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The Category L-FCyc

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Abstract

In this paper we introduce L-fuzzy homomorphism of L-fuzzy subgroups of finite cyclic groups through an embedding of lattices. We then form a category whose objects are L-fuzzy subgroups of finite cyclic groups and morphisms are L-fuzzy homomorphisms. We name this category **L-FCyc** and discuss some of its properties.

Key words : Category, L-fuzzy subgroups, L-fuzzy homomorphism.

1. Introduction

Category Theory took shape in 1945, when MacLane and and Eilenberg published the paper 'General Theory of Natural Equivalences'[5]. *Lattice Theory* has its origin in the works of Richard Dedekind in the early 1890's. Garrett Birkhoff [2] developed it into a subject of its own right. *Fuzzy Set Theory* was introduced by L.A. Zadeh in 1965 [15]. J.A. Gougen[7] considered a complete and distributive lattice L as the membership set ,instead of the interval [0,1] used by Zadeh. In 1971, A.Rosenfeld[10] introduced fuzzy groups and later on it was generalized into L-fuzzy groups.

We have presented results obtained in our studies on categories of L-fuzzy groups in[11]. In that paper, we have formed four categories of L-fuzzy groups and discussed some relations between them. In[12], we have discussed maximal lattices of cyclic groups and developed a method to construct it for finite cyclic groups. The method was further extended to the case of infinite cyclic groups in [13]. As a continuation of these works, in this paper, we introduce L-fuzzy homomorphism of L-fuzzy subgroups of finite cyclic groups through an embedding of lattices. We then form a category whose objects are L-fuzzy subgroups of finite cyclic groups and morphisms are L-fuzzy homomorphisms. We name this category as **L-FCyc** and discuss some categorical properties enjoyed by it.

Throughout this paper L denotes a

complete and distributive lattice and L_i denotes a sublattice of L. We represent the greatest element of L_i by I_i and the least element by O_i. Terms and notations in Lattice Theory used in this paper are as found in Bernard Kolman[1], Davey B.A.[4] and Vijay K.Khanna[14].

2. Basic concepts :

A relation $\leq$ on a set A is called a *partial order* if $\leq$ is reflexive, antisymmetric and transitive. The set A together with the partial order $\leq$ is called a *partially ordered set* (or a *poset*) and is denoted as $(A,\leq)$ or simply A. The elements a and b of a poset A are said to be *comparable* if if $a\leq b$ or $b\leq a$. If every pair of elements in a poset A is comparable, we say that A is a *linearly (or totally) ordered set* and the partial order in this case is called a *linear (total) order*. We also say that such an A is a *chain*. For any set S, its power set $\mathcal{P}$ (S) together with set inclusion $\subseteq$ is a poset. If $|S|\geq 2$, it is not totally ordered.The set Z^+ of positive integers together with the usual order $\leq$ is a totally ordered set(toset). An element $a\in A$ is called a *maximal element* of A if there is no element c in A such that $a<c$ and an element element $b \in$ A is called a *minimal element* of A if there is no element c in A such that $c<b$. An element $a\in$ A is called a *greatest element* of A if $x \leq a$ for all $a \in$A. An element $a \in$A is called a *least element* of A if $a \leq x$ for all $a \in$ A. The greatest element of a poset, if it exists, is denoted by I and is called the *unit element*. The least element of a poset, if it exists, is denoted by O and is called the *zero element*.

Consider a poset A and a subset B of A. An element $a \in$A is called an *upper bound* of B if $b \leq a$ for all $b \in$ B. An element $a \in$ A is called a *lower bound* of B if $a \leq b$ for all $b \in$ B. An element $a \in A$ is called a *least upper bound* of B or *supremum of B* denoted as lub(B) or sup B or $\vee$ B if a is an upper bound of B and $a \leq a'$, whenever a' is an upper bound of B. An element $a \in$ A is called a *greatest lower bound* of B or *infimum* of B denoted as glb(B) (or inf B or $\wedge$ B), if a is a lower bound of B and $a' \leq a$, whenever a' is a lower bound of B. A *lattice* is a poset $(L, \leq)$ in which every subset {a, b} consisting of two elements has a least upper bound and a greatest lower bound. We denote lub $(\{a,b,\})$ and glb $(\{a,b\})$ by $a\vee b$ and $a\wedge b$ respectively and call them the *join* and *meet* respectively of a and b.

For any non-empty set S,$(\mathcal{P}(S),\subseteq)$ is a poset. For any A, $B\in\mathcal{P}$ (S), $A\vee B = A \cup B$ and $A \wedge B = A \cap B$ exist and so it is a lattice. Another example of a lattice is $(Z^+,\leq)$ where $\leq$ is defined by $a \leq b$ iff a divides b. Here $a\vee b$ =lcm(a,b) and $a\wedge b$ =gcd(a,b). Now for any positive integer n, let D_n denote the set of all positive divisors of n. Then D_n together with the relation 'divisibility' is a lattice.

