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Fuzzy Dynamic System Approach to Multistage Decision Making Problems

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Abstract

Many decisions involve multiple stages of choices and events. In this paper to develops fuzzy dynamic system approach for solving multistage decision-making problems. Fuzzy dynamic system is a promising tool for dealing with multistage decision-making and optimization problems under fuzziness. The cases of deterministic, stochastic, fuzzy dynamic system, fuzzy criterion set dynamic programming, fuzzy termination time's are analyzed. Lastly, summarize the main results of the computational complexity.

Key words: Multistage decision process, decision under fuzziness, multistage optimization problem, fuzzy dynamic system, dynamic programming.

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1. Introduction

Decision making problems usually arise when decision are made in the sequential manner over time, where earlier decisions may affect the feasibility and performance of later decisions. The multistage decision making or multistage control problems may be summarized and purposes of our discussion. The problem is dynamic in that of its characteristics, exemplified by state evolve in time as a result of their present values and some decisions made. There are some stages, exemplified by time instants, discrete or continuous decisions are made[2]. A decision made at a given stage, and at a given state, induces a change in the output of the process. The performance of the process is measured over some planning horizon, and is expressed by an aggregate of partial scores that express the performance of the particular stage decisions. An optimal sequence of decisions or controls at the consecutive stages over the planning horizon is then sought. Dynamic programming is a powerful formal tool for dealing with a large spectrum of multistage decision making and control problems. In mid

1950's [2], dynamic programming has become a standard tool in many fields including operation research, control theory and engineering, engineering, computer science, *etc*. Dynamic programming has been one of the earliest general techniques to which fuzzy sets theory has been applied[5,8].

The multistage decision making process is finite if there are only a finite number of stages in the process and a finite number of states associated with each stage. Many multistage decision process have returns associated with each decision, and these returns may vary with both the stage and state of the process. The multistage decision process is deterministic if the outcome of each decision is known exactly[10].

The mathematical program:

$$\left.\begin{array}{l} \text{optimize } z=\sum_{i=1}^{n} f_i(x_i) \\ \text{subject to } \sum_{i=1}^{n} x_i \le b;\ \mathrm{b}\ge 0,\ x_i\ge 0 \text{ and } i=1,2,\ldots,n \end{array}\right\} \quad (1)$$

2. Principle of optimality in dynamic programming :

Dynamic programming is an approach for optimizing multistage decision making processes. It is based on Bellman's principle of optimality. This principle, begin with the last stage of an n-stage process and determine for each state the best policy for leaving that state and completing the process, assuming that all preceding stages have been completed.

Let u is the state variable, whose values specify the states, $m_j(u)$ is the optimum return from completing the process beginning at stage j in state u, $d_j(u)$ is the decision taken at stage j that achieves $m_j(u)$. The entries corresponding to the last stage of the process, $m_n(u)$ and $d_n(u)$ are generally straightforward to compute. The remaining entries are obtained recursively; *i.e.* the entries for j th stage $(j=1,2,\ldots,n-1)$ are determined as functions of the entries for the (j + 1) th stage[19,20].

The dynamic programming approach is well specified processes modeled by Eq.(1) in which each decision pays off separately, independent of other decisions. For Eq.(1), the value of $m_n(u)$ for u = 0,1,....,b are given by,

$$m_n(u)=\underset{0\le x\le u}{optimum}\left\{f_j(x)\right\} \qquad (2)$$

The recursion formula is

$$m_j(u)=\underset{0\le x\le u}{optimum}\left\{f_j(x)+m_{j+1}(u-x)\right\} \qquad (3)$$

for j = n-1, n-2,....1. In Eq.(2), the decision variable x (which is denoted x_n in Eq.(1)) runs through integral values, as x $(\equiv x_j)$ in Eq.(3). That value of x which yields the optimum in Eq.(2) is taken as $d_n(u)$ and that value of x which yields the optimum in Eq.(3) is taken as $d_j(u)$. If more than one values of x yields either optimum, arbitrarily choose one as the optimal decision. The optimal solution to program Eq.(1) is $z^*=m_1(b)$ the optimal return from completing the process beginning at state1 with b units available for allocation, with z^* determined, the optimal decisions $x_1^*, x_2^*,\ldots,x_n^*$ are found sequentially from

$$x_1^* = d_1(b), x_2^* = d_2(b - x_1^*), x_3^* = d_3(b - x_1^* - x_2^*), \dots, \quad x_n^* = d_n(b - x_1^* - \dots - x_{n-1}^*) \quad (4)$$

