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Einstein constant for almost hyperbolic Hermitian manifold on the product of two Sasakian manifolds

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Abstract

In 1981, Tsukada worked on the isospectral problem with respect to the complex Laplacian for a two-parameter family of Hermitian structures on the Calabi-Eckmann manifold $S_{2p+1} \times S_{2q+1}$ including the canonical one. In this paper, we define a two-parameter family of almost hyperbolic Hermitian structures on the product manifold $\overline{M} = M \times M'$ of a $(2p + 1)$-dimensional Sasakian manifold M and a $(2q + 1)$-dimensional Sasakian manifold M' similarly to the method used in[11], and show that any almost hyperbolic Hermitian structure on $\overline{M}$ belonging to the two parameter family is integrable and again find necessary and sufficient condition for a hyperbolic Hermitian manifold in the family to be Einstein.

Key words : Einstein, Hermitian structure, Sasakian manifold

Mathematics Subject Classification 53C25, 53B35

1. Introduction

Let us consider a differential manifold M_{2n} of class C^∞ endowed with a tensor field of type (1, 1) F such that for an arbitrary vector field X.

$$\overline{\overline{X}} = X \tag{1.1}$$

where $\overline{\overline{X}} \overset{def}{=} F(X)$

then F is called an almost hyperbolic Hermitian structure, and the differential manifold M_{2n} is called almost hyperbolic Hermitian manifold.

On almost hyperbolic Hermitian manifold M_{2n}, if there exists a symmetric metric tensor g such that,

$$g(\overline{X}, \overline{Y}) + g(X, Y) = 0, \tag{1.2}$$

Then we say that g is compatible with almost complex structure and {F, g} is called an almost hyperbolic Hermitian structure. The manifold

M_{2n} with an almost hyperbolic Hermitian structure is said to be an almost hyperbolic Hermitian manifold.

2. Definitions:

Let M = (M, F, g) be a 2n($\geq$ 4)-dimensional almost hyperbolic Hermitian manifold with almost Hermitian structure (F, g). We denote by ∇, K, ρ and τ the Riemannian connection, curvature tensor, t Ricci tensor and scalar curvature of M, respectively[3-6].

The curvature tensor K is defined by
$K(X,Y)\,Z = [\nabla_X,\nabla_Y]\,Z - \nabla_{[X,\,Y]}\,Z$
for arbitrary vector fields X,Y, Z on M.

Ricci $*$-tensor $\rho*$ of M is defined by
$$\rho*(X,Y) = \mathrm{tr}\,(Z \to K(X, F Z) F Y)$$
$$= \frac{1}{2}\mathrm{tr}\,(Z \to K(X,F Y) F Z)), \quad (2.1)$$
for arbitrary vector fields X,Y, Z on M.

Obviously the equality $\rho^* = \rho$ holds on M if M is Kahler.

The $*$-scalar curvature of M, τ^*, which is the trace of the Ricci $*$-operator Q^* is defined by $g(Q^*X,Y) = \rho^*(X,Y)$. A 4-dimensional almost Hermitian manifold is known as an almost hyperbolic Hermitian surface. For any almost hyperbolic Hermitian surface M, the Ricci and Ricci $*$-tensor are related by
$$\rho^*(X,Y) + \rho^*(Y,X) - \{\rho(X,Y) + \rho(FX,F\,Y)\}$$
$$= \frac{\tau^* - \tau}{2}\,g(X,Y), \quad (2.2)$$
for arbitrary vector fields X,Y, Z on M.

A 2n-dimensional almost Hermitian manifold (M, F, g) is called a weakly $*$-Einstein manifold if the equality $\rho^* = \dfrac{\tau}{2n}\,g$ holds on M. If $*$-scalar curvature τ^* of a weakly $*$-Einstein manifold M is constant, then M is said to be $*$-Einstein. It is known that there exist weakly $*$-Einstein manifolds which are not $*$-Einstein[7, 9]. The Nijenhuis tensor N is defined by

$$N(X,Y)=[FX,FY]- [X,Y] - F[FX,Y] - F[X,FY] \quad (2.3)$$
for arbitrary vector fields X,Y on M.

The almost hyperbolic complex structure F is integrable if and only if the Nijenhuis tensor N vanishes identically on M. An almost hyperbolic Hermitian manifold (M, F, g) with integrable almost hyperbolic complex structure F is called a hyperbolic Hermitian manifold.