A non-empty subset S of a lattice L is called a *sublattice* of L if $a, b \in S \Rightarrow a \wedge b, a \vee b \in S$. If $(A,\leq)$ and $(B,\leq)$ are posets, then $(A X B,\leq)$is a poset, with partial order $\leq$ defined by $(a,b) \leq (a',b')$ if $a\leq a'$ in A and $b\leq b'$ in B. The partial order $\leq$ defined on the Cartesian product A X B is called the *product partial order.* If $(L_1, \leq)$ and $(L_2,\leq)$ are lattices, then $(L_1 X L_2,\leq)$ is a lattice called *the product lattice of* L_1 and L_2 where the partial order $\leq$ of L is the product partial order. Here $(a_1,b_1) \vee(a_2,b_2)=(a_1\vee a_2,b_1\vee b_2)$ and $(a_1,b_1)\wedge(a_2,b_2)$

$=(a_1 \wedge a_2, b_1 \wedge b_2)$.

A lattice L is called *distributive* if for any elements *a,b* and *c* in L, we have the following distributive properties:

1. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
2. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

A lattice L is said to be *complete* if every non-empty subset of it has a glb and lub

If L_1 and L_2 are complete lattices, then $L_1 X L_2$ is also a complete lattice with joins and meets being formed co-ordinate wise.

2.1. *Definition*[6]. Given a universal set X, a *Fuzzy Set* on X (or *a fuzzy subset of X*) is defined as a function $A:X \to [0,1]$. Its range is denoted as *Im(A)*.

2.2. *Example*. Let X={a,b,c,d,e}. Define $A: \to [0,1]$ by A(a)=0.1; A(b)=0.2; A(c)=0.3; A(d)=0.4; A(e)=0.5. Then A is a fuzzy subset of X and Im(A)={0.1,0.2, 0.3,0.4,0.5}.

2.3. *Definition*[7]. If L is a lattice and X is a universal set, then an *L-fuzzy set A on X* (or*an L-fuzzy subset A* of X) is a function $A : X \to L$. We shall write $A \in L^X$ for *A is an L-fuzzy set on X*.

2.4. *Example*. Let X={a,b,c,d} and $L= D_6=\{1,2,3,6\}$ under the relation divisibility. Define $A:X \to L$ by A(a)=1, A(b)=2, A(c)=3 and A(d)=6. Then A is an L-fuzzy set on X.

2.5. *Definition*[10]. A fuzzy subset A of a multiplicative group G is said to be a *fuzzy subgroup* of G or a *fuzzy group on G* if for every x, $y \in G$
(1) $A(xy) \geq \min\{A(x), A(y)\}$ and (ii) $A(x^{-1}) = A(x)$.

2.6. *Example*. Let G = {1,-1,i,-i} under multiplication of complex numbers. Define $A: G \to [0,1]$ by A(1)=1, A(-1) = 0.5, A(i) = A(-i) =0.25. Then A is a fuzzy subgroup of G.

2.7. *Definition*[9]. An L-fuzzy subset A of G is called an *L-fuzzy subgroup* of G (or *an L-fuzzy group on* G) if

(i) $A(xy) \geq A(x) \wedge A(y)$, $\forall$ x, y $\in$ G, and
(ii) $A(x^{-1}) \geq A(x)$, $\forall x \in G$.

It may be recalled that for any positive integer n, $Z_n=\{0,1,2,\ldots,n-1\}$ is a group with respect to addition modulo n.

2.8. *Example*. Let G be the group $Z_6=\{0,1,2,3,4,5\}$ under addition modulo 6.Take $L=D_6=\{1,2,3,6\}$. Define $A: Z_6 \to D_6$ by A(0)=6, A(2)=A(4)=2, A(3)=3 and A(1)=A(5)=1. Then A is an L-fuzzy group on Z_6.

2.9. *Example*. Let V={e,a,b,c} be the *Klein-4 group*, whose composition table is as follows:

Table (i): Definition of binary operation * on V.

*	e	a	b	c
e	a	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Let $L=D_6=\{1,2,3,6\}$. Define $A:V\to D_6$ by $A(e)=6, A(a)=2, A(b)=3$ and $A(c)=1$. Then A is an L-fuzzy group on V.

2.10. *Lemma*[9]. Let G be a group and $A\in L^G$. Then A is an L-fuzzy group on G iff $A_a=\{x/x\in G,\ A(x)\geq a\}$ is a subgroup of G, $\forall a A(G)\cup\{b\in L/b\leq A(e)\}$ ■

2.11. *Notation*[9]. If G is a group and L is a lattice, then (G,L) shall denote the collection of all L-fuzzy groups on G. Hence *A∈(G,L)* means that *A is an L-fuzzy group on G*.

2.12. *Example*. Consider the group $<Z_{18},+_{18}>$ and the lattice $D_{12}=\{1,2,3,4,6,12\}$. Define $A:Z_{18}\to D_{12}$ by $A:0\mapsto 12$, $\{3,15\}\mapsto 2$, $\{6,12\}\mapsto 6$, $9\mapsto 4$, $\{2,4,8,10,14,16\}\mapsto 3$, $\{1,5,7,11,13,17\}\mapsto 1$. Then A is an L-Fuzzy subgroup of Z_{18} as can be verified using the lemma 2.10.

