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Total Neighborhood Number of a Graph

V.R. KULLI[*1] and D.K. PATWARI[2]

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Abstract

A set S of vertices of a graph G is a total neighborhood set of G if G is the union of the subgraphs induced by the closed neighborhoods of the vertices in S and for every vertex $u \in V$ there exists a vertex $v \in S$ such that u is adjacent to v. The total neighborhood number $n_t(G)$ of G is the minimum cardinality of a total neighborhood set of G. A total neighborhood nomatic partition of G is a partition $\{V_1, V_2, ..., V_k\}$ of V in which each V_i is a total neighborhood set of G. The total neighborhood nomatic number $n_{tn}(G)$ of G is the maximum order of a partition of the vertex set of G into total neighborhood sets. In this paper, we obtain results about two parameters, the total neighborhood number and total neighborhood nomatic number.

Key words: graph, total neighborhood number, total neighborhood nomatic number.

Mathematics Subject Classification: 05*C*.

1. Introduction

All graphs considered here are finite, undirected without loops and multiple edges. Let $G=(V, E)$ be a graph with p vertices and q edges. For graph theoretic terminology we refer to Harary[3].

A subset D of V is called a dominating set of G if every vertex in $V - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. A recent survey of $\gamma(G)$ is given in the book by Kulli[4].

A dominating set D of a graph G without isolated vertices is called a total dominating set of G if the induced subgraph $\langle D \rangle$ has no isolated vertices. The total domination number

$\gamma_t(G)$ of G is the minimum cardinality of a total dominating set of G. This concept was introduced by Cockayne *et al.*[1].

For any vertex $v \in V$, the open neighborhood of V is the set $N(v)=\{u \in V, uv \in E\}$ and the closed neighborhood of v is the set $N[v]=N(v) \cup \{v\}$.

A set S of vertices of G is a neighborhood set if $G = \bigcup_{v \in S} N[v]$. The neighborhood number $n_0(G)$ is the minimum cardinality of a neighborhood set of G, see[7].

A set S of vertices of G is a total neighborhood set if $G = \bigcup_{v \in S} \langle N[v] \rangle$ and for every vertex u in V, there exists a vertex v in S such that u is adjacent to v. The total neighborhood number $n_t(G)$ of G is the minimum cardinality of a total neighborhood set of G. This concept was introduced by Kulli and Patwari in[5].

Let $\delta(G)$ denote the minimum degree and $\lceil x \rceil (\lfloor x \rfloor)$ the least (greatest) integer greater (less) than or equal to x.

We note that $n_t(G)$ is only defined for G with $\delta(G) \geq 1$.

We need the following result.

Theorem A[6]. For a graph $\overline{C}_p$, $p \geq 5$,

$$n_0\left(\overline{C}_p\right) = 3.$$

2. Total Neighborhood Number:

Exact values of $n_t(G)$ for some standard graphs are given below.

Proposition 1. For any complete graph K_p with $p \geq 2$ vertices,

$$n_t(K_p) = 2.$$

Proposition 2. For any complete bipartite graph $K_{m,n}$ with $1 \leq m \leq n$,

$$n_t(K_{m,n}) = m + 1.$$

Proposition 3. For any cycle C_p with $p \geq 3$ vertices,

$$n_t\left(C_p\right) = \left\lceil \frac{2p}{3} \right\rceil.$$

Proposition 4. For any path P_p with $p \geq 3$ vertices,

$$n_t\left(P_p\right) = \left\lceil \frac{2p}{3} \right\rceil.$$

Proposition 5. For any wheel W_p with $p \geq 4$ vertices,

$$n_t(W_p) = 2.$$

The following results give lower bounds for $n_t(G)$.

Theorem 6. For any graph G without isolated vertices,

$$n_0(G) \leq n_t(G).$$

Theorem 7. For any graph G without isolated vertices,

$$\gamma(G) \leq \gamma_t(G) \leq n_t(G).$$

Theorem 8. If a nontrivial connected

graph G with a γ_t-set S and $V–S$ is independent, then

$$\gamma_t(G) = n_t(G).$$

Proof: Let S be a γ_t-set of G. If $V – S$ is independent, then $G = \bigcup_{u \in s} \langle N[u] \rangle$. Hence S is a n_t-set of G. Thus $n_t(G) \leq \gamma_t(G)$. By Theorem 7, we have $\gamma_t(G) \leq n_t(G)$. Thus $\gamma_t(G) = n_t(G)$.

Remark 9. If G is a tree with $m \geq 2$ cutvertices and every cutvertex is adjacent with at least one endvertex, then

$$n_0(G) = \gamma(G) = \gamma_t(G) = n_t(G).$$

Theorem 10. For any connected graph G with $p \geq 3$ vertices,

$$2 \leq n_t(G) \leq p – 1.$$

Furthermore, both inequalities hold if G is C_3 or P_3.