The condition N = 0 is equivalent to:
$$g((\nabla_X F)\,Y, Z) - g((\nabla_{FX} F)\,F\,Y,Z) = 0 \quad (2.4)$$
for arbitrary vector fields X,Y, Z on M.

An almost contact metric manifold M = (M, ϕ, ξ, η, g) is called a contact metric manifold if it satisfies $d\eta(X, Y) = g(X, \phi Y)$,

for any X, Y $\in$X(M). Further, a normal contact metric manifold is called a Sasakia nmanifold. It is well-known that a Sasakian manifold is characterized as an almost contact metric manifold satisfying the condition
$$(\nabla_X\phi)Y = g(X, Y)\xi - \eta(Y)X, \quad (2.5)$$
for arbitrary vector fields[2] X,Y on M.

On a (2n+1)-dimensional Sasakian manifold (M, ϕ, ξ, η, g), we have the following identities:

$\nabla_X\xi = -\phi X$, $(\nabla_X\eta)(Y) = -g(\phi X, Y)$,
$K(X, Y)\xi = \eta(Y)X - \eta(X)Y$,
$\rho(\xi,X) = 2n\eta(X)$, (2.6)
for arbitrary vector fields X,Y on M [2].

Curvature identity on a Sasakian manifold[2] is as follows:

$$K(X, Y, \phi Z,W) - K(\phi Z,X, Y,W) = -g(X, Y)g(\phi Z,W) - 2g(Z, \phi Y)\, g(X,W) + g(Z, \phi X)g(Y,W), \quad (2.7)$$

for arbitrary vector fields X, Y, Z, W $\in$ M.

3. Almost hyperbolic Hermitian manifold on the product of two sasakian manifold:

Let us define a two parameter family of almost hyperbolic Hermitian structures on the product of two Sasakian manifolds , to find the integrability conditions and necessary and sufficient condition for a hyperbolic Hermitian structure belonging to the family to be Einstein one.

Let (M, ϕ, ξ, η, g) and resp. $(M', \phi', \xi', \eta', g')$ be a (2p + 1)-dimensional Sasakian manifold and resp. a (2q +1)-dimensional Sasakian manifold. Here ∇, K and ρ and resp. ∇', K' and ρ' are the Riemannian connection, the curvature tensor and the Ricci tensor on M and resp. of M'. Let $\overline{M} = M \times M'$ be the product manifold of M and M'. Then define a Riemannian metric $g=g_{a,b}$ (a, b $\in$ R) on $\overline{M}$ by

$$\bar{g}_{a,b}=g+a(\eta\otimes \eta' + \eta' \otimes \eta) + (a^2 + b^2 - 1)\eta' \otimes \eta' + g' \quad (3.1)$$ [11].

Again, define an almost hyperbolic complex structure $F = F_{a,b}$ (a, b $\in$ R, b$\neq$0) on M as follows.

$$F_{a,b}(X + X') = -\phi(X) + \{ \frac{a}{b}\eta(X) + \frac{a^2+b^2}{b}\eta'(X')\}\xi-\phi'(X') - \{\frac{1}{b}\eta(X) + \frac{a}{b}\eta'(X')\}\xi', \quad (3.2)$$

for any tangent vector X of M and any tangent vector of X' M'. Here obviously $F^2 = -I$ holds and $(F, \bar{g})$ is an almost hyperbolic Hermitian structure on M. Since X(M) and X(M') are regarded as the Lie subalgebra of X(M), we may rewrite (3.1) and (3.2) as:

$$\bar{g}(X, Y) = g(X, Y),\ \bar{g}(X, Y') = a\eta(X)\eta'(Y'),$$
$$\bar{g}(X',Y') =g'(X',Y') + (a^2 + b^2 - 1)\eta'(X')\eta'(Y'), \quad (3.3)$$

$$F(X) =\phi(X) - \frac{a}{b}\eta(X)\xi + \frac{1}{b}\eta(X)\xi',$$

$$F(X') =\phi'(X') - \frac{a^2+b^2}{b}\eta'(X')\xi + \frac{a}{b}\eta'(X')\xi', \quad (3.4)$$

for arbitrary vector fields X,Y on M and X’, Y’ $\in$ M’.