3. L-fuzzy subgroups of finite cyclic groups:

For the sake of completeness, we shall give a summary of the relevant terms and results presented in[12], as they form the basis of the ideas developed in the present paper.

3.1. *Definition*[12]. Let $L=(\{a_1,a_2,\ldots\ldots.a_n\},\leq)$ be a lattice. We say that L is a *finite lattice* containing n points and write $|L|=n$.

3.2. *Example*[12]. $D_6=\{1,2,3,6\}$ is a lattice under divisibility. It is a finite lattice containing four points and so $|D_6|=4$.

3.3. *Definition*[12]. Let G be a group, L be a finite lattice and $A: G\mapsto L$ be an L-fuzzy group. A is said to *saturate* L if $Im(A)=L$. If there is an L-fuzzy group A on G which saturates L, then we say that *G saturates L*.

3.4. *Example*[12]. Consider $G=<Z,+>$ and $L=(\{0,1/3,1/2,1\},\leq)$. Define $A: G\mapsto L$ by $A(0)=1$, $A(x)=1/2$ if $x\in 4Z-\{0\}$, $A(x)=1/3$ if $x\in 2Z-4Z$ and $A(x)=0$ if $x\in Z-2Z$. Then A is an L-fuzzy group on G with $Im(A)=L$. Hence, A as well as G saturates L.

3.5. *Example*[12]. Let $G=<Z_4,+_4>$ and $L=(\{0,1\},\leq)$. Define $A: G\to L$ by $A(0)=1$; $A(x)=0$, if $x\neq 0$. Then A is an L-fuzzy group on G which saturates L and so G also saturates L.

3.6. *Example*[12]. Let $G=<Z_4,+_4>$ and $L=(\{0,1/2,1\},\leq)$. Define $A: G\to L$ by $A(0)=1$; $A(x)=0$, if $x\neq 0$. Here A is an L-fuzzy group on G with $Im(A)\neq L$. Hence A does not saturate L. But if we define $B: G\to L$ by $B(0)=1$; $B(2)=1/2$ and $B(1)=B(3)=0$, then B is an L-fuzzy group on G which saturates L and hence G also saturates L.

3.7. *Definition*[12]. Let G be a group and L be a finite lattice. A sublattice L_1 of L is said to be a *maximal lattice* saturated by G if there is an $A\in(G,L)$ which saturates L_1 and there is no $B\in(G,L)$ which saturates a sublattice L_2 of L with $|L_2|>|L_1|$.

3.8. *Example*[12]. Consider the sublattices $L_1=\{0,1\}$ and $L_2=\{0,1/2,1\}$ of $L=\{0,1/3,1/2,1\}$ and let $G=<Z_4,+_4>$. Define $A: G\to L$ by $A(0)=1$; $A(x)=0$, if $x\neq 0$. Also define $B: G\to L$ by

B(0)=1; B(2)=1/2 and B(1)=B(3)=0. Then L_1 is not a maximal lattice of G, because there is L_2 with $|L_2|>|L_1|$ and B: G→L which saturates L_2. It can be shown that L_2 is a maximal lattice for G.

We may recall that a group G is said to be of prime power order if $|G|= p^n$, for some prime number p and positive integer n.

3.9. *Theorem*[14]. Let G be a cyclic group of prime power order. Then the lattice of all subgroups of G is a chain■

3.10. *Theorem*[12]. Let G be a cyclic group of prime power order. Then a maximal lattice L_G for G is a chain isomorphic to the chain of all subgroups of G■

It is well-known that every finite cyclic group of order n is isomorphic to Z_n. So, henceforth we represent cyclic groups of order n by Z_n.

3.11. *Proposition*[8]. The group Z_m X Z_n is isomorphic to Z_{mn} if and only if m and n are relatively prime■

3.12. *Proposition*[8]. The group $\prod_{i=1}^{n} Z_{m_i}$ is cyclic and isomorphic to $Z_{m_1 m_2 m_3 \ldots\ldots m_n}$ if and only if the number m_i, for i=1,2…n are pairwise relatively prime■

3.13. *Theorem*[12]. Suppose n = $p_1^{n_1} p_2^{n_2} \ldots \ldots . . p_r^{n_r}$, where p_i's are distinct primes. Then the maximum lattice for $Z_n = Z_{p_1 n_1}$ *X* $Z_{p_2^{n_2}} \ldots \ldots X\ Z_{p_r^{n_r}}$ is the product lattice of the maximum chains for the factors $Z_{p_i^{n_i}}$.■

3.14. *Example*. $Z_6 \cong Z_2$ X Z_3. The maximum chain for both Z_2 and Z_3 is D_2 ={0,1}. Take Z_6={0,1} X {0,1,2}={00,01,02, 10,11,12}. Its composition table is given below in table (ii): Addition on the first digit is modulo 2 and that on second is modulo 3.

Table (ii). Composition table for Z_2XZ_3.

