_{i\bar{k}}}{\partial z^{\bar{j}}} \tag{1.12}$$

It has been shown in[2] that an analytic supersurface of a Kaehler manifold is also a Kaehler manifold. The hypersurface and embedding space are denoted by K^c_n and K^c_{n+1} respectively.

We shall use the following fundamental formulae[6] in the later part of the paper. The metric tensor of K^c_n is given by

$$g_{\alpha\bar{\beta}} = g_{i\bar{j}}\ \mathcal{B}^i_\alpha\ \mathcal{B}^{\bar{j}}_{\bar{\beta}}\,, \tag{1.13}$$

where ,

$$\mathcal{B}^i_\alpha = \frac{\partial z^i}{\partial u^\alpha}\,, \qquad \mathcal{B}^{\bar{i}}_{\bar{\alpha}} = \frac{\partial z^{\bar{i}}}{\partial u^{\bar{\alpha}}}$$

If $(N^i, N^{\bar{i}})$ be the components of a unit vector normal to the hypersurface, then

$$2\, g_{i\bar{j}}\ N^i N^{\bar{j}} = 1\,, \tag{1.14}$$

$$g_{i\bar{j}}\ N^i \mathcal{B}^{\bar{j}}_{\bar{\beta}} = 0,\ g_{i\bar{j}}\ N^{\bar{j}} \mathcal{B}^i_\alpha = 0 \tag{1.15}$$

and so its complex conjugate .

The unit vector $(\xi^i, \xi^{\bar{i}})$ orthogonal to

$$\left(\frac{dz^i}{ds}, \frac{dz^{\bar{i}}}{ds}\right)$$

is given by

$$g_{i\bar{j}}\, \frac{dz^i}{ds}\, \xi^j + g_{\bar{i}j}\, \frac{dz^{\bar{i}}}{ds}\, \xi^j = 0, \tag{1.16}$$

and

$$2 g_{i\bar{j}}\ \xi^i\ \xi^{\bar{j}} = 1\,. \tag{1.17}$$

Let $\quad C : z^i = z^i(s), \qquad z^{\bar{i}} = z^{\bar{i}}(s)$

(where s is real arc length) be a curve of K^c_n.

The components $\left(\frac{dz^i}{ds}, \frac{dz^{\bar{i}}}{ds}\right)$ and $\left(\frac{du^\alpha}{ds}, \frac{du^{\bar{\alpha}}}{ds}\right)$ of the unit tangent vectors of C with respect to the enveloping space and the hypersurface are related by

$$dz^i = \mathcal{B}^i_\alpha \frac{du^\alpha}{ds}, \tag{1.18}$$

and its conjugate.

If $(q^i, q^{\bar{i}})$ and $(p^\alpha, p^{\bar{\alpha}})$ are the component of the first curvature vectors with respect to K^c_{n+1} and K^c_n respectively, then we have from[3]

$$q^i = \mathcal{B}^i_\alpha p^\alpha + {}^*K_n N^i, \tag{1.19}$$

and its conjugate

where the normal curvature $({}^*K_n, {}^*\overline{K}_n)$ of the hypersurface given by

$${}^*K_n = \Omega_{\alpha\beta}\left(\frac{du^\alpha}{ds}\right)\left(\frac{du^\beta}{ds}\right)$$

$${}^*\overline{K}_n = \Omega_{\overline{\alpha}\overline{\beta}}\left(\frac{du^{\overline{\alpha}}}{ds}\right)\left(\frac{du^{\overline{\beta}}}{ds}\right)$$

and

$$\mathcal{B}^i_{\alpha,\beta} = \Omega_{\alpha\beta} N^i, \qquad \mathcal{B}^{\bar{i}}_{\overline{\alpha}\overline{\beta}} = \Omega_{\overline{\alpha}\overline{\beta}} N^{\bar{i}}$$

$(\Omega_{\alpha\beta}, \Omega_{\overline{\alpha}\overline{\beta}})$ are components of second fundamental tensors of the hypersurface.

Two vectors $(u^\alpha, u^{\overline{\alpha}})$ and $(v^\alpha, v^{\overline{\alpha}})$ of the hypersurface are said to be conjugate, if the relation $\Omega_{\alpha\beta} u^\alpha v^\beta = 0$ and its conjugate hold.