3. Decision making under fuzziness :

The basic elements[3] of general approach are: a fuzzy goal G in X, a fuzzy constraint C in X, and a fuzzy decision D in X, X is a (non-fuzzy) space of options. The fuzzy relation

$$\mu_D(x) = \mu_C(x) \wedge \mu_G(x), \quad \forall x \in X \quad (5)$$

which expresses the "goodness" of an $x \in X$ as a solution to the decision making problem. Considered, for definitely good (perfect) to for definitely bad (unacceptable), through all intermediate values. The "$a \wedge b = \min(a,b)$" operations is commonly used and is assumed throughout this paper though many other operations are also employed[14,16].

For an optimal (non-fuzzy) solution, an $x \in X$ such that

$$\mu_D(x^*) = \sup_{x \in X} \mu_D(x) = \sup_{x \in X} (\mu_C(x) \wedge \mu_G(x)) \quad (6)$$

The multiple fuzzy constraints and fuzzy goals are defined in different spaces. Suppose that the fuzzy constraint C is defined as a fuzzy set in $X=\{x\}$ the fuzzy goal G is defined as a fuzzy set in $Y=\{y\}$ and a function $f: X \to Y$, $y = f(y)$ is known X and Y may be sets of decisions and their outcomes, respectively.

Now, the induced fuzzy goal, both G' and C are defined as

$$\mu_{G'}(x) = \mu_G[f(x)] \text{ for each } x \in X \quad (7)$$

and the (min-type) fuzzy decision is

$$\mu_D(x) = \mu_{G'}(x) \wedge \mu_C(x) = \mu_G[f(x)] \wedge \mu_C(x), \text{ for each } x \in X \quad (8)$$

The n fuzzy constraints defined in X, $C_1, C_2, \dots, C_n$, m fuzzy goals defined in Y, $C_1, C_2, \dots, C_m$, and a function $y=f(x)$, then the (min-type) fuzzy decision is

$$\mu_D(x) = \mu_{G_1}[f(x)] \wedge \dots \wedge \mu_{G_n}[f(x)] \wedge \mu_{C_1}(x) \wedge \dots \wedge \mu_{C_m}(x), \text{ for each } x \in X \quad (9)$$

Now we proceed the multistage decision making case within the above general framework[6,24].

4. Multistage decision making (control) under fuzziness :

We assume, the dynamics of the deterministic dynamic system under control is described by a state transition equation

$$x_{t+1} = f(x_t, u_t), \quad t = 0,1,\dots \quad (10)$$

where $x_t, x_{t+1} \in X = \{x\} = \{s_1, s_2, \dots, s_n\}$ are the states at time t and $t+1$, respectively and $u \in U = \{u\} = \{c_1, c_2, \dots, c_m\}$ is the decision (control or input) at t; X and U are assumed finite.

The state transitions are given Eq.(10), the consecutive decision u_t are subjected to fuzzy constraints C^t and on the states x_{t+1} fuzzy goals G^{t+1} are imposed, $t = 0,1,\dots,N-1$. The multistage decision making process is evaluated by the fuzzy decision which is assumed a decomposable fuzzy set in $U \times X \times \dots \times U \times X$, is

$$\mu_D(u_0, u_1, \dots, u_{N-1} | x_0) = \bigwedge_{t=0}^{N-1} [\mu_{C^t}(u_t) \wedge \mu_{G^{t+1}}(x_{t+1})] \quad (11)$$

where N is termination time. The basic case, the optimal sequence of decisions $u_0^*, u_1^*, \ldots, u_{N-1}^*$, such that

$$\mu_D\left(u_0^*, u_1^*, \ldots, u_{N-1}^* \middle| x_0\right)$$

$$= \max_{u_0, u_1, \ldots, u_{N-1}} \bigwedge_{t=0}^{N-1} \left[\mu_{C^t}(u_t) \wedge \mu_{G^{t+1}}(x_{t+1})\right] \quad (12)$$