+	00	01	02	10	11	12
00	00	01	02	10	11	12
01	01	02	00	11	12	10
02	02	00	01	12	10	11
10	10	11	12	00	01	02
11	11	12	10	01	02	00
12	12	10	11	02	00	01

Its proper subgroups are Z_2={00,10} and Z_3={00,01,02}. Its subgroup lattice is given in figure(i):

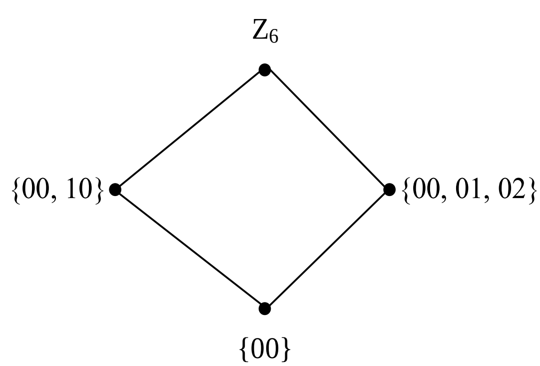

Figure (i): Subgroup lattice of Z_6

Take L=D_2XD_2

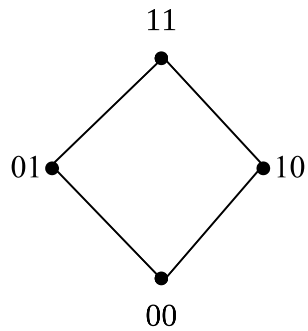

Figure (ii): The product lattice $L=D_2XD_2$

Define $A:Z_6 \to L$ by $A(00) = 11$, $A(10) = 01$, $A(01) = A(02) = 10$, $A(11) = A(12) = 00$. This is an L-Fuzzy group and $D_2 \times D_2$ is the maximum lattice for $Z_2 \times Z_3 = Z_6$.

4. L-fuzzy homomorphisms :

4.1. *Definition*. Let $A\in(G,L)$. Then for $a\in L$, the set $A^{-1}(a) = \{x\in G/A(x) = a\}$ is called the *A-pre image* of a.

4.2. *Definition*. Let $A\in(G, L)$. Then the collection of all A-pre images

$$Pr(L) = \{A^{-1}(a)/a\in L\}$$

is called the *A-pre image set* of L

4.3. *Definition*[3]. Let L and M be lattices. A mapping $f: L\to M$ is called a *lattice homomorphism* if for all $a, b \in L$

$$f(a \wedge b) = f(a) \wedge f(b) \text{ and}$$
$$f(a \vee b) = f(a) \vee f(b).$$

If, in addition, the mapping f is one-to-one and onto, we call f a *lattice isomorphism*. If $f: L\to M$ is an isomorphism, we say that L is *isomorphic* to M.

4.4. *Definition*[3]. Let L and M be lattices. If $f: L\to M$ is a one-to-one homomorphism, then f is said to be an *embedding* of L into M. In this case, L is isomorphic to the sublattice f (L) of M and we say that L is *embeddable* on M.

4.5. *Example*. Consider the lattices D_2 and D_6.

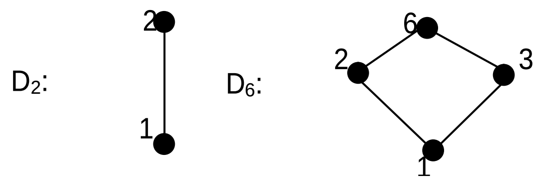

Fig.(iii): Lattice diagrams of D_2 and D_6

Define $f:D_2\to D_6$ by $f(1)=2$ and $f(2)=6$. Then f is one-to-one. Moreover

$f(1\wedge 2)=f(1)=2=\wedge 6=f(1) \wedge f(2)$ and

$f(1 \vee 2) = f(2) = 6 = 2 \vee 6 = f(1) \vee f(2)$.

This shows that D_2 is embeddable on D_6 .

4.6. *Definition*. Let G_1 and G_2 be two finite cyclic groups with maximum lattices L_1 and L_2 respectively, where L_1 is embeddable on L_2. Let $A\in (G_1, L_1)$ and $B\in(G_2, L_2)$, each of them saturating its respective lattice. Let $f: L_1\to L_2$ be a map such that

(1) $f(I_1)=I_2$,where I_1 and I_2 are maximum elements of L_1 and L_2 respectively, and
(2) f defines an embedding of L_1 on L_2.

Then a map $F: Pr(L_1)\to(L_2)$ is said to be an *L-fuzzy homomorphism* from A to B through f if

$F(A^{-1}(a)) = B^{-1}(f(a))$ for every a in L_1.

We shall denote this by writing *F_f: $A \to B$ is a homomorphism*.