Theorem 11. Let G be a graph without isolated vertices. Then

$$n_t(G) = p$$

if and only if $G = mK_2$, $m \geq 1$.

Proof: Suppose $n_t(G)=p$. Let $G_1, G_2, \ldots, G_m$ be the components of G. Each G_i, $1 \leq i \leq m$, has $p_i – 1$ vertices and $G_i \neq K_2$. By Theorem 10,

$$n_t(G_i) \leq p_i – 1, \; 1 \leq i \leq m.$$

We have

$$n_t(G) = \sum_{i=1}^{m} n_t(G_i)$$

$$\leq \sum_{i=1}^{m} (p_i - 1)$$

$$\leq p – m$$

which is a contradiction. Thus $G_i = K_2$, $1 \leq i \leq m$ and hence $G = mK_2$, $m \geq 1$.

Conversely, suppose $G = mK_2$, $m \geq 1$. Obviously $n_t(G) = p$.

Theorem 12. Let S be a total neighborhood set of G. If $V – S$ is independent, then

$$\alpha_0(G) \leq n_t(G) \leq 2\alpha_0(G).$$

Proof: Suppose S is a total neighborhood set of G and $V – S$ is independent. Then

$$|V – S| \leq \beta_0(G)$$

or $\quad p – n_t(G) \leq \beta_0(G)$.

Thus $\quad p – \beta_0(G) \leq n_t(G)$.

or $\quad \alpha_0(G) \leq n_t(G)$.

We now prove the upper bound. Suppose D is a minimum vertex cover of G. Since G has no isolated vertices, there exists a vertex v_i for every vertex $u_i \in D$ such that u_iv_i is an edge.

Let $\quad S = D \cup \{v_1, v_2, \ldots v_{\alpha_o}\}$.

Then S is a total neighborhood set of G. Thus

$$n_t(G) \leq |S|$$

or $\quad n_t(G) \leq 2\alpha_0(G)$.

Theorem 13. Let G be an r-regular graph with p vertices and $r \geq 1$. Then

$$n_t(G) \leq p – r + 1$$

and this bound is sharp.

Proof: Let $u\in V$. A set $D=(V–N(u)) \cup \{w\}$ is a total neighborhood set of G where $w \in N(u)$. Thus

$n_t(G) \leq |D| = |(V - N(u)) \cup \{w\}|$

or $n_t(G) \leq p - r + 1$

The complete graphs K_p, $p\geq2$, achieve this bound.

Theorem 14. If T is a tree with $m \geq 2$ cutvertices, then

$$n_t(\overline{T}) = 2.$$

Proof: Let u, v be two endvertices of T. Then u, v are adjacent vertices of degree $p - 1$ in $\overline{T}$. Hence $S = \{u, v\}$ is a total neighborhood set of $\overline{T}$. Thus

$$n_t(\overline{T}) \leq |S| = 2$$

and since $n_t(G) \geq 2$, we see that

$$n_t(\overline{T}) = 2.$$

Theorem 15. If T is a tree with $m \geq 2$ cutvertices, then

$$n_t(T) + n_t(\overline{T}) \leq p.$$

Furthermore, equality holds if and only if T is P_4 or P_5 or P_6.

Proof: If T is a tree with $m\geq2$ cutvertices, then clearly $n_t(T) \leq m$. By Theorem 14, $n_t(\overline{T}) = 2$. Thus

$$n_t(T) + n_t(\overline{T}) \leq m + 2 \leq p.$$

We now prove the second part.

Suppose $n_t(T) + n_t(\overline{T}) = p$. Since $n_t(\overline{T}) = 2$, we have $n_t(T) = p–2$. Assume T has at least 3 endvertices. Then $n_t(T) \leq p - 3$; a contradiction. Thus T has exactly 2 endvertices. Then T is a path.

Suppose $T=P_p$, $p\geq7$. By Proposition 4, $n_t(P_p) = \left\lfloor \frac{2p}{3} \right\rfloor \neq p - 2$, which is a contradiction.

Suppose $T=P_2$ or P_3. Then $\overline{T}$ has isolated vertices and $n_t(\overline{T})$ does not exist.

Hence $G = P_4$ or P_5 or P_6.

Converse is obvious.