For simplicity, a fuzzy goal, $\mu_{G^N}(x_N)$ is only imposed on the final state x_N. Then the fuzzy decision is

$$\mu_D(u_0, u_1, \ldots, u_{N-1} | x_0) = \{\mu_{C^0}(u_0) \wedge \mu_{G^1}(x_1)\}$$
$$\wedge \ldots \wedge \{\mu_{C^{N-1}}(u_{N-1}) \wedge \mu_{G^N}(x_N)\} \quad (13)$$

and the problem is to find $u_0^*, u_1^*, \ldots, u_{N-1}^*$, such that

$$\mu_D\left(u_0^*, u_1^*, \ldots, u_{N-1}^* \middle| x_0\right) = \max_{u_0, u_1, \ldots, u_{N-1}} \left[\mu_{C^0}(u_0)\right.$$
$$\left. \wedge\, \mu_{G^1}(x_1) \wedge \ldots \wedge \mu_{C^{N-1}}(u_{N-1}) \wedge \mu_{G^N}(x_N)\right] \quad (14)$$

This general problem formulation may be extended, mainly with respect[14,16]:

- ▶ The type of termination time: fixed and specified in advance, fuzzy, implicity given infinite.
- ▶ The type of the dynamic system: deterministic, stochastic, fuzzy and fuzzy stochastic.
- ▶ The type of objective function: cost minimization, profit/benefit maximization and fuzzy criterion set based satisfactory degree maximization, and virtually all cases a dynamic programming type algorithm can be devised.

5. Fuzzy dynamic system for the case of a fixed and specified termination time :

The case of a fixed and specified (in advance) termination time is basic, and provides a suitable point of departure for extensions. Here we discussed a deterministic, stochastic, fuzzy dynamic system and fuzzy criterion set dynamic program[4,22].

5.1 The case of a deterministic dynamic system :

A deterministic system is described by its state transition Eq.(10), *i.e.* $x_{t+1} = f(x_t, u_t)$; $x_t, x_{t+1} \in X = \{s_1, s_2, \ldots, s_n\}$; $u \in U = \{c_1, c_2, \ldots, c_m\}$; $t = 0,1,\ldots, N-1$; $x_0 \in X$ is the initial state and $N < \infty$ is a fixed and specified termination time. The fuzzy constraints are $\mu_{C^0}(u_0), \ldots, \mu_{C^{N-1}}(u_{N-1})$ and the fuzzy goal is $\mu_{G^N}(x_N)$. The fuzzy decision is

$$\mu_D(u_0, u_1, \ldots, u_{N-1} | x_0) = \mu_{C^0}(u_0) \wedge \ldots \wedge$$
$$\mu_{C^{N-1}}(u_{N-1}) \wedge \mu_{G^N}(x_N) \quad (15)$$

and to find $u_0^*, u_1^*, \ldots, u_{N-1}^*$, such that

$$\mu_D\left(u_0^*, u_1^*, \ldots, u_{N-1}^* \middle| x_0\right) = \max_{u_0, u_1, \ldots, u_{N-1}} \left[\mu_{C^0}(u_0)\right.$$
$$\left. \wedge \ldots \wedge \mu_{C^{N-1}}(u_{N-1}) \wedge \mu_{G^N}(x_N)\right] \quad (16)$$

Clearly, the last two terms are $\mu_{C^{N-1}}(u_{N-1}) \wedge \mu_{G^N}\left(f(x_{N-1}, u_{N-1})\right)$ depend on u_{N-1}, hence Eq.(16) can be rewritten as

$$\mu_D\left(u_0^*, u_1^*, \ldots, u_{N-1}^* \middle| x_0\right) = \max_{u_0, u_1, \ldots, u_{N-1}} \left[\mu_{C^0}(u_0)\right.$$

$$\wedge \ldots \wedge \mu_{C^{N-1}}(u_{N-1}) \wedge \mu_{G^N}(x_N)\Big]$$

$$= \max_{u_0, u_1, \ldots, u_{N-1}} \Big[\mu_{C^0}(u_0) \wedge \ldots \wedge \mu_{C^{N-2}}(u_{N-2})$$

$$\wedge \max_{u_{N-1}} \big(\mu_{C^{N-1}}(u_{N-1}) \wedge \mu_{G^N}(f(x_{N-1}, u_{N-1})) \big) \Big] \tag{17}$$

