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Some generating functions of biorthogonal polynomials suggested by the Laguerre polynomials

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Abstract

In this note, we have obtained some novel generating functions(both bilateral and mixed trilateral) involving Konhauser biorthogonal polynomials $Y_n^\alpha(x;k)$ by group theoretic method. As special cases, we obtain the corresponding results on generalized Laguerre polynomials.

Key words: Laguerre polynomials, biorthogonal polynomials, generating functions.

AMS-2000 Subject Classification Code: 33C 47.

1. Introduction

In[1,2], J.D.E. Konhauser discussed the biorthogonality and some other properties of $Y_n^\alpha(x;k)$ and $Z_n^\alpha(x;k)$ for any positive integer k, where $Y_n^\alpha(x;k)$ is a polynomials in x and $Z_n^\alpha(x;k)$ is a polynomials in $x^k, \alpha > -1, k$ is a positive integer. For k=1, these polynomials reduce to the generalized Laguerre polynomials $L_n^\alpha(x)$[3]. In the present paper we are interested only on $Y_n^\alpha(x;k)$. An explicit representation for the polynomials $Y_n^\alpha(x;k)$ was given by Carlitz[4] in the following form:

$$Y_n^\alpha(x;k) = \frac{1}{n!}\sum_{i=0}^{n}\frac{x^i}{i!}\sum_{j=0}^{i}(-1)^j\binom{i}{j}\left(\frac{j+\alpha+1}{k}\right)_n,$$

where $(a)_n$ is the pochhammer symbol[5].

It may be noted that a good number of generating functions involving $Y_n^\alpha(x;k)$ have been found derived in[6-10] by different methods. The aim at presenting this paper is to obtain some novel bilateral and mixed trilateral generating functions for the polynomial $Y_n^{\alpha+n}(x;k)$, a modification of Konhauser

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biorthogonal polynomials, $Y_n^{\alpha}(x;k)$ by group-theoretic method. Several special cases of interestare also discussed in this paper. The main results of our investigation are stated in the form of the following theorems:

Theorem 1 : If there exists a unilateral generating relation of the form

$$G(x,w) = \sum_{n=0}^{\infty} a_n\, Y_n^{\alpha+n}(\mathrm{x};\mathrm{k})\, w^n \tag{1.1}$$

then $(1-kw)^{\frac{-(1+\alpha)}{k}} \exp\left(x - \frac{x}{(1-kw)^{\frac{1}{k}}}\right)$

$$G\left(\frac{x}{(1-kw)^{\frac{1}{k}}},\ \frac{wv}{(1-kw)^{1+\frac{1}{k}}}\right)$$

$$= \sum_{n=0}^{\infty} w^n\, \sigma_n(x,v), \tag{1.2}$$

where

$$\sigma_n(x,v) = \sum_{m=0}^{n} a_m\, k^{n-m} \binom{n}{m} Y_n^{\alpha+m}(\mathrm{x};\mathrm{k})\, v^m.$$

Theorem 2 : If there exists a generating relation of the form

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n Y_n^{\alpha+n}(\mathrm{x};\mathrm{k})\, g_n(u)\, w^n, \tag{1.3}$$

where $g_n(u)$ is an arbitrary polynomial of degree n, then

$$(1-kw)^{\frac{-(1+\alpha)}{k}} \exp\left(x - \frac{x}{(1-kw)^{\frac{1}{k}}}\right)$$

$$G\left(\frac{x}{(1-kw)^{\frac{1}{k}}},\ u,\ \frac{wv}{(1-kw)^{1+\frac{1}{k}}}\right)$$

$$= \sum_{n=0}^{\infty} w^n\, \sigma_n(x,u,v), \tag{1.4}$$

where

$$\sigma_n(x,u,v) = \sum_{m=0}^{n} a_m\, k^{n-m} \binom{n}{m} Y_n^{\alpha+m}(\mathrm{x};\mathrm{k})\, g_m(u)\, v^m.$$

The importance of the above theorems lies in the fact that whenever one knows a generating relation of the form (1.1, 1.3) then the corresponding bilateral and mixed trilateral generating function can at once be written down from (1.2, 1.4). So one can get a large number of bilateral and mixed trilateralgenerating functions by attributing different suitable values to a_n in (1.1, 1.3).

