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Z_{4p}- Magic labeling for some more special graphs

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Abstract

For any non-trivial abelian group A, a graph G is said to be A-magic if there exists a labeling f of the edges of G with non zero elements of A such that the vertex labeling f^+ defned as $f^+(v) = \Sigma f(uv)$ taken over all edges uv incident at v is a constant[4]. A graph is said to be A-magic if it admits A- magic labeling. In this paper we consider (moduloZ_4,+) as abelian group and we prove Z_4 - magic labeling for various graphs and generalize Z_{4p} -magic labeling for those graphs. The graphs which admit Z_{4p}-magic labeling are called as Z_{4p}-magic graphs.

Mathematics Subject Classifcation 2000:05C78

Key words : A - magic labeling, Z_4 - magic labeling, Z_{4p}-magic labeling, Z_{4p}-magic graphs.

1 Introduction

By a graph $G(V,E)$ we mean G is a Finite, simple, undirected graph. The concept of magic labelings were introduced by Sedlacek in 1963, Kong, Lee and Sun[3] used the term magic labeling for the labeling of edges with non negative integers such that for each vertex v the sum of the labels of all edges incident at vis same for all v. In particular the edge labels need not be distinct.

For any non-trivial abelian group A under addition a graph G is said to be A-magic if there exists a labeling f of the edges of G with non zero elements of A such that, the vertex labeling f^+ defned as $f^+(v)= \Sigma f$

(uv) over all edges uv incident at v is a constant. Throughout this paper, we choose Z_4 which is additive modulo 4 as the abelian group and we prove some graphs such as Ladder, step-graph, $C_n^{(t)}$ and Helm graph (H_n) are Z_4-magic graphs. At the end, we prove that they are all Z_{4p}-magic graphs. Throughout this paper by a path P_n we mean, it is a path of length $n - 1$, and having n vertices[5].

2 Main Results

Defnition 2.1.[2] The cross product $G_1 \times G_2$ has its vertex set $V_1 \times V_2$ and two points $u=(u_1,u_2)$ and $v=(v_1,v_2)$ are adjacent in $G_1 \times G_2$ whenever ($u_1=v_1$ and u_2 adjacent to v_2) or ($u_2 = v_2$ and u_1 adjacent to v_1). The product $P_m \times P_n$ is called plannar grids and $P_2 \times P_n$ is called ladder. it is denoted as L_n.

Theorem 2.2. The Ladders L_n are Z_4-magic for $n \geq 2$.

Proof. Let $V(L_n)$ be the vertex set and $E(L_n)$ be the edge set of the graph L_n.

Then $V(L_n) = \{v_i/1 \leq i \leq n\} \cup \{u_i/1 \leq i \leq n\}$ and $E(L_n) = \{x_1,x_2,\ldots x_{n-1}, y_1,y_2,\ldots,y_{n-1},e_1,e_2,\ldots,e_n\}$

where $x_i = v_i\, v_{i+1}$

$y_i = u_i\, u_{i+1} \quad \forall i = 1,2,\ldots,n-1.$

and $e_i = u_i v_i\ \forall i = 1,2,\ldots,n$

Let $f : E(L_n) \to Z_4 - \{0\}$ defned as

$f(x_i) = 1$ and $f(y_i) = 1,\ 1 \leq i \leq n-1$

$f(e_1) = f(e_n) = 2$

$f(e_j) = 1, \quad \forall j=2,3,\ldots n-1$

Now, $f^+ : V(L_n) \to Z_4$

$f^+(v_i)=f(e_i)+f(x_{i-1})+f(x_i)$ for $i=2,3,\ldots n-1$

Similarly $f^+(u_i) = f(e_i) + f(y_{i-1}) + f(y_i)$ for $i=2,3,\ldots,n-1$

$f^+(v_i)=1+1+1 \equiv 3 \pmod 4$ for $i=2,3,\ldots,n-1$

$= 3$

$f^+(u_i) = 1 + 1 + 1 \equiv 3 \pmod 4$

$= 3$ for $i=2,3,\ldots,n-1$

$f^+(v_1) = f(e_1) + f(x_1)$

$f^+(v_1) = 2 + 1 \equiv 3 \pmod 4 = 3$

sameway $f^+(v_n) = 2 + 1 \equiv 3 \pmod 4 = 3$

$f^+(u_1) = 2 + 1 \equiv 3 \pmod 4 = 3$

$f^+(u_n) = 2 + 1 \equiv 3 \pmod 4 = 3$

Hence, $f^+(V(L_n))$ is a constant and it is equal to 3 where $3 \in Z_4$.

