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Prime Labeling of Some Product Graphs

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Abstract

Prime labeling originated with Entringer and was introduced by Tout, Dabboucy and Howalla[4]. A Graph G(V,E) is said to have a **prime labeling** if its vertices are labeled with distinct integers 1,2,3,…,|V(G)| such that for each edge xy the labels assigned to x and y are relatively prime. A graph admits a prime labeling is called a prime graph. In this paper, we prove that $(P_n \otimes S_2)^k$, $(P_n \otimes S_3)^k$ and $C_n \otimes S_m$ are prime graphs.

1. Introduction

A simple graph G(V,E) is said to have a prime labeling (or called prime) if its vertices are labeled with distinct integers 1,2,3,…,|V(G)|, such that for each edge $xy \in E(G)$, the labels assigned to x and y are relatively prime[1].

We begin with listing a few definitions\notations that are used.

- A graph G= (V,E) is said to have order |V| and size |E|.
- A vertex $v \in V(G)$ of degree 1 is called pendant vertex.
- P_n is a path of length n-1.
- A star S_n with n-spokes is given by (V,E) where $V(S_n)=\{v_0,v_1,v_2,\ldots,v_n\}$ and $E(S_n)=\{v_0v_i\ /\ i=1,2,\ldots,n\}$. V_0 is called the centre of the star.
- Let G be any graph and S_m be a star with m spokes. We denote by $G \otimes S_m$ the graph obtained from G by identifying one vertex of G with any vertex of S_m other than the centre[3] of S_m.
- A regular bamboo tree is one point union of $(P_n \otimes S_m)^k$ where k is the number of copies[3] of $P_n \otimes S_m$.

2. Main Results

Theorem 2.1 $(P_n \otimes S_2)^k$ is prime.

Proof:

The graph $(P_nS_2)^k$ has kn+1 vertices and kn edges.

$V= \{u, v_i^{(j)}, x_j, y_j / 1 \le i \le n-2, 1 \le j \le k\}$

$E = \{uv_1^{(j)}/1\le j\le k\} \cup \{v_i^{(j)}v_i^{(j+1)}/1\le i\le n\text{-}3, 1\le j\le k\} \cup \{v_{n-2}^{(j)}x_j/1\le j\le k\} \cup \{v_{n-2}^{(j)}y_j/1\le j\le k\}$

Define $f:V\to\{1,2,3,\ldots,kn+1\}$ by $f(u)=1$

Case1: **n is even**

$f(v_i^{(j)})=n(j-1)+i+1, 1\le i\le n-2, 1\le j\le k$

$f(x_j) = f(v_{n-2}^{(j)})+1, 1\le j\le k$

$f(y_j) = f(v_{n-2}^{(j)})+2, 1\le j\le k$

$GCD(f(v_i^{(j)}), f(v_{i+1}^{(j)})) = GCD(n(j-1)+i+1, n(j-1)+i+2)=1, 1\le i\le n-3, 1\le j\le k$

$GCD(f(v_{n-2}^{(j)}), f(x_j))=GCD(n(j-1)+n-1, n(j-1)+n) =1, 1\le j\le k$

$GCD(f(v_{n-2}^{(j)}), f(y_j))=GCD(n(j-1)+n-1, n(j-1)+n+1) =1, 1\le j\le k$

Case 2: **n is odd**

$f(v_i^{(j)}) = n(j-1)+i+1, 1\le i\le n-3, 1\le j\le k, j \equiv 1 \pmod 2$

$f(v_i^{(j)}) = n(j-1)+i+1, 1\le i\le n-2, 1\le j\le k, j \equiv 0 \pmod 2$

$f(v_{n-2}^{(j)}) = f(v_{n-3}^{(j)})+2, 1\le j\le k, j\equiv 1 \pmod 2$

$f(x_j) = f(v_{n-2}^{(j)})-1, 1\le j\le k, j\equiv 1 \pmod 2$

$f(x_j) = f(v_{n-2}^{(j)})+1, 1\le j\le k, j\equiv 0 \pmod 2$

$f(y_j) = f(v_{n-2}^{(j)})+1, 1\le j\le k, j\equiv 1 \pmod 2$

$f(y_j) = f(v_{n-2}^{(j)})+2, 1\le j\le k, j\equiv 0 \pmod 2$

$GCD(f(v_i^{(j)}), f(v_{i+1}^{(j)})) = GCD(n(j-1)+i+1, n(j-1)+i+2) =1, 1\le i\le n-4, 1\le j\le k$