2. *Operator and extended form of the group:*

At first, we seek a linear partial differential operator R of the form:

$$R = A_1(x,y,z)\frac{\partial}{\partial x} + A_2(x,y,z)\frac{\partial}{\partial y} + A_3(x,y,z)\frac{\partial}{\partial z} + A_0(x,y,z)$$

where each $A_i(i=0,1,2,3)$ is a function of x,y, and z which is independent n, α, of such that

$$R[Y_n^{\alpha+n}(x;k)y^{\alpha}z^n] = c(n,\alpha)Y_{n+1}^{\alpha+n}(x;k)y^{\alpha-1}z^{n+1}, \tag{2.1}$$

where (n,α) is a function of n, α and is independent of x,y and z.

Using (2.1) and with the help of the differential recurrence relation:

$$x\frac{d}{dx}[Y_n^{\alpha+n}(x;k)] = k(n+1)Y_{n+1}^{\alpha+n}(x;k) - (kn+n+\alpha-x+1)Y_n^{\alpha+n}(x;k) \tag{2.2}$$

we easily obtain the following linear partial differential operator

$$R = x\,y^{-1}z\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + (k+1)z^2y^{-1}\frac{\partial}{\partial z} + (1-x)y^{-1}z$$

such that

$$R\left(Y_n^{(\alpha+n)}(\mathrm{x};\mathrm{k})y^\alpha z^n\right) = k(n+1)Y_{n+1}^{(\alpha+n)}(\mathrm{x};\mathrm{k})y^{\alpha-1}z^{n+1}. \tag{2.3}$$

The extended form of the group generated by R is given by

$$e^{wR}f(x,y,z) = (1-kwy^{-1}z)^{-\frac{1}{k}}\exp\left(x - \frac{x}{(1-kwy^{-1}z)^{\frac{1}{k}}}\right)$$

$$\times f\left(\frac{x}{(1-kwy^{-1}z)^{\frac{1}{k}}}, \frac{y}{(1-kwy^{-1}z)^{\frac{1}{k}}}, \frac{z}{(1-kwy^{-1}z)^{1+\frac{1}{k}}}\right), \tag{2.4}$$

where $f(x,y,z)$ is an arbitrary function and w is an arbitrary constant.

3. Derivationof generating function :

Now writing $f(x,y,z) = Y_n^{\alpha+n}(x;k)y^\alpha z^n$ in (2.4), we get

$$e^{wR}(Y_n^{\alpha+n}(x;k)y^\alpha z^n) = (1-kwy^{-1}z)^{-\frac{(1+\alpha+n+nk)}{k}}y^\alpha z^n$$

$$\times \exp\left(x - \frac{x}{(1-kwy^{-1}z)^{\frac{1}{k}}}\right)Y_n^\alpha\left(\frac{x}{(1-kwy^{-1}z)^{\frac{1}{k}}};k\right) \tag{3.1}$$

Again, on the other hand, with the help of (2.3) we have

$$e^{wR}(Y_n^{\alpha+n}(x;k)y^\alpha z^n) = \sum_{m=0}^{\infty}\frac{w^m}{m!}k^m(n+1)_m Y_{n+m}^{\alpha+n}(x;k)y^{\alpha-m}z^{n+m} \tag{3.2}$$

Equating (3.1) and (3.2) and then substituting $wy^{-1}z = t$, we get

$$(1-kt)^{-\frac{(1+\alpha+n+nk)}{k}}\exp\left(x - \frac{x}{(1-kt)^{\frac{1}{k}}}\right)Y_n^{\alpha+n}\left(\frac{x}{(1-kt)^{\frac{1}{k}}};k\right)$$

$$= \sum_{m=0}^{\infty}k^m\binom{n+m}{m}Y_{n+m}^{\alpha+n}(x;k)t^m, \tag{3.3}$$

whichdoes not seem to have appeared in the earlier works.