Therefore, L_n admits Z_4-magic.

The following figure illustrates the above theorem. 2.2 for $n = 4$. □

Example 2.3.

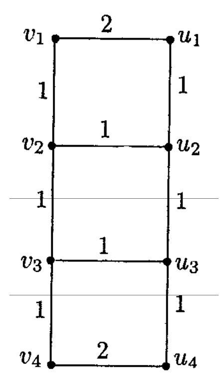

Figure 1 Z_4-Magic Labeling of L_4

Definition 2.4.[1] Let P_n be the path of n vertices and n-1 edges. Let us define each vertex as(1,1) (1,2) (1,3),..., (1,n). The edge

$e_{1,i} = (1,i)\,(1,i+1)$, where $i=1,2,...n-1$. On each edge we errect a ladder with $n-(i-1)$ steps including the edge $e_{1,i}$.

This graph is called Step graph and denoted as ST_n where n denotes the number of vertices in the base.

Example 2.5.

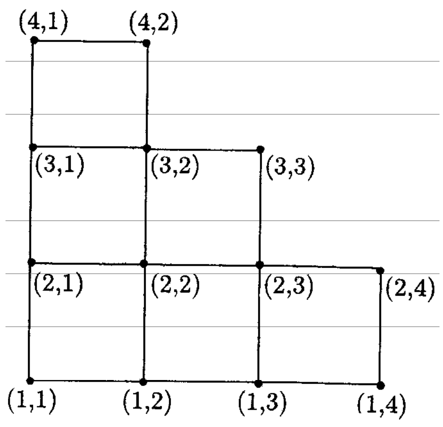

Figure 2 ST_4

Theorem 2.6. ST_n is Z_4-magic $n\geq 4$

Proof. The graph ST_n has $(n^2+3n-2)/2$ vertices and n^2 edges. Any vertex is given by (i,j), where i denotes the row and j denotes the column.

Let $f : E(ST_n)\rightarrow Z_4-\{0\}$ defined as

$f\,[(1,j-1)(1,j)]=2=f\,[(j-1,1)(j,1)]$ for $j=2,n$

$f\,[(1,j)(1,\, j+1)] = 1 = f\,[(j+1,1)(j,1)]$ for $j=2,3,\ldots,n-2$

$f\,[(n,2)(n-1,2)]=2=f\,[(2,n)(2,n-1)]$

$f\,[(n-i,2)(n-i-1,2)]=f\,[(2,n-i)(2,n-i-1)]=3$, $i=1,3,\ldots,n-4$ if n is odd

(or) $i=1,3,\ldots,n-3$ if n is even

$f\,[(n-k,2)(n-k-1,2)]=f\,[(2,n-k)(2,n-k-1)]=1$, $k=2,4,\ldots,n-3$ if n is odd

(or) $k=2,4,\ldots,n-4$ if n is even

For $j=3,4,\ldots,n/2$ for even n (or) $\frac{n+1}{2}$ for odd n

$f\,[(n-i,j)(n-i-1,j)] = f\,[(j,n-i)(j,n-i-1)] = 2$, $i=j-2,j-1,\ldots,n-(j+1)$

For a fixed j, $j=2,3,4\ldots(n-1)/2$ when n is odd (or) $j=2,3,4\ldots n/2$ when n is even

$f\,[(i,j)(i,j+1)]=f\,[(j,i)(j+1,i)]=2$ where $i=j+1,\, j+2,\ldots,n-(j-1)$

For $j = 1$

$f\,[(i,j)(i,j+1)]=f\,[(j,i)(j+1,i)]=2$ where $i=3,4,\ldots,n-2,n$

$f\,[(n-1,1)(n-1,2)]=f\,[(1,n-1)(2,n-1)]=1$ and $f\,[(2,1)(2,2)]=f\,[(1,2)(2,2)]=1$

Let $f^+ : V(ST_n) \rightarrow Z_4$

$f^+(1,1)=(2+2) \equiv 0 \pmod 4=0$

similarly

$f^+(1,n)=f^+(n,1)=(2+2) \equiv 0 \pmod 4=0$

$f^+(i,n-(i-2)) = f^+(n-(i-2),i)=(2,2) \equiv 0 \pmod 4 = 0$ $2 \leq i \leq \frac{n+1}{2}$ for odd n (or) $2 \leq i \leq \frac{n+2}{2}$ for even n

