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Combinatorial aspects of the generalized unitaroy Eulers's totient

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Abstract

A Generalized unitary Euler's totient is defined as a Dirichlet convolution of a power function & a product of the Souriau –Hsu-Mobius function with a completely multiplicative – function. Two combinatorial aspects of the generalized unitary Euler's totient, namely its connect totients and its relations with counting formulas are investigated.

Introduction

Let A be the Unique Factorization of arithmetic functions[1, 2] equipped with addition and (Direchlet) convolution defined, respectively by

$$(f+g)(n)=f(n)+g(n);\quad (f*g)(n)=\sum_{d//n} f(d)\,g\left({}^{n}/_{d}\right) \tag{1.1}$$

The convolution identity $I \in A$ is defined by

$$I = \begin{cases} 1; if\ \ n = 1 \\ 0; if\ \ n > 1 \end{cases} \tag{1.2}$$

For $f \in A$ write f^{-1} for convolution inverse whenever it exists. A non zero arithmetic function f is said be multiplicative if f(m n) = f(m) .f(n) whenever gcd (m n) =1, and is called completely multiplicative if this equality holds for all m, n $\in$ N. For $\alpha \in c$, the Souriou – Hsu- Mobius (SHM) function is defined by

$$\mu^*{}_{\alpha}{}^{(n)} = \prod_{p//n} \binom{\alpha}{v_p(n)} (-1)^{v_{p(n)}} \tag{1.3}$$

Where $n = \prod p^{v(n)}$ denotes the Unique prime factorization of n $\in$ N, $v_{p(n)}$ being the largest exponent of the prime P that unitary divides n. This function generalizes that usual Mobius function μ^*, because $\mu^*{}_1=\mu^*$ note that

$\mu^*{}_0$=I; $\mu^*{}_{-1}=\mu^*$ the arithmetic unit function defined by

$$\mu^*(n)=1(n \in N) \tag{1.4}$$

and for α, $\beta \in c$, we have

$$\mu^*_{\alpha+\beta} = \mu^*_\alpha + \mu^*_\beta \qquad (1.5)$$

It easily checked that μ^*_α is multiplicative, there are exactly two SHM functions that are completely multiplicative, namely $\mu^*_0 = I$ & $\mu^*_{-1} = \mu^*$ and there is exactly one SHM function whose convolution inverse is completely multiplicative, namely $\mu^*_1 = \mu^{*-1}$ for general reference on the Mobius function and its generalizations.

The classical Unitary Euler's totient $\varphi^*(n)$ is defined as the number of positive integers $a \le n$

Such that gcd (a, n) = 1, it well known that

$$\varphi^*(n) = \sum_{d//n} d\mu\left(\frac{n}{d}\right) = n\prod_{p//n}\left(1-\frac{1}{p}\right) \quad (1.6)$$

For a general reference about Unitary Euler's totient, its many facts and generalizations of Unitary Euler's totient has been a good deal of generalizations of interest to us here is the one due to Wang & HSU defined for k,r $\in$ N and completely multiplicative $f \in A$ by

$$\varphi_\tau^{*(k)}(n) = \sum_{d//n}\left(\frac{n}{d}\right)^n f(d)\mu_r^*(d) \qquad (1.7)$$

Where $\tau = \mu_r^* f$. It is shown that $\varphi^{*(k)}_\tau$ possesses properties extending those of the classical unitary Euler's totient, such as the following

(p_1) $\varphi^{*(k)}_\tau(\mathrm{n}) = n^k \prod_{p//n}\left(1-\frac{f(p)}{p^k}\right)^r$ where n is r-powerful, that is $v_p(n) \ge r$ for each Prime Factor p of n.

(p_2) Let $\vec{a} = (a_1, a_2, a_3, \text{---}\ a_k) \in Z^k$. Then for prime p, there uniquely exists an r x k matrix

$B_p(\vec{a})$ Over $(Z_p)=\{0,1,2,3 \ldots p-1\}$ such that $\vec{a} \equiv (1,p,p^2 \ldots p^{r-1})\ B_p(\vec{a})\ (\mathrm{mod}\ p^r)$. Let A_p be a subset of Z^k_p. Then there uniquely exists a completely multiplicative $f \in A$ with f(p) being defiend by the number of vectors in A_p for $\vec{a} \in n$, we write $(\vec{a}, n)=1$; if no row of $B_p(\vec{a})$ is in A_p for every prime unitor divisor p of n. Then for n being r-powerful, φ^*_τ(n) counts the number of k-vectors $\vec{a} = (a_1, a_2, a_3, \text{---}\ a_k) \in Z^k_n$. such that $(\vec{a}, n)=1$. We take off from the work of Wang & HSU by defining our generalized unitary Euler totient as

$$\varphi^{*f}{}_{s,\alpha}(n) = (\zeta_s, \mu_\alpha^* f)(n) = \sum_{d//n}\left(\frac{n}{d}\right)^s f(d)\mu_\alpha^*(d) \qquad (1.8)$$