Let ∇, K and ρ be the Riemannian connection, the curvature tensor and the Ricci tensor of M, and X, Y, Z,W be any smooth vector field on M and X', Y', Z',W' on M' respectively. Then, from (3.3) and (3.4), by making use of (2.5) and (2.6), we have the following:

$$\bar{g}(\bar{\nabla}_X Y, Z) =g(\nabla_X Y, Z),\ \bar{g}(\bar{\nabla}_{X'} Y, Z) = -a\eta'(X')g(\phi Y, Z),$$

$$\bar{g}(\bar{\nabla}_X Y',Z)=- a\eta'(Y')g(\phi X, Z),\ \bar{g}(\bar{\nabla}_X Y, Z') = a\eta(\nabla_X Y)\eta'(Z'),$$

$$\bar{g}(\bar{\nabla}_{X'} Y', Z) =a\eta'(\nabla'_{X'}Y')\eta(Z),\ \bar{g}(\bar{\nabla}_{X'} Y, Z')$$

$= -a\eta(Y)g'(\phi'X',Z')$,

$\bar{g}\,(\nabla_X Y',Z') = -\,a\eta(X)g'(\phi'Y',Z')$,

$\bar{g}\,(\bar{\nabla}_{X'}Y',Z') = g'(\nabla'_{X'}Y',Z') + (a^2+b^2-1)(\eta'(\nabla'_{X'}Y')\eta'(Z') - \eta'(X')g'(\phi'Y',Z') - \eta'(Y')g'(\phi'X',Z')$. (3.5)

By (3.3), (3.4) and (3.5), we have the following:

$\bar{g}\,((\bar{\nabla}_X F)Y,Z) = \eta(Z)g(X,Y) - \eta(Y)g(X,Z)$,

$\bar{g}\,((\bar{\nabla}_{X'} F)Y,Z) = 0$,

$\bar{g}\,((\bar{\nabla}_X F)Y',Z) = b\eta'(Y')g(\phi X,Z) - a\eta'(Y')(g(X,Z) - \eta(X)\eta(Z))$,

$\bar{g}\,((\bar{\nabla}_{X'} F)Y',Z) = -a\eta(Z)(\eta'(X')\eta'(Y') - g'(X',Y') + b\eta(Z)g'(\phi'X',Y')$,

$\bar{g}\,((\bar{\nabla}_X F)Y',Z') = 0$,

$\bar{g}\,((\bar{\nabla}_{X'} F)Y',Z') = (a^2 + b^2)(g'(X',Y')\eta'(Z') - g'(X',Z')\eta'(Y'))$. (3.6)

Then, from (3.4) and (3.6), we have

$$\bar{g}\,((\bar{\nabla}_{\bar{X}} F)\bar{Y},\bar{Z}) - \bar{g}\,((\bar{\nabla}_{\bar{F}\bar{X}}\bar{F})\bar{F}\,\bar{Y},\bar{Z}) = 0 \quad (3.7)$$

holds for any $\bar{X}, \bar{Y}, \in \bar{M}$. So, from (2.4), the almost hyperbolic complex structure F is integrable and hence, (M, F, $\bar{g}$) is a hyperbolic Hermitian manifold. And from (3.6), it is obvious that $\bar{M}$ is never Kahler.

Theorem 1: Let (M, ϕ, ξ, η, g) and (M$'$, ϕ', ξ', η', g$'$) be a (2p+ 1)-dimensional Sasakian manifold and a (2q +1)-dimensional Sasakian manifold. Let $\bar{M}$ = M × M$'$ be the product manifold of M and M$'$. Then $\bar{M} = (\bar{M}, F, \bar{g})$ is a hyperbolic Hermitian manifold equipped with the hyperbolic Hermitian structure $(\bar{g}, F)$ defined by (3.1) and (3.2).