4.7. *Example*. Let $G = \langle Z_2, +_2 \rangle$ whose maximum lattice is $L_1 = D_2 = \{1, 2\}$. If $A: G \to L_1$ is defined by $A: 0 \mapsto 2,\ 1 \mapsto 1$, then A is a saturating L_1-fuzzy subgroup of G. Let $H = \langle Z_6, +_6 \rangle$. Its maximum lattice is $L_2 = D_6 = \{1, 2, 3, 6\}$. Define $B: H \mapsto L_2$ by $B: 0 \mapsto 6$, $\{2,4\} \mapsto 2$, $3 \mapsto 3$, $\{1,5\} \mapsto 1$. Then B is a saturating L_2-fuzzy subgroup of H. Define $f: L_1 \to L_2$ by $f(2) = 6$, $f(1) = 2$. Then f maps the maximum element of L_1 to that of L_2 and also is an embedding of L_1 on L_2 as showed in example 4.5. We have, $A^{-1}(1) = \{1\}$ and $A^{-1}(2) = \{0\}$. Also $B^{-1}(6) = \{0\}$, $B^{-1}(2) = \{2,4\}$, $B^{-1}(3) = \{3\}$ and $B^{-1}(1) = \{1, 5\}$. Hence, $(L_1) = \{\{0\}, \{1\}\}$ and $(L_2) = \{\{0\}, \{2, 4\}, \{3\}, \{1, 5\}\}$.

Define $F: Pr(L_1) \to Pr(L_2)$ by $F(\{0\}) = \{0\}$ and $F(\{1\}) = \{2,4\}$. Now
$F(A^{-1}(1)) = F(\{1\}) = \{2,4\} = B^{-1}(2) = B^{-1}(f(1))$ and
$F(A^{-1}(2)) = F(\{0\}) = \{0\} = B^{-1}(6) = B^{-1}(f(2))$.

Hence $F(A^{-1}(a)) = B^{-1}(f(a))$ for every a in L_1 and so F defines an L-fuzzy homomorphism from A to B through f. *i.e.* $F_f: A \to B$ is a homomorphism.

5. *The Category L-FCyc :*

In this section we introduce and discuss a new category whose objects are L-fuzzy subgroups of finite cyclic groups saturating their respective maximum lattices and morphisms are L-fuzzy homomorphisms. We shall start with the definition of a category.

5.1. *Definition*[3]. A *category* $\mathcal{C}$ consists of:

(1) a *class of objects*, denoted as *Ob*$\mathcal{C}$ and whose members are denoted as A, B, C,…

(2) a family of mutually disjoint sets {Mor(*A*,*B*)} for all objects A, B in whose elements f,g,h… ∈ Mor(A,B) are called *morphisms* and

(3) a family of maps called *composition* {Mor (A,B)X Mor(B,C)→ Mor(A,C)} in which

$$(f, g) \mapsto gf \text{ for all } A,B,C \in Ob\mathcal{C}.$$

satisfying the following axioms:

(1) Associativity: For all $A,B,C,D \in Ob\mathcal{C}$ and all $f \in Mor(A,B)$, $g \in Mor(B,C)$ and $h \in Mor(C,D)$, we have, $h(gf) = (hg)f$

(2) Identity: For each $A \in Ob\mathcal{C}$ there is a morphism $I_A \in Mor(A,A)$, called the *identity*, such that we have, $fI_A = f$ and $I_A g = g$ for all $B,C \in Ob\mathcal{C}$, and all $f \in Mor(A,B)$ and $g \in Mor(C,A)$.

5.2. *Example*[3]. All sets together with the set maps and their composition form a category. This category is denoted by **Set.**

5.3. *Example*[3]. All groups together with group homomorphisms and their composition form a category which will be denoted by **Grp.**

5.4. *Example*[3]. Topological spaces

together with continuous functions and their composition form a category denoted as **Top.**

5.5. *Example*[3]. A map f between two posets is said to be *order–preserving* if $a \leq b$ implies $f(a) \leq f(b)$. All posets together with order-preserving maps and their composition is a category. This category is denoted by **Poset.**

5.6. *Example*[3].All lattices together with order-preserving maps and their composition is a category. This category is denoted by **Lat.**

5.7. *Theorem.* L-fuzzy subgroups of finite cyclic groups which saturate their respective maximum lattices together with L-fuzzy homomorphism of L-fuzzy groups through an embedding form a category.

Proof: *Objects* are L-fuzzy subgroups of finite cyclic groups which saturate their respective maximum lattices. *Morphisms* are L-fuzzy homomorphisms of L-fuzzy groups through an embedding.

Composition of morphisms: Let G_1, G_2 and G_3 be finite cyclic groups with maximum lattices L_1, L_2 and L_3 respectively where L_1 is embeddable on L_2 and L_2 is embeddable on L_3. Let $A \in (G_1, L_1)$, $B \in (G_2, L_2)$ and $C \in (G_3, L_3)$, each of them saturating its respective maximum lattice. Let $f: L_1 \to L_2$ and $g: L_2 \to L_3$ be maps such that $f(I_1)=I_2$, $g(I_2)=I_3$ and f is an embedding of L_1 on L_2 and g is an embedding of L_2 on L_3. Then gf is an embedding of L_1 on L_3. Let $F_f: A \to B$ and $G_g: B \to C$ be homomorphisms. Then $F:Pr(L_1) \to Pr(L_2)$ and $G:Pr(L_2) \to Pr(L_3)$ through f and g respectively are given by

$$F(A^{-1}(a))=B^{-1}(f(a)) \text{ for all a in } L_1 \text{ and}$$
$$G(B^{-1}(b))=C^{-1}(g(b)) \text{ for all b in } L_2.$$

Now, $G(B^{-1}(f(a)))=C^{-1}(g(f(a))$ for all a in L_1. *i.e.*, $G(F(A^{-1}(a))= C^{-1}((gf)(a))$ for all a in L_1. Define $GF : Pr(L_1) \to (L_3)$ by $(GF)(A^{-1}(a))= G(F(A^{-1}(a))$ for all a in L_1. Then GF is an L-fuzzy homomorphism from A to C through gf. *i.e.*, $GF_{gf}:A \to C$ is a homomorphism. This is the composite of F_f and G_g.