On repeating this backward iteration for $u_{N-2}, u_{N-1}, \ldots, u_0$, we obtain the set of dynamic programming recurrence equations:

$$\left.\begin{aligned} &\mu_{G^{N-i}}(x_{N-i}) = \max_{u_{N-i}} \{ \mu_{C^{N-i}}(u_{N-i}) \wedge \mu_{G^{N-i+1}}(x_{N-i+1}) \} \\ &x_{N-i+1} = f(x_{N-i}, u_{N-i}), \; i = 1, 2, \ldots, N \end{aligned}\right\} \tag{18}$$

where $\mu_{G^{N-i}}(\cdot)$ is a fuzzy goal at $t = N - i$ induced by a fuzzy goal at $t = N - i + 1$.

An optimal sequence of decision sought, $u_0^*, u_1^*, \ldots, u_{N-1}^*$, is given by the successive maximizing values of u_{N-i} in Eq.(18). It is convenient to represent the solution u_t^*, by an optimal policy $a_t^* : X \to U$, such that $u_t^* = a_t^*(x_t)$, $t = 0, \ldots, N-1$, *i.e.* relating an optimal decision to the current state.

5.2 The case of a stochastic dynamic system:

The stochastic system is assumed to be a Markov chain whose temporal evolution is described by a conditional probability $P(x_{t+1} | x_t, u_t)$ such that $x_t, x_{t+1} \in X = \{s_1, s_2, \ldots, s_n\}$, $u_t \in U = \{c_1, c_2, \ldots, c_m\}$; $x_0 \in X$ is the initial state, $t = 0, 1, \ldots, N-1$ and $N < \infty$ is a fixed and specified termination time. Now the following two problem formulations:

- ▶ due[3] to find an optimal sequence of decisions $u_0^*, u_1^*, \ldots, u_{N-1}^*$ to maximize the probability of attainment of the fuzzy goal subject to the fuzzy constraints, *i.e.*

$$\mu_D(u_0^*, u_1^*, \ldots, u_{N-1}^* | x_0) = \max_{u_0, u_1, \ldots, u_{N-1}} \big[\mu_{C^0}(u_0)$$

$$\wedge \ldots \wedge \mu_{C^{N-1}}(u_{N-1}) \wedge \mu_{G^N}(x_N) \big] \tag{19}$$

where the fuzzy goal is viewed to be a fuzzy event in X whose (non-fuzzy) probability[23] is,

$$E\mu_{G^N}(x_N) = \sum_{x_N \in X} P(x_N | x_{N-1}, u_{N-1}) . \mu_{G^N}(x_N) \tag{20}$$

- ▶ due[17] to find an optimal sequence of decisions to maximize the expectation of the fuzzy decisions membership function, *i.e.*

$$\mu_D(u_0^*, u_1^*, \ldots, u_{N-1}^* | x_0) = \max_{u_0, u_1, \ldots, u_{N-1}} E\big[\mu_{C^0}(u_0)$$

$$\wedge \ldots \wedge \mu_{C^{N-1}}(u_{N-1}) \wedge \mu_{G^N}(x_N) \big] \tag{21}$$

These function are clearly not equivalent.

5.2.1 Bellman and Zadeh's approach[3]: Since in Eq.(19) $\mu_{C^{N-1}}(u_{N-1}) \wedge E\mu_{G^N}(f(x_{N-1}, u_{N-1}))$ depend only on , the next two right-most terms depend only on u_{N-1} *etc.*, the structure of Eq.(19) is essentially the same as that of Eq.(17), and the set of fuzzy dynamic programming recurrence equation is

$$\begin{cases} \mu_{G^{N-i}}(x_{N-i}) = \max_{u_{N-i}} \{\mu_{C^{N-i}}(u_{N-i}) \wedge E\mu_{G^{N-i+1}}(x_{N-i+1})\} \\ E\mu_{G^{N-i+1}}(x_{N-i+1}) = \sum_{x_{N-i+1}\in X} P(x_{N-i+1}|x_{N-1}, u_{N-1}).\mu_{G^{N-i+1}}(x_{N-i+1}); i = 1,2,..., N \end{cases} \quad (22)$$

and we consecutively obtain u^*_{N-i} or optimal policies a^*_{N-i}, such that $u^*_{N-i} = a^*_{N-i}(x_{N-i})$.