4. Necessary and sufficient condition for almost hyperbolic Hermitian manifold on product of two sasakian manifold to be Einstein :

From (3.3) and (3.5), by taking account of (2.5) and (2.6), we have the formulas for the curvature tensor of $\bar{M}$:

$\bar{g}\,(\bar{K}(X, Y)Z, W) = g(K(X, Y)Z, W)$,

$\bar{g}\,(\bar{K}(X, Y')Z, W) = -\,a\eta'(Y')(g(X, Z)\eta(W) - g(X,W)\eta(Z)\_$,

$\bar{g}\,(\bar{K}(X', Y')Z, W) = 2ag'(\phi'X', Y')\,g(\phi Z,W)$,

$\bar{g}\,(\bar{K}(X, Y')Z, W') = ag(\phi X, Z)g'(\phi'Y',W') - a^2\eta'(Y')\eta'(W')(g(X, Z) - \eta(X)\eta(Z)) - a^2\eta(X)\,\eta(Z)(g'(Y',W') - \eta'(Y')\eta'(W'))$,

$\bar{g}\,(\bar{K}(X', Y')Z', W) = a\,(a^2 + b^2)\,\eta(W)(\eta'(X)\,g'(Y', Z') - \eta'(Y')\,g'(X', Z'))$,

$\bar{g}\,(\bar{K}(X, Y)Z, W') = a\eta(K(X, Y)Z)\eta'(W')$, (3.8)

$\bar{g}\,(\bar{K}(X', Y')Z', W') = g'(K'(X', Y')Z',W') + 2(a^2 + b^2 - 1)\{\eta'(X')\eta'(W')g'(Y', Z') - \eta'(Y')\eta'(W')g'(X', Z') - \eta'(X')\eta'(Z')\,g'(Y',W') + \eta'(Y')\eta'(Z')g'(X',W')\} - (a^2 + b^2 - 1)^2\{\eta'(X')\eta'(Z')g'(Y',W') - \eta'(Y')\eta'(Z')g'(X',W') + \eta'(Y')\eta'(W')g'(X', Z') - \eta'(X')\eta'(W')g'(Y', Z') + (a^2 + b^2 - 1)\{2g'(\phi'X',Y')g'(\phi'Z',W') + g'(\phi'X',Z')\,g'(\phi'Y',W') - g'(\phi'Y', Z')\,g'(\phi'X',W')$.

From (3.8), we have the following,

$\bar{\rho}\,(Y, Z) = \rho(Y, Z) + 2a^2q\eta(Y)\,\eta(Z)$,

$\bar{\rho}\,(Y, Z') = 2a\,(p + q\,(a^2 + b^2))\eta(Y)\eta'(Z')$,

$$\bar{\rho}(Y', Z') = \rho'(Y', Z') - 2(a^2 + b^2 - 1)g'(Y', Z') + 2(pa^2 + a^2 + b^2 - 1 + q(a^2 + b^2 - 1)(a^2 + b^2 + 1))\eta'(Y')\eta'(Z') \quad (3.9)$$

for $Y, Z \in M$, $Y', Z' \in M'$.

Thus, from (3.3) and (3.9), $(M, \bar{g})$ is Einstein if and only if there is a constant λ satisfying the following conditions:

$$\rho(Y, Z) + 2a^2q\eta(Y)\eta(Z) = \lambda g(Y, Z) \quad (3.10)$$

$$2a(p+q(a^2+b^2))\eta(Y)\eta'(Z') = a\lambda\eta(Y)\eta'(Z') \quad (3.11)$$

$$\rho'(Y', Z') - 2(a^2 + b^2 - 1)g'(Y',Z') + 2(pa^2 + a^2 + b^2 - 1 + q(a^2 + b^2 - 1)(a^2 + b^2 + 1))\eta'(Y')\eta'(Z') = \lambda(g'(Y', Z') + (a^2 + b^2 - 1)\eta'(Y')\eta'(Z')) \quad (3.12)$$

for $Y, Z \in M$, $Y', Z' \in M'$.

(3.10) and (3.12) may be rewritten as:

$$\rho(Y,Z) = \lambda g(Y,Z) - 2a^2q\eta(Y)\eta(Z) \quad (3.13)$$

$$\rho'(Y', Z') = (\lambda + 2(a^2 + b^2 - 1))g'(Y', Z') + \{\lambda(a^2 + b^2 - 1) - 2(pa^2 + a^2 + b^2 - 1 + q(a^2+b^2-1)(a^2+b^2+1)\}\eta'(Y')\eta'(Z') \quad (3.14)$$

for $Y, Z \in M$, $Y', Z' \in M'$.

As M and M' are both Sasakian, so we have

$$\rho(Y,\xi) = 2p\eta(Y),$$
$$\rho'(Y',\xi') = 2q\eta'(Y')$$
(3.15)

for $Y \in M$, $Y' \in M'$.