Associativity: Let G_4 be a finite cyclic group with maximum lattice L_4 where L_3 is embeddable on L_4. Let $D \in (G_4, L_4)$, saturating its maximum lattice. Let $h: L_3 \to L_4$ be a map such that $h(I_3)=I_4$ and h is an embedding of L_3 on L_4.Let $H_h:C \to D$ be another morphism in the category. Then, $H(C^{-1}(c))=D^{-1}(h(c))$ for all c in L_3. Now, $H(GF(A^{-1}(a))=H(C^{-1}((gf)(a))= D^{-1}(h(gf)(a))$. Also, $(HG)(F(A^{-1}(a)))=(HG)(B^{-1}(f(a)))=H(G(B^{-1}(f(a))= H(C^{-1}(gf)(a)))=D^{-1}(h(gf)(a)))$. It follows that $H_h(G_gF_f)= (H_hG_g)(F_f)$.

Identity: For each A in the category and the corresponding lattice L_1, let $e: L_1 \to L_1$ be defined by e (a) =a for all a in L_1 and $I:Pr(L_1) \to Pr(L_1)$ be an identity mapping on $Pr(L_{-1})$. Then $I_e: A \to A$ is an isomorphism. We shall show that I_e is the identity of A. Suppose that, $F_f: A \to B$ is a homomorphism. Then $F(A^{-1}(a))=B^{-1}(f(a))$ for all a in L_1. Also $I_e (A^{-1}(a)) =A^{-1} (e (a)) =A^{-1} (a)$ for all a in L_1. Hence,

$(F_fI_e)(A^{-1}(a)) = F_f((I_e)(A^{-1}(a)) = F_f(A^{-1}(a))$. Thus $F_fI_e=F_f$.

Similarly, $I_eG_g=G_g$ for G_g: $B\to A$. Hence, I_e is the identity of A .That is,$1_A=I_e$■

5.8. *Notation*. We shall denote the above category by **L-FCyc.** Whenever we say that A is an object in **L-FCyc,** we shall mean that there is a finite cyclic group G_1 and a corresponding maximum lattice L_1 such that $A\in(G_1, L_1)$. Similarly, the statement F_f: $A\to B$ is a morphism in **L-FCyc** shall imply that A $\in(G_1, L_1)$, $B\in(G_2, L_2)$, F: Pr $L_1\to$Pr L_2 and that f: $L_1\to L_2$ is an embedding such that $F(A^{-1}(a)) = B^{-1}(f(a))$ for every point a in L_1.

5.9. *Proposition.* Let F_f: $A\to B$ and F_g: $A\to B$ be two morphisms in **L-FCyc**. Then f = g.

Proof: Given F_f: $A\to B$. Hence, $F(A^{-1}(a)) = B^{-1}(f(a))$ for all a in L_1. Also F_g: $A\to B$. Hence, $F(A^{-1}(a))=B^{-1}(g(a))$ for all a in L_1. $\therefore$ $B^{-1}(f(a)) =B^{-1}(g(a))$ for all a in L_1. $\therefore$ f(a)=g(a) for all a in L_1. $\therefore$ f=g■

5.10. *Proposition* Let F_f: $A\to B$ and G_f: $A\to B$ be two morphisms in **L-FCyc.** Then F = G.

Proof: Given F_f: $A\to B$. Hence, $F(A^{-1}(a)) = B^{-1}(f(a))$ for all a in L_1. G_f: $A\to B$. Hence, $G(A^{-1}(a)) = B^{-1}(f(a))$ for all a in L_1. $\therefore$ $F(A^{-1}(a))=G(A^{-1}(a))$ for all a in L_1. $\therefore$F=G■

6. Properties of L-FCyc :

In this section we discuss some properties of the category **L-FCyc.** As this requires some more results from category, we shall present them first.

6.1. *Definition*[3]. Let $\mathcal{B}$ and $\mathcal{C}$ be categories. Let $\mathcal{F}$ consist of

(1) a map ob $\mathcal{B} \ni A \mapsto \mathcal{F}(A) \in ob\mathcal{C}$

(2) a family of maps $\{Mor_{\mathcal{B}}(A,B) \ni f \mapsto \mathcal{F}(f) \in Mor_{\mathcal{C}}(\mathcal{F}(A),\mathcal{F}(B))\}$ for all A,B $\in ob\mathcal{B}$

Then $\mathcal{F}$ is called a *covariant functor* (or simply a *functor*) if $\mathcal{F}$ complies with the following axioms:

