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Some new integrals involving I-function of one variable and E-operator

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Abstract

The aim of this paper is to establish some new integrals involving I-function of one variable and E-operator.

1. Introduction

The I-function of one variable is defined by Saxena [1, p.366-375] and we will represent here in the following manner:

$$I^{m,n}_{p_i,q_i:r}\left[x\left|\begin{matrix}[(a_j,\alpha_j)_{1,n}],[(a_{ji},\alpha_{ji})_{n+1,p_i}]\\ [(b_j,\beta_j)_{1,m}],[(b_{ji},\beta_{ji})_{m+1,q_i}]\end{matrix}\right.\right] = (1/2\pi\omega)\int_L \theta(s)\, x^s\, ds \quad (1)$$

where $\omega = \sqrt{(-1)}$,

$$\theta(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+\beta_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}s)\right\}},$$

integral is convergent, when ($B>0, A\le 0$), where

$$B = \sum_{j=1}^{n}\alpha_j - \sum_{j=n+1}^{p_i}\alpha_{ji} + \sum_{j=1}^{m}\beta_j - \sum_{j=m+1}^{q_i}\beta_{ji}, \quad (2)$$

$$A = \sum_{j=1}^{p_i}\alpha_{ji} - \sum_{j=1}^{q_i}\beta_{ji}\ , \quad (3)$$

$$|\arg x| < \tfrac{1}{2} B\pi,\ \forall\ i \in (1, 2, \ldots, r). \quad (4)$$

2. Formula Used:

In the present investigation we require the following formulae:

From Sharma [2, p 27 (2.3)], we have

$$\int_0^1 x^{\rho-1}(1-x)^{\rho-2}\ {}_2F_1\left[\alpha,\beta;\frac{\alpha+\beta}{2};\frac{x(1+a)}{1+ax+b(1-x)}\right]$$

$$.[1+ax+b(1-x)]^{-2\rho+1}dx$$

$$= \frac{2^{-2\rho+\alpha+\beta-1}\Gamma(\rho-1)\Gamma(\alpha/2+\beta/2-1)\Gamma(\rho-\alpha/2-\beta/2-1)}{(1+a)^{\rho}(1+b)^{\rho-1}\Gamma(\alpha)\Gamma(\beta)}$$

$$.\left[\frac{(2\rho-\alpha+\beta-2)\Gamma(\alpha/2+1/2)\Gamma(\beta/2)}{\Gamma(\rho-\alpha/2)\Gamma(\rho-\beta/2-1/2)} + \frac{(2\rho+\alpha-\beta-2)\Gamma(\alpha/2)\Gamma(\beta/2+1/2)}{\Gamma(\rho-\beta/2)\Gamma(\rho-\alpha/2-1/2)}\right], \quad (5)$$

where Re(ρ) > 1, Re(2ρ –α – β) > 2, a and b are such that none of the expression 1 + a, 1+ b, 1 + ax + b(1 – x); 0 ≤ x ≤ 1, is zero.

3. Main Integral:

$$\int_0^1 x^{\rho-1}(1-x)^{\rho-2}\ {}_2F_1\left[\alpha,\beta;\frac{\alpha+\beta}{2};\frac{x(1+a)}{1+ax+b(1-x)}\right]$$

$$.\,[1+ax+b(1-x)]^{-2\rho+1} I^{m,n}_{p_i,q_i:r}\left[\frac{zx^{-\lambda}(1-x)^{-\lambda}}{[1+ax+b(1-x)]^{-2\lambda}}\right]dx$$

$$= \frac{2^{-2\rho+\alpha+\beta-1}\Gamma(\alpha/2+\beta/2-1)}{(1+a)^{\rho}(1+b)^{\rho-1}\Gamma(\alpha)\Gamma(\beta)}(\Gamma(\alpha/2+1/2)\Gamma(\beta/2)$$

$$.\,I^{m+3,n}_{p_i+3,q_i+3:r}\left[\frac{z2^{2\lambda}}{(1+a)^{-\lambda}(1+b)^{-\lambda}}\middle|^{\ldots\ldots,(\rho-\alpha/2,\lambda),(-1/2-\beta/2+\rho,\lambda),(-2+2\rho-\alpha+\beta,2\lambda)}_{(\rho-1,\lambda),(\rho-\alpha/2-\beta/2-1,\lambda),(2\rho-\alpha+\beta-1,2\lambda),\ldots\ldots}\right]$$

$$+\ \Gamma(\alpha/2)\Gamma(\beta/2+1/2)$$

$$.\,I^{m+3,n}_{p_i+3,q_i+3:r}\left[\frac{z2^{2\lambda}}{(1+a)^{-\lambda}(1+b)^{-\lambda}}\middle|^{\ldots\ldots,(\rho-\beta/2,\lambda),(-1/2-\alpha/2+\rho,\lambda),(-2+2\rho+\alpha-\beta,2\lambda)}_{(\rho-1,\lambda),(\rho-\alpha/2-\beta/2-1,\lambda),(2\rho+\alpha-\beta-1,2\lambda),\ldots\ldots}\right]), \quad (6)$$

where Re(ρ) > 1, Re (2ρ – α – β) > 2, a and b are such that nons of the expressions 1 + a, 1 + b, 1 + ax + b(1 - x); 0 ≤ x ≤ 1, is zero and λ ≥ 0, |argz| < ½πB, where B is given by equation (2).