Corollary 1: Replacing α by α – *n* in both sides of (3.3), we get the following generating relation:

$$(1-kt)^{-\frac{(1+\alpha+nk)}{k}}\exp\left(x-\frac{x}{(1-kt)^{\frac{1}{k}}}\right)Y_n^{\alpha}\left(\frac{x}{(1-kt)^{\frac{1}{k}}};k\right)$$

$$=\sum_{m=0}^{\infty}k^m\binom{n+m}{m}Y_{n+m}^{\alpha}(x;k)t^m, \quad (3.4)$$

which is also found derived in[7] by the classical[8] method.

Corollary 2 : Putting $n = 0$ in (3.3), we get the following generating relation:

$$(1-kt)^{-\frac{(1+\alpha)}{k}}\exp\left(x-\frac{x}{(1-kt)^{\frac{1}{k}}}\right)=\sum_{m=0}^{\infty}Y_m^{\alpha}(x;k)(kt)^m, (3.5)$$

which is found derived in [7,8].

Special case 1: If we put $k = 1$, then $Y_n^{\alpha}(x;k)$ reduces to the generalized Laguerre polynomials, $L_n^{\alpha}(x)$. Thus Putting k= 1 in (3.3) we get the following generating relation on Laguerre polynomials:

$$(1-t)^{-(1+\alpha+2n)}\exp\left(\frac{-xt}{1-t}\right)L_n^{\alpha+n}\left(\frac{x}{1-t}\right)$$

$$=\sum_{m=0}^{\infty}\binom{n+m}{m}L_{n+m}^{\alpha+n}(x), \quad (3.6)$$

whichdoes not seem to have appeared in the earlier works.

Sub Case 1: Replacing α by α – *n* in both sides of (3.6), we get the following generating relation:

$$(1-t)^{-(1+\alpha+n)}\exp\left(\frac{-xt}{1-t}\right)L_n^{(\alpha)}\left(\frac{x}{1-t}\right)$$

$$=\sum_{m=0}^{\infty}\binom{n+m}{m}L_{n+m}^{(\alpha)}(x)t^m, \quad (3.7)$$

whichis found derived in [3,11,12].

Sub Case 2 : Putting n =0 in the above relation, we get the following generating relation:

$$(1-t)^{-(1+\alpha)}\exp\left(\frac{-xt}{1-t}\right)=\sum_{m=0}^{\infty}L_m^{\alpha}(x)\,t^m, (3.8)$$

whichare found derived in [3,12].

Now we proceed to prove the theorems (Theorem 1 & 2)

4. Proof of theorem 1 :

Let us consider the generating relation of the form:

$$G(x,w)=\sum_{n=0}^{\infty}a_nY_n^{\alpha+n}(x;k)\,w^n \quad (4.1)$$

Replacing w by wvz and then multiplying both sides of (4.1) by y^{α}, we get

$$y^{\alpha}\,G(x,\,wvz)=\sum_{n=0}^{\infty}a_n(Y_n^{\alpha}(x;k)y^{\alpha}z^n)\,(wv)^n (4.2)$$

Operating both sides of (4.2) by e^{wR}, we get

$$e^{wR}\big(y^{\alpha}\,G(x,wvz)\big)=e^{wR}\left(\sum_{n=0}^{\infty}a_n(Y_n^{\alpha}(x;k)y^{\alpha}z^n)(wv)^n\right) \quad (4.3)$$

Now the left member of (4.3), with the help of (2.4), reduces to

$$(1-kwy^{-1}z)^{-\frac{(1+\alpha)}{k}}\exp\left(x-\frac{x}{(1-kwy^{-1}z)^{\frac{1}{k}}}\right)y^{\alpha}$$

$$\times G\left(\frac{x}{(1-kwy^{-1}z)^{\frac{1}{k}}},\quad \frac{wvz}{(1-kwy^{-1}z)^{1+\frac{1}{k}}}\right). \quad (4.4)$$

The right member of (4.3), with the help of (2.3), becomes

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n} a_{n-m}\, w^n k^m \binom{n}{m} Y_n^{\alpha+n-m}(x;k) y^{\alpha-m} z^n v^{n-m} \quad (4.5)$$

Now equating (4.4) and (4.5) and then substituting $y = z = 1$, we get

$$(1-kw)^{-\frac{(1+\alpha)}{k}}\exp\left(x-\frac{x}{(1-kw)^{\frac{1}{k}}}\right)G\left(\frac{x}{(1-kw)^{\frac{1}{k}}},\right.$$

$$\left.\frac{wv}{(1-kw)^{1+\frac{1}{k}}}\right)=\sum_{n=0}^{\infty} w^n\, \sigma_n(x,v),$$

where

$$\sigma_n(x,v)=\sum_{m=0}^{n} a_m k^{n-m}\binom{n}{m} Y_n^{\alpha+m}(x;k) v^m.$$

This completes the proof of the theorem and does not seem to have appeared in the earlier works.