$GCD(f(v_{n-3}^{(j)}), f(v_{n-2}^{(j)})) = GCD(n(j-1)+n-2, n(j-1)+n) =1, 1\le j\le k, j\equiv 1 \pmod 2$

$GCD(f(v_{n-3}^{(j)}), f(v_{n-2}^{(j)})) = GCD(n(j-1)+n-2, n(j-1)+n-1) =1, 1\le j\le k, j\equiv 0 \pmod 2$

$GCD(f(v_{n-2}^{(j)}), f(x_j)) = GCD(n(j-1)+n, n(j-1)+n-1) =1, 1\le j\le k, j\equiv 1 \pmod 2$

$GCD(f(v_{n-2}^{(j)}), f(x_j)) = GCD(n(j-1)+n-1, n(j-1)+n) =1, 1\le j\le k, j\equiv 0 \pmod 2$

$GCD(f(v_{n-2}^{(j)}), f(y_j)) = GCD(n(j-1)+n, n(j-1)+n+1) =1, 1\le j\le k, j\equiv 1 \pmod 2$

$GCD(f(v_{n-2}^{(j)}), f(y_j)) = GCD(n(j-1)+n-1, n(j-1)+n+1) =1, 1\le j\le k, j\equiv 0 \pmod 2$

Therefore, $(P_n\otimes S_2)^k$ admits prime labeling and hence, $(P_n\otimes S_2)^k$ is a prime graph[2].

Example 2.2

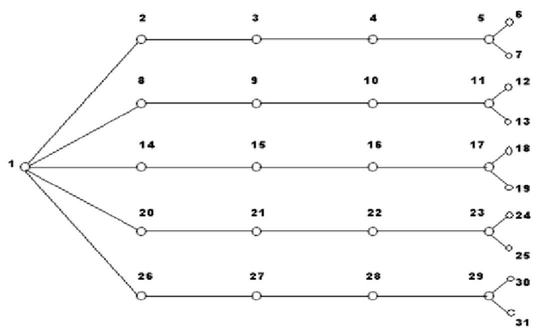

Fig 1. $(P_6\otimes S_2)^5$ – n even

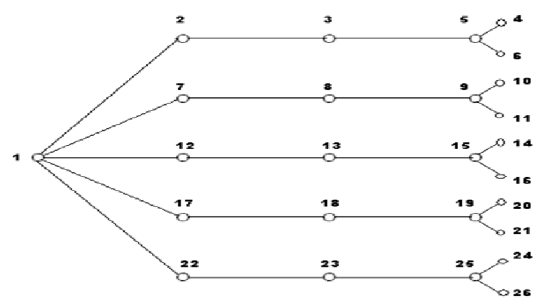

Fig 2. $(P_5\otimes S_2)^5$ – n odd

Theorem 2.3 $(P_n\otimes S_3)^k$ is prime

Proof:

Let u be the initial vertex. The graph has k(n+1)+1 vertices and k(n+1) edges.