(i) $\mathcal{F}(1_A) = 1_{\mathcal{F}(A)}$ for all A $\in$ ob$\mathcal{B}$

(ii) $\mathcal{F}(fg) = \mathcal{F}(f)\mathcal{F}(g)$ for all f $\in \mathrm{Mor}_{\mathcal{B}}(B,C)$, g $\in \mathrm{Mor}_{\mathcal{B}}(A,B)$ and for all A,B,C $\in$ ob$\mathcal{B}$

6.2. *Definition*[3]. Let $\mathcal{B}$ and $\mathcal{C}$ be categories. Let $\mathcal{F}$ consist of

(1) a map ob$\mathcal{B} \ni A \mapsto \mathcal{F}(A) \in ob\mathcal{C}$

(2) a family of maps $\{Mor_{\mathcal{B}}(A,B) \ni f \mapsto \mathcal{F}(f) \in Mor_{\mathcal{C}}(\mathcal{F}(B),\mathcal{F}(A))\}$ for all A,B $\in ob\mathcal{B}$

Then $\mathcal{F}$ is called a *contravariant functor* if $\mathcal{F}$ complies with the following axioms:

(i) $\mathcal{F}(1_A) = 1_{\mathcal{F}(A)}$ for all A $\in$ ob$\mathcal{B}$

(ii) $\mathcal{F}(fg) = \mathcal{F}(g)\mathcal{F}(f)$ for all f $\in \mathrm{Mor}_{\mathcal{B}}(B,C)$, g $\in \mathrm{Mor}_{\mathcal{B}}(A,B)$ and for all A,B,C $\in ob\mathcal{B}$.

6.3. *Example*[3]. Consider the category **Grp.** Its objects are sets with an additional structure. The morphisms are maps compatible with the structure of the sets. The composition is juxtaposition. If one assigns to every object, the underlying set and to every morphism the underlying set map, then this defines a covariant functor from **Grp** to **Set.** This kind of functors are called *forgetful functors.*

6.4. *Example.* Let A: $Z_m \to L_m$ and B: $Z_n \to L_n$ be objects in **L-FCyc.** If $F_f \in$ Mor(A,B), then f : $L_m \to L_n$ is an embedding such that $f(I_m)=I_n$ and for $F:Pr(L_m) \to Pr(L_n)$, $F(A^{-1}(a))=B^{-1}(f(a))$ for every a in L_m. Consider the category **Lat** with lattices as objects and lattice homomorphisms as morphisms. Define $\mathcal{F}$: **L-FCyc** $\to$ **Lat** as follows:

$\mathcal{F}(A)=L_m$ and $\mathcal{F}(F_f)=f$.

Assume that I:Pr $(L_m) \to$ Pr(L_m) is the identity mapping on Pr(L_m) and e:$L_m \to L_m$ is the identity mapping on L_m. Then $1_A = I_e$ and 1_{L_m} =e. Further, we get

(1) $\mathcal{F}(1_A) = \mathcal{F}(I_e) =$ e=$1_{L_m} = 1_{\mathcal{F}(A)}$ and

(2) $\mathcal{F}(F_f G_g) = \mathcal{F}(FG_{fg})$ =fg= $\mathcal{F}(F_f)\,\mathcal{F}(G_g)$

Hence, $\mathcal{F}$ is a covariant functor from **L-FCyc** to **Lat.**

6.5. *Definition*[3]. Let $\mathcal{C}$ be a category. Given A ϵ Ob $\mathcal{C}$ and f ϵ Mor(B,C), we define a map *Mor (A,f)*: Mor(A,B) $\to$ Mor(A,C) by

Mor(A,f)(g)=fg, for all g ϵ Mor (A,B).

We also define a map *Mor (f,A)*: Mor(C,A) $\to$ Mor(B,A) by

Mor(f,A)(h)=hf ,for all h ϵ Mor (C,A).

6.6. *Proposition*[3]. Let $\mathcal{C}$ be a category and A ϵ Ob$\mathcal{C}$. Then Mor(A,-):$\mathcal{C} \to$ **Set** with

Ob $\mathcal{C} \ni$ B $\longmapsto$ Mor(A,B) $\in$ Ob **Set**

Mor (B,C) $\ni$ f $\longmapsto$ Mor(A,f) $\in$ Mor(Mor(A,B), Mor(A,C))

is a covariant functor. Furthermore, Mor(-, A): $\mathcal{C} \to$ **Set** with

Ob $\mathcal{C} \ni$ B $\longmapsto$ Mor(B,A) $\in$ Ob **Set**

Mor (B,C) $\ni$ f $\longmapsto$ Mor(f,A) $\in$ Mor (Mor (C,A), Mor(B,A))

is a contravariant functor■

6.7. *Remark*[3]. The above proposition says that corresponding to any object A in a category $\mathcal{C}$, one can form a covariant functor Mor(A,-) and a contravariant functor Mor (-, A). Of these, Mor(A,-) is called *covariant representable functor* and Mor(-,A) is called *contravariant representable functor.*

6.8. *Definition*[3]. Let $\mathcal{C}$ be a category and f a morphism in $\mathcal{C}$. f is called a *monomorphism* if the map Mor(B, f) is injective (*i.e.* one-to-one) for all B$\in$Ob$\mathcal{C}$.