$f^+(2,n-j) = f^+(n-j-2)$, $1 \leq j \leq n-3$

$f^+(2,n-j) = f\,[(2,n-j)(2,n-j+1)]+ f\,[(2,n-j)(2,n-j-1)]+f\,[(2,n-j)(1,n-j)]+ f\,[(2,n-j)(3,n-j)]$, $1 \leq j \leq n-3$

$\equiv 0 \pmod 4 = 0$

$f^+(n-j,2) = f\,[(n-j,2)(n-j+1,2)]+f\,[(n-j,2)(n-j-1,2)]+ f\,[(n-j,2)(n-j,1)]+f\,[(n-j,2)(n-j,3)]$, $1 \leq j \leq n-3$

$\equiv 0 \pmod 4 = 0$

$f^+(2,2)=1+3+3+1\equiv 0 \pmod 4=0$ for even n

$f^+(2,2)=1+1+1+1\equiv 0 \pmod 4=0$ for odd n

For $3 \leq i \leq \frac{n+1}{2}$ when n is odd and $3 \leq i \leq \frac{n}{2}$ when n is even

$f^{+}(i,n\text{-}j)=f^{+}(n\text{-}j,i),\ i\text{-}1\le j\le n\text{-}i$

$f^{+}(i,n\text{-}j)=f\,[(i,n\text{-}j)(i,n\text{-}j+1)]+f\,[(i,n\text{-}j)(i,n\text{-}j\text{-}1)]+f\,[(i,n\text{-}j)(i\text{-}1,n\text{-}j)]+f\,[(i,n\text{-}j)(i+1,n\text{-}j)]$

$\equiv 2+2+2+2 \pmod 4$

$=0,\ i\text{-}1\le j\le n\text{-}i$

$f^{+}(n\text{-}j,i)=f\,[(n\text{-}j,i)(n\text{-}j+1,i)]+f\,[(n\text{-}j,i)(n\text{-}j\text{-}1,i)]+f\,[(n\text{-}j,i)(n\text{-}j,i\text{-}1)]+f\,[(n\text{-}j,i)(n\text{-}j,i+1)]$

$\equiv 2+2+2+2 \pmod 4$

$=0,\ i\text{-}1\le j\le n\text{-}i$

$f^{+}(1,i)=f^{+}(i,1)=2+1+1\equiv 0 \pmod 4=0,\ i=2,3,\ldots n\text{-}1$

Here f^{+} is a constant and it is equal to zero for all $v\in V(ST_n)$

Hence ST_n is Z_4 magic □

Example 2.7.

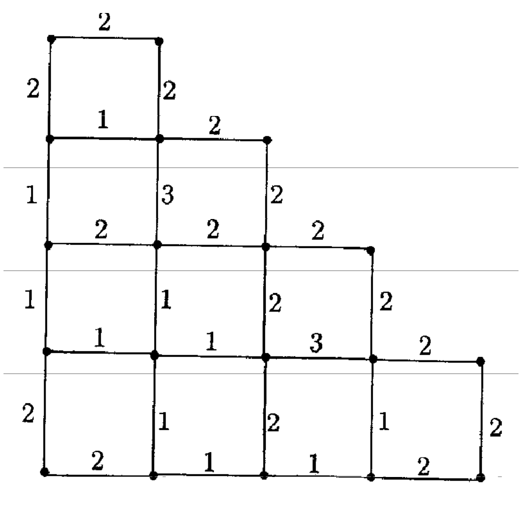

Figure 3 Z_4-Magic Labeling of ST_5

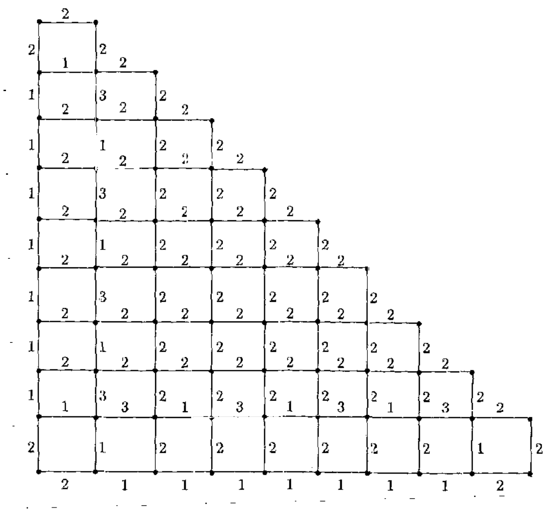

Figure 4 Z_4-Magic Labeling of ST_{10}

Defnition 2.8.[2] Let $C_n^{(t)}$ denote the one point union of 't' cycles C_n of length n. This graph has $t(n\text{-}1)+1$ vertices and tn edges.