2. Union and Special Curves :

Assuming that $\frac{\delta}{\delta s}$ is the usual covariant differentiation along the curve $C: z^i = z^i(s)$; $z^{\bar{i}} = z^{\bar{i}}(s)$ (where s is real arc length) of K^c_n, where C is non-geodesic and non-asymptotic curve, we have the first two Frenet's formulae in K^c_{n+1} as follows

$$\left.\begin{aligned} \frac{\delta n^i_{(0)}}{\delta s} &= K_{(1)} n^i_{(1)} \\ \frac{\delta n^i_{(1)}}{\delta s} &= -K_{(1)} n^i_{(0)} + K_{(2)} n^i_{(2)} \end{aligned}\right\} \tag{2.1}$$

and their conjugate , where

$(n^i_{(0)}, n^{\bar{i}}_{(0)}) = \left(\frac{dz^i}{ds}, \frac{dz^{\bar{i}}}{ds}\right)$, $(n^i_{(1)}, n^{\bar{i}}_{(1)})$ and $(n^i_{(2)}, n^{\bar{i}}_{(2)})$ are the components of unit tangent vector, unit principal normal vector and unit bi-normal vector, $K_{(1)}$ and $K_{(2)}$ are the first and second curvatures of the curve on the embedding space.

Considering two congruence λ and μ of curves given at any point of K^c_n by

$$\lambda^i = t^\alpha \mathcal{B}^i_\alpha + CN^i \tag{2.2}$$

where t^α and C are parameters and $\overline{t}^\alpha$, $\overline{C}$ are their complex conjugate. Since $(\lambda^i, \overline{\lambda}^i)$ represents a unit vector, $2 g_{i\bar{j}} \lambda^i \lambda^{\bar{j}} = 1$ and it follows by using equations (1.14), (1.15) and (2.2) that

$$2 g_{\alpha\overline{\beta}} t^\alpha t^{\overline{\beta}} = 1 - |c|^2 \tag{2.3}$$

$$\mu^i = s^\alpha \mathcal{B}^i_\alpha + DN^i$$

and its conjugate, where s^α is a real parameter.

Let Γ be a special curve relative to the congruence $\underline{\lambda}$ and an union curve relative to the congruence $\underline{\mu}$. A curve of the congruence is said to be a special curve relative to the congruence $\underline{\lambda}$, if the vector $(\lambda^i, \lambda^{\bar{i}})$ lies in a variety spanned by the vectors $(p^{\alpha}\mathcal{B}_{\alpha}^{i}, p^{\bar{\alpha}}\mathcal{B}_{\bar{\alpha}}^{\bar{i}})$ and $(q^{i}, q^{\bar{i}})$, and a curve of the congruence is said to be union curve relative to the congruence $\underline{\mu}$ if the vector $(\mu^{i}, \mu^{\bar{i}})$ lies in a variety spannend by the vectors $\left(\frac{dz^{i}}{ds}, \frac{dz^{\bar{i}}}{ds}\right)$ and $(q^{i}, q^{\bar{i}})$. In other words

$$\left(\lambda^{i}, \lambda^{\bar{i}}\right) = u\left(p^{\alpha}\mathcal{B}_{\alpha}^{i}, p^{\bar{\alpha}}\mathcal{B}_{\bar{\alpha}}^{\bar{i}}\right) + \omega(q^{i}, q^{\bar{i}})$$

and

$\left(\mu^{i}, \mu^{\bar{i}}\right) = u\left(\frac{dz^{i}}{ds}, \frac{dz^{\bar{i}}}{ds}\right) + z\left(q^{i}, q^{\bar{i}}\right)$, which yield

$$\lambda^{i} = \mu p^{\alpha}\mathcal{B}_{\alpha}^{i} + \omega q^{i} \tag{2.4}$$

and its Conjugate

$$\mu^{i} = u\frac{dz^{i}}{ds} + z\, q^{i} \tag{2.5}$$

and its conjugate.