$V = \{u, v_i^{(j)}, x_j, y_j, z_j/1\le i\le n-2, 1\le j\le k\}$

$E = \{uv_1^{(j)}/1\le j\le k\} \cup \{v_i^{(j)}v_{i+1}^{(j)}/1\le i\le n-3, 1\le j\le k\} \cup \{v_{n-2}^{(j)}\ x_j/1\le j\le k\} \cup \{v_{n-2}^{(j)}\ y_j/1\le j\le k\} \cup \{v_{n-2}^{(j)}z_j/1\le j\le k\}$

Define $f:V\to\{1,2,3,\ldots,k(n+1)+1\}$ by $f(u)=1$

Case 1 Suppose $n\equiv 1(\mathrm{mod}2)$

$f(v_i^{(j)}) = (n+1)(j-1)+i+1,\ 1\le i\le n-3,\ 1\le j\le k$

$f(v_{n-2}^{(j)}) = f(v_{n-3}^{(j)})+2,\ 1\le j\le k$

$f(x_j) = f(v_{n-2}^{(j)})-1,\ 1\le j\le k$

$f(y_j) = f(v_{n-2}^{(j)})+1,\ 1\le j\le k$

$f(z_j) = f(v_{n-2}^{(j)})+2,\ 1\le j\le k$

$GCD(f(v_i^{(j)}), f(v_{i+1}^{(j)})) = GCD((n+1)(j-1)+i+1, (n+1)(j-1+i+2) = 1, 1\le i\le n-4, 1\le j\le k$

$GCD(f(v_{n-3}^{(j)}), f(v_{n-2}^{(j)})) = GCD((n+1)(j-1)+n-2,(n+1)(j-1)+n) = 1, 1\le j\le k$

$GCD(f(v_{n-2}^{(j)}), f(x_j)) = GCD((n+1)(j-1)+n, (n+1)(j-1)+n-1) = 1, 1\le j\le k$

$GCD(f(v_{n-2}^{(j)}, f(y_j)) = GCD((n+1)(j-1)+n, (n+1)(j-1)+n+1) = 1, 1\le j\le k$

$GCD(f(v_{n-2}^{(j)}), f(z_j) = GCD((n+1)(j-1)+n, (n+1)(j-1)+n+2) = 1, 1\le j\le k$

Case 2 Suppose $n\equiv 0(\mathrm{mod}2)$

Subcase2(i) **$n\equiv 4(\mathrm{mod}6)$**

$f(v_i^{(j)}) = (n+1)(j-1)+i+1,\ 1\le i\le n-3,\ 1\le j\le k$

$f(v_{n-2}^{(j)}) = f(v_{n-3}^{(j)})+2,\ 1\le j\le k,\ j\equiv 0(\mathrm{mod}2)$

$f(x_j) = f(v_{n-2}^{(j)})-1,\ 1\le j\le k,\ j\equiv 0(\mathrm{mod}2)$

$f(y_j) = f(v_{n-2}^{(j)})+1,\ 1\le j\le k,\ j\equiv 0(\mathrm{mod}2)$

$f(z_j) = f(v_{n-2}^{(j)})+2,\ 1\le j\le k,\ j\equiv 0(\mathrm{mod}2)$

$f(v_{n-2}^{(1)}) = f(v_{n-3}^{(1)})+3$

$f(x_1) = f(v_{n-2}^{(1)})-1$

$f(y_1) = f(v_{n-2}^{(1)})-2$

$f(z_1) = f(v_{n-2}^{(1)})+1$

$f(v_{n-2}^{(j)}) = f(v_{n-3}^{(j)})+1,\ 1\le j\le k,\ j\equiv 1(\mathrm{mod}2),\ j\ne 1,\ 1\le j\le k$

$f(x_j) = f(v_{n-2}^{(j)})+1,\ 1\le j\le k,\ j\equiv 1(\mathrm{mod}2),\ j\ne 1$

$f(y_j) = f(v_{n-2}^{(j)})+2,\ 1\le j\le k,\ j\equiv 1(\mathrm{mod}2),\ j\ne 1$

$f(z_j) = f(v_{n-2}^{(j)})+3,\ 1\le j\le k,\ j\equiv 1(\mathrm{mod}2),\ j\ne 1$

$GCD(f(v_{n-3}^{(1)}), f(v_{n-2}^{(1)})) = GCD(n-2, n+1) = 1$

$GCD(f(v_{n-2}^{(1)}, f(y_1)) = GCD(n+1, n-1) = 1$

$GCD(f(v_{n-3}^{(j)}), f(v_{n-2}^{(j)})) = GCD((n+1)(j-1)+n-2,(n+1)(j-1)+n) = 1, 1\le j\le k, j\equiv 0(\mathrm{mod}\ 2)$