5.2.2 Kacprzyk and Staniewski's approach[17] : To solve problem Eq.(21), first introduce a sequence of functions $h_i : X \times \underset{j=1}{\overset{i}{X}} U \to [0,1]$ and $g_j : X \times \underset{l=1}{\overset{j+1}{X}} U \to [0,1]$; $i = 0,1.2,..., N$; $j = 1,2,..., N-1$, such that

$$\begin{cases} h_N(x_N, u_0,..., u_{N-1}) = \mu_{C^0}(u_0) \wedge ... \wedge \mu_{C^{N-1}}(u_{N-1}) \wedge \mu_D(u_0,..., u_{N-1}|x_0) \\ ... \quad ... \quad ... \quad ... \quad ... \quad ... \\ g_k(x_k, u_0,..., u_k) = \sum_{i=1}^{n} h_{k+1}(s_i, u_0,..., u_k).P(s_i|x_k, u_k) \\ h_k(x_k, u_0,..., u_{k-1}) = \max_{u_k} g_k(x_k, u_0,..., u_k) \\ ... \quad ... \quad ... \quad ... \quad ... \quad ... \\ h_0(x_0) = \max_{u_0} g_0(x_0, u_0) \end{cases} \quad (23)$$

The consecutive decisions and states are $u_0,...,u_j$ and $x_0,...,x_j$, respectively, then g_j is the expected value of $\mu_D(.|x_0)$ provided that the next decisions are optimal, *i.e.* $u^*_{j+1}, u^*_{j+2},...., u^*_{N-1}$. It can be shown[16,17]], that there exist functions $\omega_k : X \times \underset{j=1}{\overset{i}{X}} U \to U$ such that $h_k(x_k, u_0,..., u_{k-1}) = g_k(x_k, u_0,..., \omega_{k-1}, \omega_k(x_k, u_0,..., u_{k-1}))$. Then, an optimal policy sought, a^*_t, $t = 0,1,2,..., N-1$ is given

$$\begin{cases} a^*_0 = \omega_0(x_0) \\ a^*_1(x_0, x_1) = \omega_1(x_1, a^*_0(x_0)) \\ ... \quad ... \quad ... \quad .... \\ a^*_{N-1}(x_0, x_1,..., x_{N-1}) = \omega_{N-1}(x_{N-1}, b^*_{N-2}(x_0,..., x_{N-2},..., b^*_0(x_0)),...) \end{cases} \quad (24)$$

It is depends not only on the current state but also on the trajectory.

5.2.3 The case of a fuzzy dynamic system: In this case the system is fuzzy and its dynamics is governed by a state transitions equation

$$X_{t+1} = F(X_t, U_t),\ t = 0,1,2,... \quad (25)$$

where X_t, X_{t+1} are fuzzy states at time (stage) t and t + 1, and U_t is a fuzzy decision at t characterized by their membership functions $\mu_{X_t}(x_t)$, $\mu_{X_{t+1}}(x_{t+1})$ and $\mu_{U_t}(u_t)$, respectively. Eq.(25) is equivalent to a conditioned fuzzy[14,16] set $\mu_{X_{t+1}}(x_{t+1}|x_t, u_t)$ by

Baldwin and Pilsworth[1] proposed a dynamic programming scheme. First, for each $t = 0,1,2,..., N-1$ a fuzzy relation $\mu_{R^t}(u_t, x_{t+1}) = \mu_{C^t}(u_t) \wedge \mu_{G^{t+1}}(x_{t+1})$ is constructed. The degree to which U_t and X_{t+1} satisfy C^t and C^{t+1} is

$$\mu_T(u_t, \mu_{R^t}(u_t, x_{t+1}), x_{t+1}) = \max_{u_t}\Big[\big(\mu_{U_t}(u_t) \wedge \mu_{C^t}(u_t)\big) \wedge \max_{x_{t+1}}\big(\mu_{X_{t+1}}(x_{t+1}) \wedge \mu_{G^{t+1}}(x_{t+1})\big)\Big] \quad (26)$$