Where $\alpha \in c, s \in R$, $\zeta_s(n)=n^s$, $\zeta_0=\mu$ and f is completely multiplicative function comparing with that terminology of Wang-HSU, we see that

$$\varphi^{*f}{}_{k,r} = \varphi^{*(K)}{}_\tau = (\tau = \mu_r^* f) \text{ for brevity } \varphi^*{}_{s1,\alpha} = \varphi^{*\mu}{}_{s,\alpha'};$$
$$\varphi^*_\alpha = \varphi^*_{1,\alpha} \qquad (1.9)$$

Proposition 1.1 :

Let $s \in R$, $\alpha \in c$, and f be a completely multiplicative function. We have the product representation

$$\varphi^{*f}{}_{s,\alpha}(n) = n^s \prod_{p//n} \sum_{i=0}^{vp(n)} (-1)^i \binom{\alpha}{i}\left(\frac{f(p)}{p^s}\right)^i \quad n \in N$$

Proof: $$\frac{f(p)}{p^s} = \sum \frac{1}{p^s} \mu^*(d) \varphi^*\left(\frac{n}{d}\right)$$

$$= \sum \frac{1}{p^s} \mu^*(d) \prod_{p//n} n\left(1-\frac{1}{p}\right)$$

$$= n \prod \sum \frac{1}{p^s} (-1)^i \binom{\alpha}{\varphi^i} \frac{f(p)}{p^s}$$

2. Connections with other Totients:

Case I: ($f = \mu^*$)

When the parameters S and α take integer values the GET does indeed represent a number of well known arithmetic functions namely,

$\varphi^*_{1,1}=\zeta*\mu*=\varphi*$ (The classical Unitary Euler totient)

$\varphi^*_{1,-1}=\mu^* * \mu^*=\sigma_0=\tau$ (The number of unitary divisor function)

$\varphi^*_{s,1} = \zeta*\mu* = \sigma_s$ (The sum of the s^{th} power of unitary divisor function)

Where $S \in N$ and $\alpha = 1$, this particular Unitary totient $\varphi^*_{s,1} = \zeta_s * \mu*$ is equivalent to quite a few classical totients. In passing let us mention that, our GET is closely to the generalized Ramanujan sum through $\varphi^*_{1,\alpha}(r)=C^{*(\alpha)}(n,r)$ whenever r/n

Case I: ($f \neq \mu^*$) :

The GET also includes a number of known totients in this case.

1) The Garcia –ligh totient defined for fixed S,d $\in N$ by $\varphi^*_{(s,d,n)} = \varphi^{*Id}_{1,1}(n)$

Where $I_d(n) = I(gcd(d,n))$ is easily shown to be completely multiplicative. This totient $\varphi^{*Id}_{1,1}(n)$ counts the number of elements in the set $\{s,s+d, \dots \dots \dots s+(n-1)d\}$ that are relatively prime to n with $\varphi^*_{(1,1n)} \equiv \varphi^*_{(n)}$

2) The Garcia – ligh totient is special case of the following totient taken from exercise. The number of integer $x \in \{1,2,3 \dots \dots n\}$ and gcd (f(x), n) =1, using our terminology above,

$$\varphi^{*vg}_{11}(n)=(\zeta_1 * v_g\mu^*)(n)=\sum_{d//n} dv_g\left(\frac{n}{d}\right)\mu^*\left(\frac{n}{d}\right)$$

Where v_g is the completely multiplicative function defined over prime p by $v_g(p)=g_p$, the number of solutions of the congurence $g(x) \equiv 0(mod\ p)$.

Propostion 1.2 :

Let f, g $\subset A$, for k $\subset N$, define the K-convolution of f & g by

$$(f *_k g)(n) = \sum_{d^k//n} f(d) g\left(\frac{n}{d}\right)$$

It is easily checked that the k-convolution is neither commutative nor associative. Yet preserves multiplicativity that is f & g are multiplicative functions, then the $f*_k g$ is also multiplicative. The k^{th} convolute of $f \in A$ is defined by

$$f^k(n) = f(x) = \begin{cases} f\left(n^{\frac{1}{k}}\right), \text{if } n \text{ is a } k^{th} \text{ power} \\ 0, \text{otherwise} \end{cases}$$

The k- convolution is connected to the usual (Dirichlet) convolution via

$$f *_k g = f^{(k)} * g$$

We list here some examples of arithmetic functions which enjoy k- convolution relations.

Conclusion

This is to say that there are wide rang further application to Mobius and totient functions. More properties are to be discussed in coming papers, also completely multiplicative functions to be discussed.

References

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