$GCD(f(v_{n-2}^{(j)}), f(x_j)) = GCD((n+1)(j-1)+n, (n+1)(j-1)+n-1) = 1, 1\le j\le k, j\equiv 0(\mathrm{mod}\ 2)$

$GCD(f(v_{n-2}^{(j)}, f(y_j)) = GCD((n+1)(j-1)+n, (n+1)(j-1)+n+1) = 1, 1\le j\le k, j\equiv 0(\mathrm{mod}\ 2)$

$GCD(f(v_{n-2}^{(j)}), f(z_j)) = GCD((n+1)(j-1)+n, (n+1)(j-1)+n+2) = 1, 1\le j\le k, j\equiv 0(\mathrm{mod}\ 2)$

$GCD(f(v_{n-3}^{(j)}), f(v_{n-2}^{(j)})) = GCD((n+1)(j-1)+n-2, (n+1)(j-1)+n-1) = 1, 1\le j\le k, j\ne 1, j\equiv 1(\mathrm{mod}\ 2)$

Subcase2(ii) **$n\equiv 0(\mathrm{mod}6)$**

$f(v_i^{(j)}) = (n+1)(j-1)+i+1,$ 1ddiddn-3, $1\le j\le k$

$f(v_{n-2}^{(j)}) = f(v_{n-3}^{(j)})+2,\ 1\le j\le k,\ j\equiv 0,2,4(\mathrm{mod}6)$

$f(x_j) = f(v_{n-2}^{(j)})-1,\ 1\le j\le k,\ j\equiv 0,2,4(\mathrm{mod}6)$

$f(y_j) = f(v_{n-2}^{(j)})+1,\ 1\le j\le k,\ j\equiv 0,2,4(\mathrm{mod}6)$

$f(z_j) = f(v_{n-2}^{(j)})+2,\ 1\le j\le k,\ j\equiv 0,2,4(\mathrm{mod}6)$

$f(v_{n-2}^{(j)}) = f(v_{n-3}^{(j)})+1,\ 1\le j\le k,\ j\equiv 1,3(\mathrm{mod}6)$

$f(x_j) = f(v_{n-2}^{(j)})+1,\ 1\le j\le k,\ j\equiv 1,3(\mathrm{mod}6)$

$f(y_j) = f(v_{n-2}^{(j)})+2,\ 1\le j\le k,\ j1,3(\mathrm{mod}6)$

$f(z_j) = f(v_{n-2}^{(j)})+3,\ 1\le j\le k,\ j\equiv 1,3(\mathrm{mod}6)$

$f(v_{n-2}^{(j)}) = f(v_{n-3}^{(j)})+3,\ 1\le j\le k,\ j\equiv 5(\mathrm{mod}6)$

$f(x_j) = f(v_{n-2}^{(j)})-1,\ 1\le j\le k,\ j\equiv 5(\mathrm{mod}6)$

$f(y_j) = f(v_{n-2}^{(j)})-2,\ 1\le j\le k,\ j\equiv 5(\mathrm{mod}6)$

$f(z_j) = f(v_{n-2}^{(j)})+1,\ 1\le j\le k,\ j\equiv 5(\mathrm{mod}6)$

GCD $(f(v_{n-2}^{(j)}), f(x_j)) = $ GCD $((n+1)(j-1) + n-1, (n+1)(j-1) + n) = 1, 1 \leq j \leq k, j \equiv 1, 3 \pmod 6$