The fuzzy decision is

$$\mu_D(U_0,...,U_{N-1}|X_0) = \max_{u_0}\begin{bmatrix}\big(\mu_{U_0}(u_0) \wedge \mu_{C^0}(u_0)\big) \wedge ... \wedge \max_{u_{N-1}}\big(\mu_{U_{N-1}}(u_{N-1}) \wedge \mu_{C^{N-1}}(u_{N-1})\big) \\ \wedge \max_{x_N}\big(\mu_{X^N}(x_N) \wedge \mu_{G^N}(x_N)\big)\end{bmatrix} \quad (27)$$

and an optimal sequence of fuzzy decisions $U_0^*,...,U_{N-1}^*$, such that

$$\mu_D(U_0^*,...,U_{N-1}^*|X_0) = \max_{U_0,...,U_{N-1}} \mu_D(U_0,...,U_{N-1}|X_0)$$

$$= \max_{U_0}\max_{u_0}\left[\big(\mu_{U_0}(u_0) \wedge \mu_{C^0}(u_0)\big) \wedge .... \wedge \max_{U_{N-1}}\max_{u_{N-1}}\begin{pmatrix}\mu_{U_{N-1}}(u_{N-1}) \wedge \mu_{C^{N-1}}(u_{N-1}) \wedge \\ \max_{X_N}\max_{x_N}\big(\mu_{X_N}(x_N) \wedge \mu_{G^N}(x_N)\big)\end{pmatrix}\right] \quad (28)$$

Hence the set of dynamic programming recurrence equation is

$$\begin{cases}\mu_{\overline{G}^N}(X_N) = \max_{x_N}\big[\mu_{X_N}(x_N) \wedge \mu_{G^N}(x_N)\big] \\ \mu_{\overline{G}^{N-i}}(X_{N-i}) = \max_{U_{N-i}}\Big[\max_{u_{N-1}}\big(\mu_{U_{N-i}}(u_{N-i}) \wedge \mu_{C^{N-i}}(u_{N-i})\big) \wedge \mu_{\overline{G}^{N-i+1}}(X_{N-i+1})\Big] \\ \mu_{X^{N-i+1}}(x_{N-i+1}) = \max_{x_{N-i}}\Big[\max_{u_{N-1}}\big(\mu_{U_{N-i}}(u_{N-i}) \wedge \mu_{X^{N-i+1}}(u_{N-i+1}|x_{N-i}, u_{N-i})\big)\Big] \wedge \mu_{X_{N-i}}(x_{N-i}) \\ i = 1,2,..., N-1\end{cases} \quad (29)$$

They redefine the problem formulation in terms of the reference fuzzy states and fuzzy decisions to finally make Eq.(29) solvable by them[14,16].

6. Fuzzy dynamic programming for the case of a fuzzy termination time :

Let $R = \{0,1,...,k-1,k,k+1,...,N\}$ be the set of decision making states. At each $t \in R$, a fuzzy constraint $\mu_t(u_t)$, and a fuzzy goal $\mu_{G^v}(x_v)$, $v \in R$ is imposed on the final state. The fuzzy termination time is given by $\mu_T(v)$, $v \in R$, which is a termination time v.

The fuzzy decision is now[11,12]

$$\mu_D(u_0,u_1,...,u_{v-1}|x_0) = \mu_{C^0}(u_0) \wedge ... \wedge \mu_{C^{v-1}}(u_{v-1}) \wedge \mu_T(v).\mu_{G^v}(x_v) \qquad (30)$$

and to find an optimal termination time v^* and an optimal sequence of decisions $u_0^*,u_1^*,...,u_{v-1}^*$ such that

$$\mu_D(u_0^*,u_1^*,...,u_{v^*-1}^*|x_0) = \max_{v,u_0,u_1,...,u_{v-1}} [\mu_{C^0}(u_0) \wedge ... \wedge \mu_{C^{v-1}}(u_{v-1}) \wedge \mu_T(v).\mu_{G^v}(x_v)] \qquad (31)$$

6.1 The case of a deterministic dynamic system :

Eq.(31) was formulated and solved by Kacprzyk[9,11]. Then, Stein[21] presented a computationally more efficient model and solution.