Theorem 2.9. $C_n^{(t)}$ is Z_4-magic where C_n is a cycle of length n and t denotes the number of cycles.

Proof. Let u_0 be the vertex shared by all the t cycles. Let $u_i^{(j)}$ $i=1,2,\ldots,n\text{-}1$ be the other vertices of j^{th} cycle where $j=1,2,3,\ldots t$

Case 1 n is even and $t\in N$

let $f: E(C_n^{(t)})\to Z_4-\{0\}$ be defned as follows

$f\,(u_0u_1^{(j)})=1, 1\le j\le t$

$f\,(u_{2i}u_{2i+1}^{(j)})=1, 1\le i\le (n-2)/2\ \ 1\le j\le t$

$f\,(u_{2i-1}^{(j)}u_{2i}^{(j)})=3,\ 1\le i\le (n-2)/2\ \ 1\le j\le t$

and

$f(u_0 u_{n-1}^{(j)}) = 3,\ 1 \le i \le (n-2)/2\ j = 1,2,\ldots,t$

Let $f^+: V(C_n^{(t)}) \to Z_4$ defned as

$$f^+(u_0) = \sum_{j=1}^{t} \left(f(u_0 u_1^{(j)}) + f(u_0 u_{n-1}^{(j)}) \right)$$
$$\equiv (1+3).t \ (\text{mod } 4)$$
$$\equiv 0 \ (mod\ 4) = 0$$

$$f^+(u_i^{(j)}) = f(u_{i-1}^{(j)}, u_i^{(j)}) + f(u_i^{(j)}, u_{i+1}^{(j)})$$
$$\equiv (1+3)\ (mod\ 4)$$
$$= 0,\ 1 \le i \le n-2,\ 1 \le j \le t$$

$$f^+(u_{n-1}^{(j)}) = f(u_{n-2}^{(j)} u_{n-1}^{(j)}) + f(u_{n-1}^{(j)} u_0)$$
$$\equiv (1+3)\ (mod\ 4)$$
$$= 0,\ 1 \le j \le t$$

Here we find $f^+(v)$ is a constant and it is equal to 0 for all $v \in V(C_n^{(t)})$.

Case 2 :
Subcase (a) n is odd $t = 1$ (*mod* 2)

Let $f: E(C_n^{(t)}) \to Z_4$-{0} defined as

$f(u_0 u_1^{(j)}) = 1 = f(u_0 u_{n-1}^{(j)}),\ 1 \le j \le t$

$f(u_i^{(j)} u_{i+1}^{(j)}) = 1,\ 1 \le j \le t$ and $1 \le i \le n-2$

$f^+: V(C_n^{(t)}) \to Z_4$

$$f^+(u_0) = \sum_{j=1}^{t} \left(f(u_0 u_1^{(j)}) + f(u_0 u_{n-1}^{(j)}) \right)$$
$$\equiv (1+1).t\ (mod\ 4) = 2$$

$$f^+(u_i^{(j)}) = f(u_{i-1}^{(j)}, u_i^{(j)}) + f(u_i^{(j)}, u_{i+1}^{(j)})$$
$$\equiv (1+1)\ (mod\ 4)$$
$$= 2\ (mod\ 4)\ 1 \le i \le n-2,\ 1 \le j \le t$$

$$f^+(u_{n-1}^{(j)}) = f(u_{n-2}^{(j)} u_{n-1}^{(j)}) + f(u_{n-1}^{(j)} u_0)$$
$$\equiv (1+1)\ (mod\ 4)$$
$$= 2,\ 1 \le j \le t$$

$f^+(v)$ is constant and it is equal to 2 for all $v \in V(C_n^{(t)})$

Subcase b n is odd and $t \equiv 0$ (*mod* 2)