Theorem 16. If C_p is a cycle with $p\geq7$ vertices, then

$$n_t(\overline{C}_p) = 3.$$

Proof: By Theorem A, $n_0(\overline{C}_p) = 3$, $p \geq 5$. By Theorem 6, $n_0(\overline{C}_p) \leq n_t(\overline{C}_p)$. Thus $3 \leq n_t(\overline{C}_p)$, $p \geq 7$. Let $u_1, u_2, ..., u_p$ be the vertices of C_p, $p \geq 7$. In $\overline{C}_p$, we see that $\langle N[u_1]\rangle \cup \langle N[u_2]\rangle = \overline{C}_p - \{u_3, u_p\}$. A set $S = \{u_1, u_2, u_r\}$ is a total neighborhood set of $\overline{C}_p$, where $N[u_r]$ contains u_1, u_2, u_3 and u_p.

Thus $n_t(\overline{C}_p) \le 3$. Hence $n_t(\overline{C}_p) = 3$.

Theorem 17. If C_p is a cycle with $p \ge 7$ vertices, then

$$n_t(C_p) + n_t(\overline{C}_p) \le \frac{2p+11}{3}.$$

Proof: By Proposition 3 and Theorem 16, the result follows.

Theorem 18. If both G and $\overline{G}$ are connected graphs with $p \ge 4$ vertices, then

$$n_t(G) + n_t(\overline{G}) \le 2(p-1)$$

$$n_t(G) \,.\, n_t(\overline{G}) \le (p-1)^2.$$

Furthermore, the bounds are attained if and only if $G = P_4$.

Proof: Suppose G and $\overline{G}$ are connected. By Theorem 10, $n_t(G) \le p - 1$ and $n_t(\overline{G}) \le p-1$. Hence both inequalities hold.

The second part is easy to prove, so we omit proof.

3. Total Neighborhood Nomatic Number:

The domatic number $d(G)$ of G is the maximum order of a partition of the vertex set of G into dominating sets. This concept was introduced by Cockayne *et. al.*[1]. The total domatic number $d_t(G)$ of G is the maximum order of a partition of the vertex set of G into total dominating sets. This concept was introduced by Cockayne *et al.*[1]. In this section, we establish some basic results on the total neighborhood nomatic number of a graph.

In[5], Kulli and Patwari introduced the concept of total neighborhood nomatic number as follows:

Let G be a graph without isolated vertices. A total neighborhood nomatic partition of G is a partition $\{v_1, v_2, \ldots, v_k\}$ of V in which each V_i is a total neighborhood set of G. The total neighborhood nomatic number $n_{tn}(G)$ of G is the maximum order of a partition of the vertex set of G into total neighborhood sets.

The total neighborhood nomatic number of some standard graphs are given below.

Proposition 19. If K_p is a complete graph with $p \ge 2$ vertices, then

$$n_{tn}(K_p) = \left\lfloor \frac{p}{2} \right\rfloor.$$

Proposition 20. If C_p is a cycle with $p \ge 3$ vertices, then

$$n_{tn}(C_p) = 1.$$

Proposition 21. If $K_{r,s}$ is a complete bipartite graph with $1 \le r \le s$, then

$$n_{tn}(K_{r,s}) = 1.$$

Proposition 22. If W_p is a wheel with $p \ge 4$ vertices, then

$$n_{tn}(W_p) = 2.$$

It is easy to see the following result.

Proposition 23. If G is a graph without isolated vertices, then

$$n_{tn}(G) \leq \delta(G).$$

Proposition 24. If G is a nontrivial tree, then $n_{tn}(G) = 1$.

Proposition 25. If G is a graph without isolated vertices, then

$$n_t(G)\, n_{tn}(G) \leq p.$$

Proof: Suppose $\{v_1, v_2, \ldots, v_{n_{tn}}\}$ is a total neighborhood nomatic partition of G. Then

$|V_i| \geq n_t(G)$ for each i.

Hence $\frac{p}{n_{tn}(G)} \geq n_t(G)$.

Thus $n_t(G)\, n_{tn}(G) \leq p$.

From Proposition 23 and Proposition 25, we have

Proposition 26. If G is a graph without isolated vertices, then

$$n_{tn}(G) \leq \min\left\{\delta(G),\ p/n_t(G)\right\}.$$

Proposition 27. If G is a graph without isolated vertices, then

$$n_{tn}(G) \leq d_t(G).$$

Proof: By Theorem 7, $\gamma_t(G) \leq n_t(G)$. Hence $n_{tn}(G) \leq d_t(G)$.

Theorem 28. If G is a graph without isolated vertices, then

$$n_t(G) + n_{tn}(G) \leq p + \delta(G).$$

Furthermore, equality holds if and only if $G = mK_2$, $m \geq 1$.

Proof: By Theorem 11 and Proposition 22, the inequality follows:

We now prove the second part.

Suppose $n_t(G) + n_{tn}(G) = p + \delta(G)$. Assume $n_t(G) < p$. Then $n_t(G) > \delta(G)$, which is a contradiction. Thus $n_t(G) = p$ and hence by Theorem 11, $G=mK_2$, $m \geq 1$.

Converse is obvious.

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