Kacprzyk's approach: In Kacprzyk's[9,12], formulation the set of possible termination times is $\{v \in R : \mu_T(v) > 0\} = \{k,k+1,...,N\} \subseteq R$, hence an optimal sequence of decisions is $u_0^*,u_1^*,...,u_{k-1}^*,u_k^*,...,u_{v-1}^*$. The part $u_{k-1}^*,u_k^*,...,u_{v-1}^*$ is determined by solving

$$\begin{cases} \mu_{G^{v-i}}(x_{v-i},v) = \max_{V_{v-i}} \{\mu_{C^{v-i}}(u_{v-i}) \wedge \mu_{G^{v-i+1}}(x_{v-i+1},v)\} \\ x_{v-i+1} = f(x_{v-i},u_{v-i}),\ v = k,k+1,...,N-1 \end{cases} \qquad (32)$$

where $\mu_{G^v}(x_v,v) = \mu_T(v).\mu_{G^v}(x_v)$, an optimal termination time v^*, found by the maximizing v in

$$\mu_{G^{k-1}}(x_v,v) = \max_v \mu_{G^{k-1}}(x_{k-1},v) \qquad (33)$$

The part $u_0^*,u_1^*,...,u_{k-2}^*$ is then determined by solving

$$\begin{cases} \mu_{G^{k-i+1}}(x_{k-i+1}) = \max_{u_{k-i-1}} \{\mu_{C^{k-i-1}}(u_{k-i-1}) \wedge \mu_{C^{k-i}}(u_{k-i})\} \\ x_{k-i} = f(x_{k-i-1},u_{k-i-1}); i = 1,2,...,k-1 \end{cases} \qquad (34)$$

Stein's approach: Stein[21] presented a computationally more efficient dynamic programming approach. At $t = N-1$, $i \in \{1,2,...,N-1\}$, and attain $\mu_{\bar{G}^{N-i}}(x_{N-i}) = \mu_T(N-i)\mu_{G^{N-i}}(x_{N-i})$ or apply u_{N-i} attain $\mu_{C^{N-i}}(u_{N-i}) \wedge \mu_{G^{N-i+1}}(x_{N-i+1})$.

The set of recurrence equation,

$$\begin{cases} \mu_{G^{N-i}}(x_{N-i}) = \mu_{\bar{G}_{N-i}}(x_{N-i}) \max_{u_{N-i}} \{\mu_{C^{N-i}}(u_{N-i}) \wedge \mu_{G^{N-i+1}}(x_{N-i+1})\} \\ x_{N-i+1} = f(x_{N-i},u_{N-i}); i = 1,2,...,N \end{cases} \qquad (35)$$

and an optimal termination time;

$$\mu_{G^{N-i}}(x_{N-i}) > \max_{u_{N-i}}\{\mu_{C^{N-i}}(u_{N-i}) \wedge \mu_{G^{N-i+1}}(x_{N-i+1})\} \quad (36)$$

6.2 The case of a stochastic dynamic system:

This stochastic dynamic system was first formulated and solved in Kacprzyk[11,12] by combining the section 4.2.1 and 5.1. An optional termination time v^* and an optional sequence of decisions $u_0^*, u_1^*, ..., u_{v^*-1}^*$ such that

$$\mu_D(u_0^*, u_1^*, ..., u_{v^*-1}^* | x_0) = \max_{v, u_0, u_1, ..., u_{v-1}} [\mu_{C^0}(u_0) \wedge ... \wedge \mu_{C^{v-1}}(u_{v-1}) \wedge E\mu_{G^v}(x_v)] \quad (37)$$

where $\mu_{G^v}(x_v) = \mu_T(v).\mu_{G^v}(x_v)$; and $\{v : \mu_T(v) > 0\} = \{k, k+1, ..., N\}$. As in section 4.2.1, we determine v^* and $u_{k-1}^*, u_k^*, ..., u_{v^*-1}^*$ by solving