Thus, from (3.13) and (3.15), we obtain

$$2p\eta(Y) = (\lambda - 2a^2q)\eta(Y). \quad (3.16)$$

And from (3.14) and (3.15), we have

$$2q\eta'(Y') = (\lambda(a^2 + b^2) - 2pa^2 - 2q(a^2 + b^2 - 1)(a^2 + b^2 + 1))\eta'(Y'). \quad (3.17)$$

From (3.16) and (3.17), we have

$$\lambda = 2p + 2a^2q \quad (3.18)$$

and

$$(a^2 + b^2)\lambda = 2pa^2 + 2q(a^2 + b^2)^2. \quad (3.19)$$

From (3.18) and (3.19), we have

$$(p + a^2q)(a^2 + b^2) = pa^2 + q(a^2 + b^2)^2 \quad (3.20)$$

and so

$$pb^2 + (a^2 + b^2)a^2q = q(a^2 + b^2)^2. \quad (3.21)$$

From (3.21), we have

$$(p - (a^2 + b^2)q)b^2 = 0. \quad (3.22)$$

Now, if $b \neq 0$, then from (3.22), we have

$$p = (a^2 + b^2)q. \quad (3.23)$$

Again, suppose that $a \neq 0$ (as $b \neq 0$), then, from (3.11), we have

$$\lambda = 4p. \quad (3.24)$$

So, from (3.18) and (3.24), we get

$$a^2q = p, \quad (3.25)$$

and hence, with (3.23), we get

$$b^2 = 0, \quad (3.26)$$

which is a contradiction. So, it must follow that $a = 0$.

Therefore, from (3.3), we have

$$\bar{g}(Y,Z') = 0, \quad (3.27)$$

and

$$\bar{g}(Y',Z') = g'(Y',Z') + (b^2 - 1)\eta'(Y')\eta'(Z'), \quad (3.28)$$

for $Y, Z \in M$, $Y', Z' \in M'$.

Again, from (3.21), we get

$$p = b^2q. \quad (3.29)$$

From (3.18), we get

$$\lambda = 2p. \quad (3.30)$$

Therefore, we have the following:

Theorem 2: Einstein constant of any almost hyperbolic Hermitian manifold on product of two (2p+1)-dimensional Sasakian manifold is equal to 2p.

From (3.13) and (3.14), taking account of (3.29) and (3.30), we have

$$\rho(Y,Z) = 2pg(Y,Z), \quad (3.31)$$

$$\rho'(Y',Z') = 2(p + b^2 - 1)g'(Y',Z') - 2(b^2 - 1)$$

$(q+1)\eta'(Y')\eta'(Z')$, (3.32)

for $Y, Z \in M$, $Y', Z' \in M'$.

Again, from (3.9), we get

$$\bar{\rho}(Y, Z') = 0, \quad (3.33)$$

for $Y \in M$, $Z' \in M'$.

Therefore, summing up the arguments, we have the following.

Theorem 3: An almost hyperbolic Hermitian manifold equipped with the hyperbolic Hermitian structure $(\bar{g}, F)$ taken on product of two sasakian manifolds (M, ϕ, ξ, η, g) and $(M', \phi', \xi', \eta', g')$ as $\bar{M} = M \times M'$ is Einstein if and only if $a = 0$ where M is an Einstein Sasakian manifold and M' is an η-Einstein Sasakian manifold with the Ricci tensor[10-12].

Corollary 1: From theorem 3 η-Einstein Sasakian manifold M' is Einstein if p = q. hence M is the Riemannian product of the same dimensional Einstein Sasakian manifolds M and M'.

5. Ricci $*$-tensor of $\bar{M} = (\bar{M}, \bar{g}, F)$:

By (2.14), (3.4) and (3.8), yields

$\bar{\rho}^*(X,Y) = (1-2aq)(g(X,Y) \text{ “ } \eta(X)\eta(Y))$,

$\bar{\rho}^*(X, Y') = 0$,

$\bar{\rho}^*(X', Y') = (1-2ap-(2q+1)(a^2+b^2-1))$

$(g'(X',Y') - \eta'(X')\eta'(Y'))$ (3.34)

For $X, Y \in M$ and $X', Y' \in M'$.

So, from (3.3) and (3.34), we get

Theorem 4: $\bar{M} = (\bar{M}, \bar{g}, F)$ is never weakly $*$-Einstein.

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