GCD $(f(v_{n-2}^{(j)}, f(y_j)) = $ GCD $((n+1)(j-1)+n-1, (n+1)(j-1)+n+1) = 1, 1 \leq j \leq k, j \equiv 1, 3 \pmod 6$

GCD $(f(v_{n-2}^{(j)}), f(z_j)) = $ GCD $((n+1)(j-1)+n-1, (n+1)(j-1)+n+2) = 1, 1 \leq j \leq k, j \equiv 1, 3 \pmod 6$

GCD $(f(v_{n-3}^{(j)}), f(v_{n-2}^{(j)})) = $ GCD $((n+1)(j-1)+n-2, (n+1)(j-1)+n+1) = 1, 1 \leq j \leq k, j \equiv 5 \pmod 6$

GCD $(f(v_{n-2}^{(j)}), f(x_j)) = $ GCD $((n+1)(j-1)+n+1, (n+1)(j-1)+n) = 1, 1 \leq j \leq k, j \equiv 5 \pmod 6$

GCD $(f(v_{n-2}^{(j)}, f(y_j)) = $ GCD $((n+1)(j-1)+n+1, (n+1)(j-1)+n-1) = 1, 1 \leq j \leq k, j \equiv 5 \pmod 6$

GCD $(f(v_{n-2}^{(j)}), f(z_j)) = $ GCD $((n+1)(j-1)+n+1, (n+1)(j-1)+n+2) = 1, 1 \leq j \leq k, j \equiv 5 \pmod 6$

Subcase 2(iii) **n≡2(mod6)**

$f(v_i^{(j)}) = (n+1)(j-1)+i+1, 1 \leq i \leq n-3, 1 \leq j \leq k$

$f(v_{n-2}^{(j)}) = f(v_{n-3}^{(j)})+1, 1 \leq j \leq k, j \equiv 1 \pmod 2$

$f(x_j) = f(v_{n-2}^{(j)})+1, 1 \leq j \leq k, j \equiv 1 \pmod 2$

$f(y_j) = f(v_{n-2}^{(j)})+2, 1 \leq j \leq k, j \equiv 1 \pmod 2$

$f(z_j) = f(v_{n-2}^{(j)})+3, 1 \leq j \leq k, j \equiv 1 \pmod 2$

$f(v_{n-2}^{(j)}) = f(v_{n-3}^{(j)})+2, 1 \leq j \leq k, j \equiv 0 \pmod 2$

$f(x_j) = f(v_{n-2}^{(j)})-1, 1 \leq j \leq k, j \equiv 0 \pmod 2$

$f(y_j) = f(v_{n-2}^{(j)})+1, 1 \leq j \leq k, j \equiv 0 \pmod 2$

$f(z_j) = f(v_{n-2}^{(j)})+2, 1 \leq j \leq k, j \equiv 0 \pmod 2$

In this case also for every edge $uv \in E(G)$, it can be verified that GCD $(f(u), f(v))=1$.

Therefore, $(P_n \otimes S_3)^k$ admits prime[4] labeling and hence, $(P_n \otimes S_3)^k$ is a prime graph.