Let $f: E(C_n^{(t)}) \to Z_4 - \{0\}$ defined as

$f(u_0 u_1^{(j)}) = f(u_0 u_{n-1}^{(j)}) = 3,\ 1 \le j \le t$

also

$f(u_{2i}^{(j)} u_{2i+1}^{(j)}) = 3,\ 1 \le i \le (n-3)/2,\ 1 \le j \le t$

$f(u_{2i-1}^{(j)} u_{2i}^{(j)}) = 1,\ 1 \le i \le (n-1)/2,\ 1 \le j \le t$

$f^+: V(C_n^{(t)}) \to Z_4$ we find

$$f^+(u_0) = \sum_{j=1}^{t} \left(f(u_0 u_1^{(j)}) + f(u_0 u_{n-1}^{(j)}) \right)$$
$$\equiv (3+3)t\ (mod\ 4)$$
$$\equiv 6(2m)\ (mod\ 4) = 0$$

$$f^+(u_i^{(j)}) = \sum_{j=1}^{t} \left(f(u_0 u_1^{(j)}) + f(u_0 u_{n-1}^{(j)}) \right)$$
$$\equiv (3+1)\ (mod\ 4)$$
$$= 0,\ 1 \le i \le n-2,\ 1 \le j \le t$$

$$f^+(u_{n-1}^{(j)}) = f(u_{n-2}^{(j)} u_{n-1}^{(j)}) + f(u_{n-1}^{(j)} u_0)$$
$$\equiv (3+1)\ (mod\ 4) = 0,\ 1 \le j \le t$$

So $f^+(v)$ is a constant for all $v \in V(C_n^{(t)})$

Hence, $C_n^{(t)}$ is Z_4-magic graph in all the cases[6] □

Example 2.10

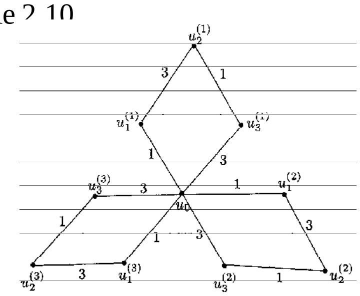

Figure 5 Z_4 -Magic Labeling of $C_4^{(3)}$

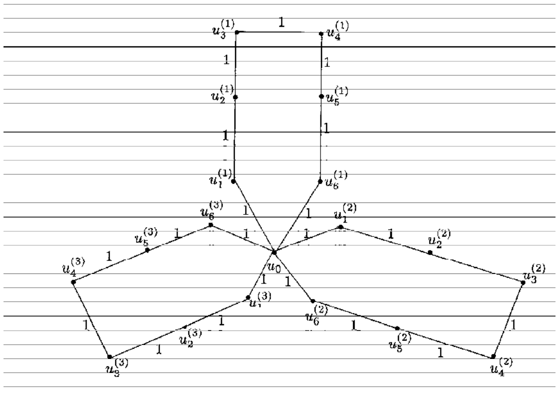

Figure 6 Z_4 -Magic Labeling of $C_7^{(3)}$

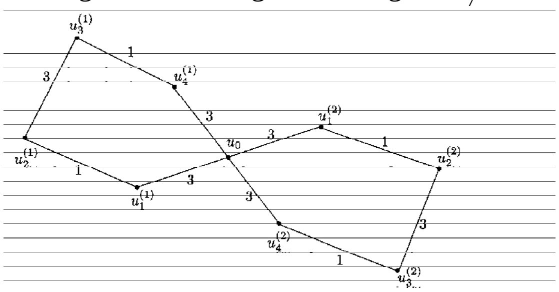

Figure 7 Z_4 -Magic Labeling of $C_5^{(2)}$

Definition 2.11.[2] The Helm is graph obtained from a wheel W_n by attaching a pendent edge at each vertex of the n-cycle of the wheel of order n. It is denoted as H_n. H_n has $2n + 1$ vertices and 3n edges.

Theorem 2.12. H_n is *Z_4-magic for* $n \geq 3$

Proof. Let the centre of the wheel be u. Let $u_1, u_2, \ldots, u_n$ be the vertices on the wheel W_n and $v_1, v_2, \ldots, v_n$ be the pendant vertices joined with $u_1, u_2, \ldots, u_n$.