Special case 2: Now putting $k = 1$ in our theorem 1 we get the following result on generalised Laguerre polynomials:

Theorem 3: If there exists a generating relation of the form

$$G(x,u,w)=\sum_{n=0}^{\infty} a_n L_n^{(\alpha+n)}(\mathrm{x})\, w^n$$

then

$$(1-w)^{-(1+\alpha)}\exp\left(\frac{-wx}{1-w}\right)G\left(\frac{x}{1-w},\ \frac{wv}{(1-w)^2}\right)=\sum_{n=0}^{\infty} w^n\,\sigma_n(x,v),$$

where

$$\sigma_n(x,v)=\sum_{m=0}^{n} a_m\binom{n}{m} L_n^{(\alpha+m)}(\mathrm{x})\, v^m,$$

whichdoes not seem to have appeared in the earlier works.

5. Proof of theorem 2 :

Again Let us assume another generating relation of the form:

$$G(x,u,w)=\sum_{n=0}^{\infty} a_n Y_n^{\alpha+n}(x;k)\, g_n(u)\, w^n \quad (5.1)$$

Replacing *w* by *wvz* and multiplying both sides of (5.1) by y^{α} and finally applying the operator e^{wR} on both sides, we easily obtain, as in section 4, the following generating relation:

$$(1-kwy^{-1}z)^{-\frac{(1+\alpha)}{k}}\exp\left(x-\frac{x}{(1-kwy^{-1}z)^{\frac{1}{k}}}\right)$$

$$y^{\alpha}G\left(\frac{x}{(1-kwy^{-1}z)^{\frac{1}{k}}},u,\frac{wvz}{(1-kwy^{-1}z)^{1+\frac{1}{k}}}\right)$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n} a_{n-m}\, w^n k^m\binom{n}{m} Y_n^{\alpha+n-m}(x;k) y^{\alpha-m} z^n\, g_{n-m}(u) v^{n-m}.$$

Now substituting $y = z = 1$, we get

$$(1-kw)^{-\frac{(1+\alpha)}{k}}\exp\left(x-\frac{x}{(1-kw)^{\frac{1}{k}}}\right)$$

$$G\left(\frac{x}{(1-kw)^{\frac{1}{k}}},u,\frac{wv}{(1-kw)^{1+\frac{1}{k}}}\right)$$

$$=\sum_{n=0}^{\infty} w^n\,\sigma_n(x,u,v),$$

where

$$\sigma_n(x,u,v) = \sum_{m=0}^{n} a_m k^{n-m} \binom{n}{m} Y_n^{\alpha+m}(x;k)\, g_m(u) v^m.$$

This completes the proof of the theorem. This result does not seem to have appeared in the earlier works.

Special case 3: Now putting $k = 1$ in the above theorem, we get the following theorem on generalised Laguerre polynomials.

Theorem 4: If there exists a generating relation of the form

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n L_n^{(\alpha+n)}(\mathrm{x})\, g_n(u)\, w^n$$

then

$$(1-w)^{-(1+\alpha)} \exp\left(\frac{-wx}{1-w}\right) G\left(\frac{x}{1-w},\right.$$

$$\left. u, \frac{wv}{(1-w)^2}\right) = \sum_{n=0}^{\infty} w^n\, \sigma_n(x,u,v),$$

where

$$\sigma_n(x,u,v) = \sum_{m=0}^{n} a_m \binom{n}{m} L_n^{(\alpha+m)}(\mathrm{x})\, g_m(u)\, v^m,$$

whichdoes not seem to have appeared in the earlier works[11-12].

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