6.9. *Definition*[3]. Let $\mathcal{C}$ be a category and f a morphism in $\mathcal{C}$. f is called an *epimorphism* if the map Mor(f,B) is injective for all B$\in$Ob$\mathcal{C}$.

6.10. *Lemma*[3]. (a). f $\in$ Mor(A,B) is a monomorphism in $\mathcal{C}$ if and only if fg=fh implies g=h for all C ϵ Ob$\mathcal{C}$ and for all g,h ϵ Mor(C,A).
(b). f $\in$ Mor(A,B) is an epimorphism in $\mathcal{C}$ if and only if gf=hf implies g=h for all C ϵ Ob$\mathcal{C}$ and for all g,h ϵ Mor(B,C)■

The above lemma enables us to roughly define a *monomorphism* as a *left cancellable morphism*; and an *epimorphism* as *a right cancellable morphism.*

6.11. *Theorem*. Every morphism in **L-FCyc** is a monomorphism

Proof: Let $F_f : A \to B$ be a morphism in **L-FCyc.** Let the groups corresponding to A and B be G_1 and G_2; and the lattices be L_1 and L_2. Then F: $Pr(L_1) \to Pr(L_2)$ and $f : L_1 \to L_2$ is the embedding of lattices. Let C be any object of **L-FCyc.** Then $C \in mk(G_3, L_3)$ for some group G_3 and the corresponding maximum lattice L_3. Also let G_g and H_h be any two morphisms in Mor(C,A). Then G, H: $Pr(L_3) \to Pr(L_1)$ and g, h: $L_3 \to L_1$. Then FG_{fg}, $FH_{fh} : C \to B$ are morphisms in Mor(C,B). Hence, $(FG)(C^{-1}(a)) = B^{-1}((fg)(a))$ for all a in L_3.
Similarly, $(FH)(C^{-1}(a)) = B^{-1}((fh)(a))$ for all a in L_3.
Suppose that $F_fG_g = F_fH_h$. *i.e.* $FG_{fg} = FH_{fh}$. Then $(FG)(C^{-1}(a)) = (FH)(C^{-1}(a))$ for all a in L_3

$\Rightarrow B^{-1}((fg)(a)) = B^{-1}((fh)(a))$ for all a in L_3

$\Rightarrow (fg)(a) = (fh)(a)$ for all a in L_3

$\Rightarrow f(g(a)) = f(h(a))$ for all a in L_3

$\Rightarrow g(a) = h(a)$ for all a in L_3

$\Rightarrow g = h$.

By assumption, $G_g, H_h : C \to A$ are homomorphisms. g = h implies that G_g, $H_g : C \to A$ are homomorphisms. Hence, by proposition 5.10, G=H. Therefore $G_g = H_h$. This shows that F_f is a monomorphism■

6.12. *Remark.* All the morphisms in **L-FCyc** are not epimorphisms. We give below an example to prove this.

6.13. *Example.* Let $Z_1=\{0\}$ with the maximum lattice, $L_1=\{(O_1=)I_1\}$, the one-point lattice; $Z_2=\{0,1\}$ with the maximum lattice, $L_2=\{O_2, I_2\}$ and $Z_6=\{0,1,2,3,4,5\}$ with the maximum lattice $L_3=\{O_3(=1), 2, 3, I_3(=6)\}$. Define $A: Z_1 \to L_1$, $B: Z_2 \to L_2$ and $C: Z_6 \to L_3$ by

$A(0)=I_1$

$B(0)=I_2, B(1)=O_2$ and

$C(0)=I_3, C(2)=C(4)=2, C(3)=3$ and $C(1)=C(5)=O_3$.

Define $F : Pr(L_1) \to Pr(L_2)$ by $F(A^{-1})(I_1)=\{0\}$; $G : Pr(L_2) \to Pr(L_3)$ by $G(B^{-1})(I_2)=\{0\}$ and $G(B^{-1})(O_2)=\{2,4\}$ and $H : Pr(L_2) \to (L_3)$ by $H(B^{-1})(I_2)=\{0\}$ and $H(B^{-1})(O_2)=\{3\}$. Define $f: L_1 \to L_2$ by $f(I_1)=I_2$; $g : L_2 \to L_3$ by $g(I_2)=I_3$, $g(O_2)=2$ and $h : L_2 \to L_3$ by $h(I_2)=I_3$, $h(O_2)=3$. Then $F_f: A \to B$, $G_g, H_h : B \to C$. Here $G_g \neq H_h$. But $G_gF_f = H_hF_f$ ■

7. Concluding remarks

In this paper, we have constructed a category **L-FCyc** whose objects are L-fuzzy subgroups of finite cyclic groups (where L in this case is the maximum lattice of the group concerned) and the morphisms are L-fuzzy homomorphisms defined through an embedding of lattices. Some properties of **L-FCyc** are discussed here. We have proved that every morphism in this category is a monomorphism. We have also constructed a counter example to show that there are morphisms in this category which are not epimorphisms. We are continuing the investigation and hope to get more interesting results.

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