$V(H_n) = \{u\} \cup \{u_i/1 \leq i \leq n\} \cup \{v_i/1 \leq i \leq n\}$

$E(H_n) = \{uu_i/1 \leq i \leq n\} \cup \{u_iv_i/1 \leq i \leq n\}$ $\{u_iu_{i+1}/1 \leq i \leq n - 1\}$

Case 1 Let $n = 1$ (*mod* 2)

Let $f : E(H_n) \to Z_4 - \{0\}$ denned as follows

$$f(uu_j) = 2, \quad 1 \leq i \leq n$$
$$f(u_ju_{j+1}) = 1 \leq i \leq n - 1$$
$$f(u_ju_n) = 1$$
$$f(u_iv_i) = 2, \quad 1 \leq i \leq n$$

Let $f^+ : V(H_n) \to Z_4$

$$f^+(u) = \sum_{i=1}^{n} f(uu_i)$$
$$\equiv (2+2+2+\ldots+2).n \ (mod\ 4)$$
$$\equiv 6\ (mod\ 4)\ 2$$
$$= 2$$

$$f^+(u_i) = f(uu_i) + f(u_{i-1}u_i) + f(u_iu_{i+1}) + f(u_iv_i)$$
$$= (2 \text{ times } 2 + 2 \text{ times } 1)$$
$$\equiv (2+2+1+1)(mod\ 4)$$
$$= 2, \quad 1 \leq i \leq n$$
$$f^+(v_i) = 2, \quad 1 \leq i \leq n$$

f^+ is constant and it is equal to 2 for all $v \in V(H_n)$

Case 2 $n \equiv 2$ (*mod* 4)

Let $f : E(H_n) \to Z_4 - \{0\}$ defined as follows

$$f(uu_i) = 1, \quad 1 \leq i \leq n$$
$$f(u_{2i-1}u_{2i}) = 2, \quad 1 \leq i \leq n/2$$
$$f(u_{2i}u_{2i+1}) = 1, \quad 1 \leq i \leq (n/2)-1$$
$$f(u_1u_n) = 1$$
$$f(u_iv_i) = 2, \quad 1 \leq i \leq n$$

Let $f^+ : V(H_n) \to Z_4$ by definition

$$f^+(u) = \sum_{i=1}^{n} f(uu_i)$$
$$= 1+1+1+\ldots+1(n) \text{ times}$$

$\equiv 2\ (mod\ 4) = 2$

$f^+(u_i) = f(uu_i) + f(u_{i-1}u_i) + f(u_iu_{i+1}) + f(u_iv_i)$

$= 2,\ 1 \le i \le n$

$f^+(v_i) = 2,\ \forall i = 1,2,\ldots n$

Therefore, f^+ is constant and it is equal to 2 for all $v \in V(H_n)$. Hence H_n is Z_4 magic.

Case 3 $n \equiv 0\ (mod\ 4)$

Subcase (3a) $n \equiv 0\ (mod\ 8)$

Let $f^+ : E(H_n) \to Z_4 - \{0\}$ defined as follows

$f(uu_i) = 2,\ 1 \le i \le \frac{n}{2} - 2$

$f(uu_i) = 1,\ \frac{n}{2} - 1 \le i \le n$

$f(u_iu_{i+1}) = 1,\ 1 \le i \le \frac{n}{2} - 2$

$f(u_iu_{i+1}) = 2,\ \frac{n}{2} - 1 \le i \le n,\ i \equiv 1\ (mod\ 2)$

$f(u_iu_{i+1}) = 1,\ \frac{n}{2} - 1 \le i \le n-1,\ i \equiv 1\ (mod\ 2)$

$f(u_1u_n) = 1$

$f(u_iv_i) = 2,\ 1 \le i \le n$

Let $f^+\ V(H_n) \to Z_4$ by definition

$$f^+(u) = \sum_{i=1}^{n} f(uu_i)$$

$$= 2+2+\ldots 2\left(\frac{n-4}{2}\right) \textit{times} + 1 + 1 + 1 + \ldots + 1 \left(\frac{n+4}{2}\right) \textit{times}$$