Proof of (6):

To establish (6), on the left hand side replace the I-function of one variable by its equivalent counter integral as given in (1), change the order of integration which is valid under the given conditions, we get

$$\frac{1}{2\pi\omega}\int_L z^{\xi}\,\theta(\xi)\{\int_0^1 x^{\rho-\lambda\xi-1}(1-x)^{\rho-\lambda\xi-2}$$

$$.\,[1+ax+b(1-x)]^{-2(\rho-\lambda\xi)+1}$$

$$.\ {}_2F_1\left[\alpha,\beta;\frac{\alpha+\beta}{2};\frac{x(1+a)}{1+ax+b(1-x)}\right]dx\}d\xi.$$

Now evaluate the inner integral with the help of (5) and finally interpret it with (1), to obtain (6).

4. Integral by means of finite difference operator E:

$$\int_0^1 x^{\rho-1}(1-x)^{\rho-2}\ {}_2F_1[\alpha,\beta;\frac{\alpha+\beta}{2};\frac{x(1+a)}{1+ax+b(1-x)}]$$

$$.[1+ax+b(1-x)]^{-2\rho+1} I^{m,n}_{p_i,q_i:r}[\frac{zx^{-\lambda}(1-x)^{-\lambda}}{[1+ax+b(1-x)]^{-2\lambda}}]$$

$$.{}_uF_v[^{e_1,\ldots\ldots,e_u;}_{f_1,\ldots\ldots,f_v;}\ c\{\frac{x(1-x)}{1+ax+b(1-x)^2}\}^{\nu}]\mathbf{dx}$$

$$=\frac{2^{-2\rho+\alpha+\beta-1}\Gamma(\alpha/2+\beta/2)}{(1+a)^{\rho}(1+b)^{\rho-1}\Gamma(\alpha)\Gamma(\beta)}.\sum_{r=0}^{\infty}\frac{\prod_{j=1}^{u}(e_j;r)c^r}{\prod_{j=1}^{v}(f_j;r)r!}[\frac{1}{4(1+a)(1+b)}]^{r\nu}$$

$$.(\Gamma(\alpha/2+1/2)\Gamma(\beta/2).\ I^{m+3,n}_{p_i+3,q_i+3:r}[\frac{z2^{2\lambda}}{(1+a)^{-\lambda}(1+b)^{-\lambda}}\,|^{\ldots\ldots.,(\rho-\alpha/2+r\nu,\lambda),(-1/2-\beta/2+\rho+r\nu,\lambda),(-2+2\rho-\alpha+\beta+2r\nu,2\lambda)}_{(\rho-1+r\nu,\lambda),(\rho-\alpha/2-\beta/2-1+r\nu,\lambda),(2\rho-\alpha+\beta-1+2r\nu,2\lambda),\ldots\ldots.}]$$

$$+\Gamma(\alpha/2)\Gamma(\beta/2+1/2).\ I^{m+3,n}_{p_i+3,q_i+3:r}[\frac{z2^{2\lambda}}{(1+a)^{-\lambda}(1+b)^{-\lambda}}\,|^{\ldots\ldots.,(\rho-\beta/2+r\nu,\lambda),(-1/2-\alpha/2+\rho+r\nu,\lambda),(-2+2\rho+\alpha-\beta+2r\nu,2\lambda)}_{(\rho-1+r\nu,\lambda),(\rho-\alpha/2-\beta/2-1+r\nu,\lambda),(2\rho+\alpha-\beta-1+2r\nu,2\lambda),\ldots\ldots.}]), \quad (7)$$

provided that (i) $Re(\rho) > 1$, $Re(2\rho - \alpha - \beta) > 2$, a and b are such that nons of the expressions $1 + a$, $1 + b$, $1 + ax + b(1 - x)$; $0 \le x \le 1$, is zero and $\lambda \ge 0$ and $|\arg z| < ½ B\pi$, where B is given by (2) (ii) $u \le v$ (or $u \le v + 1$ and $|c| < 1$) and no one of $f_1, \ldots, f_v$ is zero or a negative integer.

Proof:

To prove (7), multiply both sides of (6) by

$$\frac{\prod_{j=1}^{u}(e_j+\delta)c^{\delta}}{\prod_{j=1}^{v}(f_j+\delta)}$$

and applying the operator ${}_eE^{\nu}_{\rho}E_{\delta}$ to get

$${}_eE^{\nu}_{\rho}E_{\delta}(\int_0^1 x^{\rho-1}(1-x)^{\rho-2}\ {}_2F_1[\alpha,\beta;\frac{\alpha+\beta}{2};\frac{x(1+a)}{1+ax+b(1-x)}]$$

$$.[1+ax+b(1-x)]^{-2\rho+1}.\ I^{m,n}_{p_i,q_i:r}[\frac{zx^{-\lambda}(1-x)^{-\lambda}}{[1+ax+b(1-x)]^{-2\lambda}}]$$