from equations (1.19),(2.2),(2.3),(2.4) and (2.5) and their conjugates, we have

$$t^{\alpha} = (u + w)p^{\alpha}, \qquad C = w^{*}K_{n} \quad (2.6)$$

$$s^{\alpha} = n\,\frac{du^{\alpha}}{ds} + z\,p^{\alpha}, \qquad D = z^{*}K_{n} \quad (2.7)$$

and their conjugate relations,
we define

$$R^{2} = 2\, g_{\alpha\bar{\beta}}\; t^{\alpha}\, t^{\bar{\beta}},\; S^{2} = 2\, g_{\alpha\bar{\beta}}\; s^{\alpha}\, s^{\bar{\beta}}, \quad (2.8)$$

and their conjugate relations

$$K_{(1)}^{2} = 2\, g_{\alpha\bar{\beta}}\; p^{\alpha}\, p^{\bar{\beta}}$$

$$\cos w = \frac{g_{\alpha\bar{\beta}}\; t^{\alpha}\, s^{\bar{\beta}} + g_{\alpha\bar{\beta}}\; s^{\alpha} s^{\bar{\beta}}}{2\sqrt{g_{\alpha\bar{\beta}}\; t^{\alpha}\, t^{\bar{\beta}}}\sqrt{g_{\alpha\bar{\beta}}\; s^{\alpha}\, s^{\bar{\beta}}}}$$

where $K_{(1)}$ is the first curvature of the curve in K_{n}^{c}. Using (1.16) , (2.6) and their conjugate, (2.7) and its conjugate, we have

$${}^{*}K_{n}\; S\; \cos w = Dk_{(1)} \tag{2.9}$$

and its conjugate.

From this equation and the equations

$$1 = 2\, g_{i\bar{j}}\; \mu^{i}\, \mu^{\bar{j}} = S^{2} + D \tag{2.10}$$

$$K_{(1)}^{2} = 2\, g_{i\bar{j}}\; q^{i}\, q^{j} \tag{2.11}$$

where

$$K_{(1)}^{2} = k_{(1)}^{2} + {}^{*}K_{n}\; {}^{*}\overline{K}_{n},$$

we obtain

$${}^{*}K_{n} = \frac{eK_{(1)}(1 - S^{2})^{\frac{1}{2}}}{(1 - S^{2}\sin^{2} w)^{\frac{1}{2}}} \tag{2.12}$$

and

$$k_{1} = \frac{eK_{(1)}\; s\; \cos w}{(1 - S^{2}\sin^{2} w)^{\frac{1}{2}}}, \tag{2.13}$$

where $e = \pm 1$ in order that e cos w be non-negative.

Now, we have the following:

Theorem (2.1): If a special curve

relative to a fixed congruence $\underline{\lambda}$ is an union curve relative to another fixed congruence $\underline{\mu}$, then the modulus of the normal and the first curvatures with respect to K_n^c at a given point of the curve are proportional to its first curvature with respect to K_{n+1}^c.

Theorem (2.2): If the components of the vector fields $\underline{\lambda}$ and $\underline{\mu}$ tangent to the hypersurface are in the same direction, then the ratio of the two first curvatures is equal to the magnitude of the tangential (to the hypersurface component of $\underline{\mu}$).

Proof : Taking $w = 0$, we get the proof of both theorems.

3. Hyper-asymptotic curves : A curve of hypersurface is said to be a hyper asymptotic curve relative to its congruence $\underline{\mu}$ if the vectors $\left(\mu^i, \mu^{\bar{i}}\right)$ lies in the variety spanned by the vectors

$$\left(\eta^i_{(0)}, \eta^{\bar{i}}_{(0)}\right) \, and \, \left(\eta^i_{(2)}, \eta^{\bar{i}}_{(2)}\right).$$

In other words

$$\left(\mu^i, \mu^{\bar{i}}\right) = u_{(1)}\left(\eta^i_{(0)}, \eta^{\bar{i}}_{(0)}\right) + z_{(1)}\left(\eta^i_{(2)}, \eta^{\bar{i}}_{(2)}\right),$$

which yields

$$\mu^i = u_{(1)}\eta^i_{(0)} + z_{(1)}\eta^i_{(2)} \tag{3.1}$$

and its conjugate.