Example 2.4

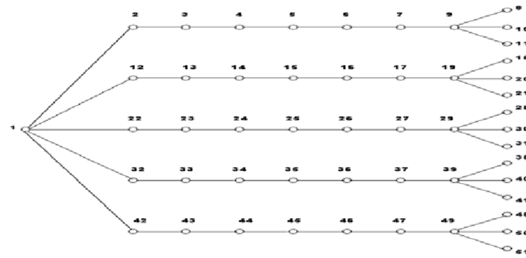

Fig 3. $(P_9 \otimes S_3)^5$ n odd

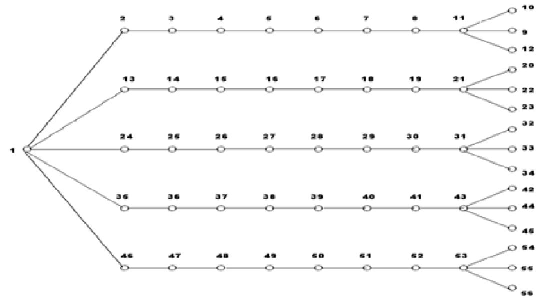

Fig 4. $(P_{10} \otimes S_3)^5$ $n \equiv 4 \pmod 6$ n even

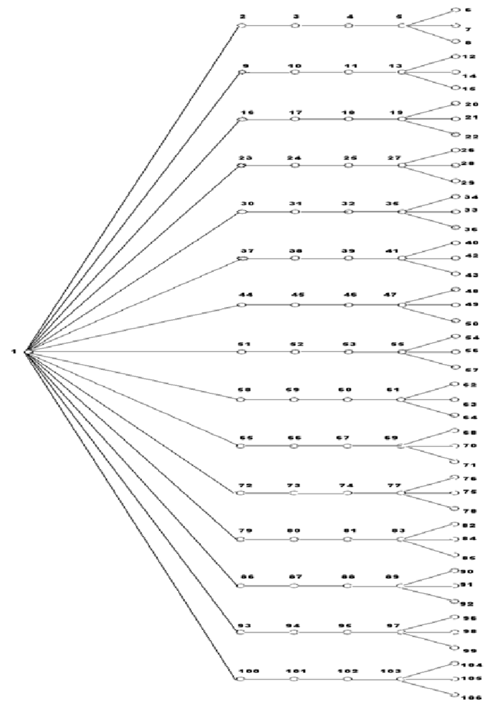

Fig 5. $(P_6 \otimes S_3)^{15}$ $n \equiv 0 \pmod 6$

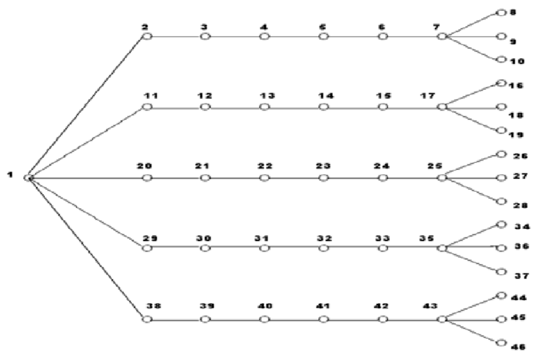

Fig 6. $(P_8 \otimes S_3)^5$ $n \equiv 2 \pmod 6$

Theorem 2.5 $C_n \otimes S_m$ is prime

Proof:

The graph $C_n \otimes S_m$ has n+m vertices and n+m edges. Let $u_1, u_2, \ldots, u_n$ be the vertices of the cycle c_n and $v_1, v_2, \ldots, v_m$ be the vertices of the star and v_1 be the central vertex.

$V = \{u_i\ v_j / 1 \le i \le n,\ 1 \le j \le m\}$

$E = \{u_i u_{i+1} / 1 \le i \le n-1\} \cup \{u_n u_1\} \cup \{u_1 v_1\} \cup \{v_1 v_i / 2 \le i \le m\}$

Case (i) n is even

Define f: $V \to \{1, 2, \ldots, n+m\}$ by

$f(v_1) = 1$, $f(u_1) = 2$, $f(u_i) = i+1$, $2 \le i \le n$

$f(v_i) = n+i-1$, $2 \le i \le m$

In this case also for every edge $uv \in E(G)$, it can be verified that GCD(f(u), f(v))=1.

Case (ii) $n \ne 3, 5$ and n is odd

Define f: $V \to \{1, 2, \ldots, n+m\}$ b y

$f(u_1) = 2$, $f(v_1) = 1$, $f(v_2) = n+1$, $f(u_n) = n+2$

$f(u_i) = i+1$, $2 \le i \le n-1$, $f(v_i) = n+i$, $3 \le i \le m$

In this case also for every edge $uv \in E(G)$, it can be verified that GCD(f(u),f(v))=1.