$\equiv 2\ (mod\ 4) = 2$

$f^+(u_i) = f(uu_i) + f(u_{i-1}u_i) + f(u_iu_{i+1}) + f(u_iv_i)$

$= (2\ \textit{times}\ 2 + 2\ \textit{times}\ 1)$

$\equiv (2+2+1+1)(mod\ 4)\ \forall i = 1,2,\ldots n$

$= 2$

$f^+(v_i) \equiv 2\ (mod\ 4)$

$= 2\ \forall i = 1,2,\ldots n$

Subcase (3b) $n \equiv 4\ (mod\ 8)$

$f(uu_i) = 2,\ 1 \le i \le \frac{n}{2}$

$f(uu_i) = 1,\ \frac{n}{2} + 1 \le i \le n$

$f(u_iu_{i+1}) = 1,\ 1 \le i \le \frac{n}{2}$

$f(u_1u_n) = 1$

$f(u_{n-2i+1}u_{n-2i}) = 1,\ 1 \le i \le \frac{n}{2} - 1$

$f(u_{n-2i+2}u_{n-2i+1}) = 2,\ 1 \le i \le \frac{n}{2}$

Let $f^+ V(H_n) \to Z_4$ by definition

$$f^+(u) = \sum_{i=1}^{n} f(uu_i)$$

$$= 2+2+\ldots 2\left(\frac{n}{2}\right) \textit{times} + 1+1+1+\ldots+1 \left(\frac{n}{2}\right) \textit{times}$$

$\equiv 2\ (mod\ 4) = 2$

$f^+(u_i) = f(uu_i) + f(u_{i-1}u_i) + f(u_iu_{i+1}) + f(u_iv_i)$

$= (2\ \textit{times}\ 2 + 2\ \textit{times}\ 1)$

$\equiv (2+2+1+1)(mod\ 4)\ \forall i = 1,2,\ldots n$

$= 2$

$f^+(v_i) \equiv 2\ (mod\ 2)$

$= 2\ \forall i = 1,2,\ldots n$

Therefore, f^+ is constant and it is equal to 2 for all $v \in V(H_n)$. So H_n admits Z_4 magic. Note $u_0 = u_n$ and $u_{n+1} = u_1$. □

Example 2.13.

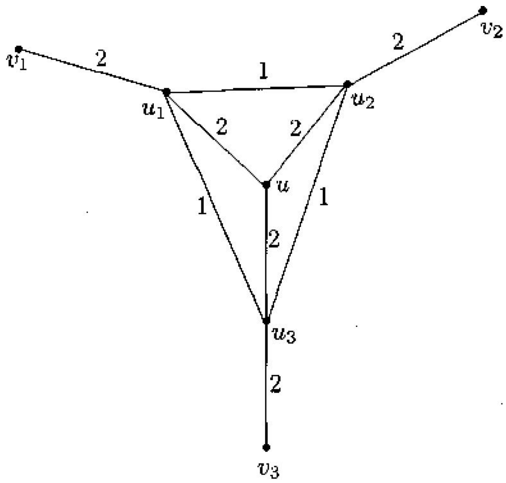

Figure 8. Z_4 -Magic Labeling of H_3

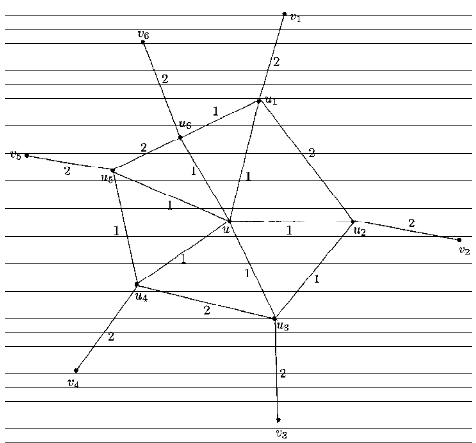

Figure 9. Z_4 -Magic Labeling of H_6

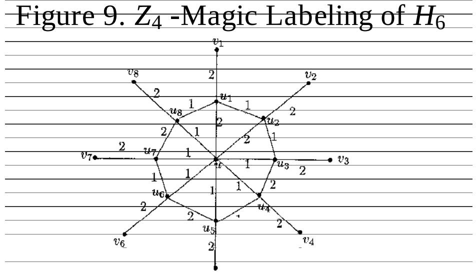

Figure 10. Z_4 -Magic Labeling of H_8

Observation 2.14.

In all the above, theorems, if we multiply the edge labeling by any positive integer p, we get the vertex labeling to be a constant and the constant value is p times (original constant occurs in the respective place of each case). Hence we can prove that all the above graphs admit Z_{4p}-magic labeling. So the graphs L_n, ST_n, $C_n^{(t)}$ and H_n are Z_{4p}-magic graphs.

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