From equation (2.1) and its conjugate we deduce

$$\frac{\delta g^i}{\delta s} = -K^2_{(1)}\eta^i_{(1)} + \frac{d}{ds}\log K_{(1)}\, q^i + K_{(1)}K_{(2)}\eta^i_{(2)}, \tag{3.2}$$

and its conjugate.

Another expression for $\left(\frac{\delta q^i}{\delta s}, \frac{\delta q^{\bar{i}}}{\delta s}\right)$ will be obtained from (1.19) and its conjugate after using

$$\frac{\delta N^i}{\delta s} = \frac{1}{2}\,\Omega_{\bar{\beta}\bar{\gamma}}\, g^{\bar{\beta}\alpha}\,\mathcal{B}^i_\alpha\,\frac{du^{\bar{\gamma}}}{ds} \tag{3.3}$$

and its conjugate.
From equations (1.19), (2.1), (3.1), (3.2) and their conjugates, we have

$$s^\alpha = u_{(1)}\frac{du^\alpha}{ds} + \nu\left(\frac{\delta p^\alpha}{\delta s} - \frac{{}^*K_n}{2}\,\Omega_{\bar{\beta}\bar{\gamma}}\, g^{\bar{\beta}\alpha}\frac{du^{\bar{\gamma}}}{ds} - p^\alpha\frac{d}{ds}\log K_{(1)} + K^2_{(1)}\frac{du^\alpha}{ds}\right) \tag{3.4}$$

$$D = \nu\left(\Omega_{\alpha\beta}p^\alpha\frac{du^\beta}{ds} + \frac{d\,{}^*K_n}{ds} - {}^*K_n\frac{d}{ds}\log K_{(1)}\right) \tag{3.5}$$

and their conjugate, where

$$\nu = \frac{Z_{(1)}}{K_{(1)}K_{(2)}}.$$

Let $\left(\xi^\alpha_{(0)}, \xi^{\bar{\alpha}}_{(0)}\right), \left(\xi^\alpha_{(1)}, \xi^{\bar{\alpha}}_{(1)}\right) and \left(\xi^\alpha_{(2)}, \xi^{\bar{\alpha}}_{(2)}\right)$ be the unit tangent vector, unit principal normal vector and unit first binormal vectors, $K_{(1)}$ and $K_{(2)}$ be the first and second curvatures of the curve with respect to the hypersurface.

We obtain from Frenet's formulae with respect to K_n^c

$$\frac{\delta p^{\alpha}}{\delta s} = -K_{(1)}^{2}\frac{du^{\alpha}}{ds} + p^{\alpha}\frac{d}{ds}\log K_{(1)} + K_{(1)}K_{(2)}\xi_{(2)}^{\alpha} \tag{3.6}$$

and its conjugate.

Equations (3.4),(3.5),(3.6), their conjugates and the definitions

$$\left.\begin{aligned}\cos\phi &= \frac{\left(g_{\alpha\bar{\beta}}\, s^{\bar{\alpha}}\, \frac{du^{\bar{\beta}}}{ds} + g_{\bar{\alpha}\beta}\, s^{\bar{\alpha}}\, \frac{du^{\beta}}{ds}\right)}{S}\ gives\\ S\cos\phi &= u_{(1)}\end{aligned}\right\} \tag{3.7}$$

and

$$\left(s^{\alpha} - s\cos\phi\frac{du^{\alpha}}{ds}\right)\left[\Omega_{\gamma\beta}p^{\gamma}\frac{du^{\beta}}{ds} + \frac{d^{*}k}{ds} - {}^{*}K_{n}\frac{d}{ds}\log K_{(2)}\right] = D\left[K_{n}\ \bar{K}_{n}\ \frac{du^{\alpha}}{ds} + p^{\alpha}\frac{d}{ds}\log\frac{K_{(1)}}{K_{(2)}} + K_{(1)}K_{(2)}\xi_{(2)}^{\alpha} - \frac{K_{n}}{2}\Omega_{\bar{\beta}\gamma}g^{\alpha\bar{\beta}}\frac{du^{\bar{\gamma}}}{ds}\right]$$

and its conjugate.

A hyper asymptotic curve relative to $\underline{\mu}$ is characterized by this equation.

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