$$\begin{cases} \mu_{G^{v-i}}(x_{v-i}) = \max_{u_{v-i}}\{\mu_{C^{v-i}}(u_{v-i}) \wedge E\mu_{G^{v-i+1}}(x_{v-i+1})\} \\ E\mu_{G^{v-i+1}}(x_{v-i+1}) = \sum_{x_{v-i+1} \in X} P(x_{v-i+1} | x_{v-1}, u_{v-1}) \times \mu_{G^{v-i+1}}(x_{v-i+1}); i = 1, 2, ..., N \end{cases} \quad (38)$$

and v^* is given by the maximizing v in

$$\mu_{G^{k-1}}(x_{k-1}) = \max_v \mu_{G^{k-1}}(x_{k-1}, v) \quad (39)$$

The remaining part $u_{k-2}^*, u_{k-3}^*, ..., u_0^*$ is obtained by solving

$$\begin{cases} \mu_{G^{k-1-i}}(x_{k-1-i}) = \max_{u_{k-1-i}}\{\mu_{C^{k-1-i}}(u_{k-1-i}) \wedge E\mu_{G^{k-i}}(x_{k-i})\} \\ E\mu_{G^{k-i}}(x_{k-i}) = \sum_{x_{k-i} \in X} P(x_{k-i} | x_{k-1-i}, u_{k-1-i}) \times \mu_{G^{k-1-i}}(x_{k-1-i}); i = 1, 2, ..., k-1 \end{cases} \quad (40)$$

In the later Stein's[21], formulation the problem is solved by the following set of recurrence equations

$$\begin{cases} \mu_{G^{N-i}}(x_{N-i}) = \mu_{\overline{G}_{N-i}}(x_{N-i}) \wedge \max_{u_{N-i}}\{\mu_{C^{N-i}}(u_{N-i}) \wedge E\mu_{G^{N-i+1}}(x_{N-i+1})\} \\ E\mu_{G^{N-i+1}}(x_{N-i+1}) = \sum_{x_{N-i} \in X} P(x_{N-i+1} | x_{N-i}, u_{N-i}) \times \mu_{G^{N-i+1}}(x_{N-i+1}); i = 1, 2, ..., N \end{cases} \quad (41)$$

where v^* occurs when $\mu_{G^{N-i}}(x_{N-i}) > \max_{u_{N-i}}\{\mu_{C^{N-i}}(u_{N-i}) \wedge E\mu_{G^{N-i+1}}(x_{N-i+1})\}$ (42)

6.3 The case of a fuzzy dynamic system:

In this case, fix some (finite and relatively small) number of reference fuzzy states (and possibly decisions), and obtain an auxiliary approximate system whose state transitions are of deterministic system type [14] and [18]. Then, Stein's [21] approach can be employed.

7. Computational complexity :

The development of efficient algorithms for processing various aspects of fuzzy dynamic programs is an important research area in dynamic programming. In this section, we will summarize main results of the computational complexity analysis. Esogbue[16] using two algorithms of Kacprzyk[9] and Stein[21] for the fuzzy termination time discussed.

The basic dynamic programming formulation of the problem proposed by Stein[21] is the same expect for the structure of the recurrence equations that require N only iterations of the optimizing process as opposed to $N(N+1)/2$ required is Kacprzyk's algorithm. The space and the time complexities are of order $O(n)$ and $O(mn)$, respectively. Essentially, the time complexity is of order $O((2m-1)(N-k))$. This is of order $O(k^2)$.

The total storage demand is $n(N-k+1)(N-k+2)/2+2n+n(k-1)$. This is order $O(k)$. It was pointed out, the difference between the earliest and the latest possible termination times is an important factor in the computationally burden of this optimization process.

So, the model proposed by Stein[21] is computationally more efficient, taking $O(k)$ memory spaces in $O(k^2)$ operations, as opposed to $O(k^2)$ memory spaces in $O(k^3)$ operations in Kacprzyk's[9] model. The superiority is exhibited both from the space and time complexity considerations.

8. Conclusion

We have shown that a brief exposition of main aspects of fuzzy dynamic system, including main problem classes and major applications in a variety of fields. For other surveys of fuzzy dynamic system and its applications, we refer the reader[7], a follow up study by [15] and then the fundamental presentation by [13] and [16].

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