Case (iii) n = 3, 5

Define f: $V \to \{1, 2, \ldots, n+m\}$

$f(v_1) = 1$, $f(v_2) = 2$, $f(u_i) = i+2$, $1 \le i \le n$

$f(v_i) = n+i+2$, $3 \le i \le m$

In this case also for every edge $uv \in E(G)$, it can be verified that GCD(f(u), f(v))=1.

Therefore, $C_n \otimes S_m$ admits prime labeling and hence, $C_n \otimes S_m$ is a prime graph.

Example 2.6

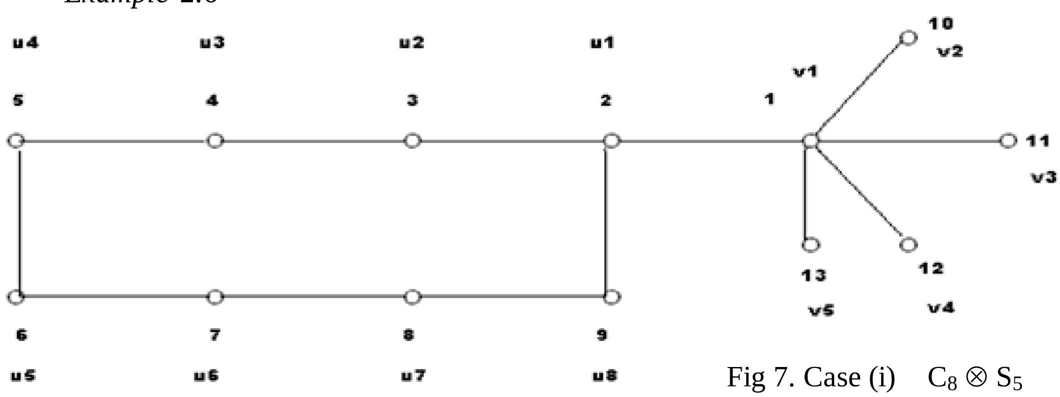

Fig 7. Case (i) $C_8 \otimes S_5$

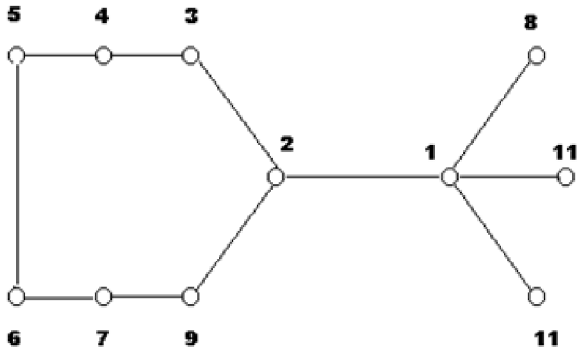

Fig 8. case(ii) $C_7 \otimes S_4$

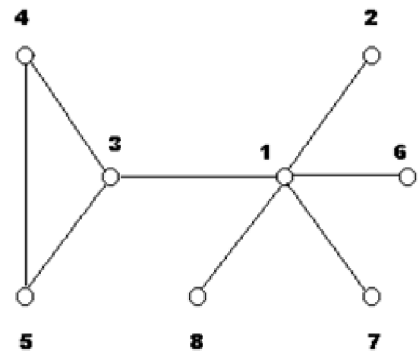

Fig 9. case(iii) $C_3 \otimes S_5$

References

1. J.A. Gallian, A Dynamic survey of graph labeling, The Electronic Journal of Combinatorics, 18(2011), #DS6.
2. S.M. Lee, I. Wui, J.yeh, On amalgamation of prime graphs, Bull. Malasian Math. Soc. (Second Series), *11*, 59-67 (1988).
3. C. Sekar, Studies in graph theory, Ph.D. thesis, Madurai kamaraj university (2002).
4. A.Tout, A.N. Dabboucy, K. Howalla, Prime labeling of graphs, Nat. Acad. Sci. Letters, *11*, 365-368 (1982).