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Product form in networks of queues with negative customers

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Abstract

In this paper, we introduce a new classes of queueing networks with "negative customers". A negative customer arriving to a queue reduces the total customers count in that queue by one negative customer donot receive service applications for such negative customer queueing networks include certain transaction models and certain neural network models. Even though the underlying equations of the models to be described can be nolinear they do have a product form solution for the equilibrium state probabilities.

Key words : Queue with negative customers, negative arrivals, product form solution, networks of queues.

1.1 Introduction

An intriguing generalization of the usual product form model involving the concept of "negative" Customers has been first developed by E. Gelenbe and co-authors in a series of papers[3,4,5,6]. In this new queueing paradiagm. There are two types of customers. Positive customers and negative customers. A positive customer acts as the normal sort of customers we have seen so far. A negative customer, however, when it enters a queue will instantly destroy one positive customer and itself. Put another way a negative and positive customer in a queue instantly cancel each other and disappear from the network.

1.2 Queueing networks with negative customers :

Let us assume that positive customers arrive at the ith queue according to a poisson process with rate Λ_i and negative customers arrive at the ith queue according to a poisson

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process of rate λ_i. A positive customer leaving queue i will enter queue j as a positive customer with probability r_{ij}^+, or will enter queue j as a negative customer with probability r_{ij}^-. A positive customer arriving at a queue will decrease, the queue length by one if there is at least one positive customer present in the queue. A negative customer arriving at any empty queue is cleared from the network. Let us also assume that all service times are independently distributed exponential random variables. The service rate of the ith queue is μ_i. Finally the probability that a customer leaving queue i leaves the network is d_i.

Naturally the sum of all the queue departure probabilities should equal one.

$$\sum_j r_{ij}^+ + \sum_j r_{ij}^- + d_i = 1, \qquad 1 \le i \le M \quad (1.1)$$

It should be clear that this model only makes sense for open networks.

1.3 Product form solutions :

In this paper, the product form solution for queueing networks with negative customers will be presented. Let us state the main result. First the effective utilization is

$$q_i = \frac{\lambda_i^+}{\mu_i + \lambda_i^-} \quad (1.2)$$

Here λ_i^+ and λ_i^- are the solutions of the following non linear system of simultaneous traffic equations.

$$\lambda_i^+ = \sum_j q_j \mu_j r_{ij}^+ + \Lambda_i \quad (1.3)$$

$$\lambda_i^- = \sum_j q_j \mu_j r_{ji}^- + \lambda_i \quad (1.4)$$

Here the variable λ_i^+ is the total mean arrival rate of positive customers to the ith queue. The variable λ_i^- is the total mean arrival rate of negative customers to the ith queue. It can be seen that each equation has two terms. The first corresponding to arrivals into the ith queue from the other queues and the second corresponding arrivals into the ith queue from outside of the network. These equations are intrinsically non linear as λ_i^- appears in the denominator of q_j. It should be noted that it is natural for the denominator of q_i to consist of $\mu_i + \lambda_i^-$ as negative arrivals to a queue effectively increase its service rate. In simple cases it is possible to obtain closed form analytical solutions for these traffic equations. In more complex cases the equations can only be solved iteratively through iteration. That is, one starts with a guess as to the solution and substitutes it into the equations to produce an improved estimate of the solution. After enough successive substitutions, the estimates will converge to the solution. This is a commonly used technique to solve simultaneous sets of non linear equations[7-9].

Theorem : If (1.3) and (1.4) have a unique nonnegative solution and $q_i < 1$ for each i = 1,2, … M. Then the equilibrium state probabilities are given by

$$p(\underline{n}) = \prod_{i=1}^{M} [1 - q_i] q_i^n \quad (1.5)$$

This solution can be seen to be analogous to the product form solution for queues with only positive customers. This is a surprising result as systems of simultaneous non linear equations usually do not have a closed form analytic solution. Often one can only solve for special cases of such systems[10-11].

To show that the theorm is indeed correct the solution of (1.5) can be shown to satisfy the global balance equations of the states. These equations equate the net flow out of state $\underline{n}$ with the net flow into the same state. The following vectors will be used.

$$\begin{aligned} \underline{n}_i^{+} &= (n_1 \ldots\ldots\ldots n_i + 1 \ldots\ldots\ldots n_M), \\ \underline{n}_i^{+} &= (n_1 \ldots\ldots\ldots n_i - 1 \ldots\ldots\ldots n_M), \\ \underline{n}_{ij}^{\pm} &= (n_1 \ldots\ldots\ldots n_i + 1 \ldots\ldots\ldots n_j - 1 \ldots\ldots\ldots n_M), \\ \underline{n}_{ij}^{++} &= (n_1 \ldots\ldots\ldots n_i + 1 \ldots\ldots\ldots n_j + 1 \ldots\ldots\ldots n_M), \end{aligned} \tag{1.6}$$

The first two vectors correspond to the external arrival and external departure from a queue, with respect to p($\underline{n}$), respectively. The third vector, in association with p($\underline{n}$), corresponds to the transfer of a positive customer from one queue to another. The last vector, in association with p($\underline{n}$), corresponds to a negative customer leaving one queue for another with at least one customer[5-8].

The global balance equation for a state in the interior of the state transition diagram, in terms of this notation, is

$$\begin{aligned} & p(\underline{n})\sum_i \ [\Lambda_i + (\lambda_i + \mu_i)1[n_i > 0]] \qquad (1.7) \\ & = \sum_i \ \left[p(\underline{n}_i^{+})\mu_i d_i + p(\underline{n}_i^{-})\Lambda_i 1[n_i > 0]\right. \\ & + p(\underline{n}_i^{+})\lambda_i \\ & + \sum_j \ (p(\underline{n}_{ij}^{\pm})\mu_i r_{ij}^{+} 1[n_j > 0] + p(\underline{n}_{ij}^{++})\mu_i r_{ij}^{-} \\ & \left. + p(\underline{n}_i^{+})\mu_i r_{ij}^{-} 1[n_j = 0])\right] \end{aligned}$$

Here 1[x] is an indicator function that is equal to one if the argument x is greater than 0 and equal to zero otherwise.

By substituting (1.5) and then (1.2 – 1.4) into (1.7) one can show, after several steps, that the product form solution satisfies the global balance equations and thus the theorem is true.

1.4 Conclusions

We have presented the application queueing network models with negative customers. We have discussed that this model only makes sense for open networks. It is impossible to get steady – state results for an analogous closed network with negative customers as the queue lengths go to zero in a finite amount of time. However, it is not so obvious that such a continuous time open model has a product form solution as the underlying equations are non linear. In fact, though, as we have seen, it does indeed have a product form solution for the equilibrium state probabilities.

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