$$.\frac{\prod_{j=1}^{u}(e_j+\delta)c^{\delta}}{\prod_{j=1}^{v}(f_j+\delta)}\mathbf{dx})$$

$$={}_eE^{\nu}_{\rho}E_{\delta}\{\frac{\prod_{j=1}^{u}(e_j+\delta)c^{\delta}}{\prod_{j=1}^{v}(f_j+\delta)}\frac{2^{-2\rho+\alpha+\beta-1}\Gamma(\alpha/2+\beta/2)}{(1+a)^{\rho}(1+b)^{\rho-1}\Gamma(\alpha)\Gamma(\beta)}$$

$$.(\Gamma(\alpha/2+1/2)\Gamma(\beta/2)$$

$$.\ I^{m+3,n}_{p_i+3,q_i+3:r}[\frac{z2^{2\lambda}}{(1+a)^{-\lambda}(1+b)^{-\lambda}}\,|^{\ldots\ldots,(\rho-\alpha/2,\lambda),(-1/2-\beta/2+\rho,\lambda),(-2+2\rho-\alpha+\beta,2\lambda)}_{(\rho-1,\lambda),(\rho-\alpha/2-\beta/2-1,\lambda),(2\rho-\alpha+\beta-1,2\lambda),\ldots\ldots}]$$

$$+\Gamma(\alpha/2)\Gamma(\beta/2+1/2)$$

$$.\ I^{m+3,n}_{p_i+3,q_i+3:r}[\frac{z2^{2\lambda}}{(1+a)^{-\lambda}(1+b)^{-\lambda}}\,|^{\ldots\ldots,(\rho-\beta/2,\lambda),(-1/2-\alpha/2+\rho,\lambda),(-2+2\rho+\alpha-\beta,2\lambda)}_{(\rho-1,\lambda),(\rho-\alpha/2-\beta/2-1,\lambda),(2\rho+\alpha-\beta-1,2\lambda),\ldots\ldots}])\}. \quad (8)$$

Further expand both sides of (8) and use the finite difference operator E [3, p.33, with w = 1] has the following operations

$E_a f(a)=f(a+1),\ E_a^n f(a) = E_a[E_a^{n-1} f(a)],$ (9)

to get

$$\sum_{r=0}^{\infty}\int_0^1 x^{\rho-1}(1-x)^{\rho-2}\ {}_2F_1[\alpha,\beta;\frac{\alpha+\beta}{2};\frac{x(1+a)}{1+ax+b(1-x)}]$$

$$.[1+ax+b(1-x)]^{-2\rho+1}.\, I^{m,n}_{p_i,q_i:r}[\frac{zx^{-\lambda}(1-x)^{-\lambda}}{[1+ax+b(1-x)]^{-2\lambda}}]$$

$$.\frac{\prod_{j=1}^{u}(e_j+\delta+r)c^{\delta+r}}{\prod_{j=1}^{v}(f_j+\delta+r)r!}[x(1-x)\{1+ax+b(1-x)\}^{-2}]^{r\nu}\mathbf{dx}$$

$$= \frac{2^{-2\rho+\alpha+\beta-1}\Gamma(\alpha/2+\beta/2)}{(1+a)^{\rho}(1+b)^{\rho-1}\Gamma(\alpha)\Gamma(\beta)}\sum_{r=0}^{\infty}\frac{\prod_{j=1}^{u}(e_j+\delta+r)c^{\delta+r}}{\prod_{j=1}^{v}(f_j+\delta+r)r!\,[4(1+a)(1+b)]^{r\nu}}.$$

$$.(\Gamma(\alpha/2+1/2)\Gamma(\beta/2).\, I^{m+3,n}_{p_i+3,q_i+3:r}[\frac{z2^{2\lambda}}{(1+a)^{-\lambda}(1+b)^{-\lambda}}|$$

$$\begin{matrix}\text{.........}{}_{1,p},\left(\rho-\frac{\alpha}{2}+r\nu,\lambda\right),\left(-\frac{1}{2}-\frac{\beta}{2}+\rho+r\nu,\lambda\right),(-2+2\rho-\alpha+\beta+2r\nu,2\lambda)\\ (\rho-1+r\nu,\lambda),\left(\rho-\frac{\alpha}{2}-\frac{\beta}{2}-1+r\nu,\lambda\right),(2\rho-\alpha+\beta-1+2r\nu,2\lambda),......\end{matrix}]$$

$$+\ \Gamma(\alpha/2)\Gamma(\beta/2+1/2).\, I^{m+3,n}_{p_i+3,q_i+3:r}[\frac{z2^{2\lambda}}{(1+a)^{-\lambda}(1+b)^{-\lambda}}|$$

$$\begin{matrix}......,(\rho-\beta/2+r\nu,\lambda),(-1/2-\alpha/2+\rho+r\nu,\lambda),(-2+2\rho+\alpha-\beta+2r\nu,2\lambda)\\ (\rho-1+r\nu,\lambda),(\rho-\alpha/2-\beta/2-1+r\nu,\lambda),(2\rho+\alpha-\beta-1+2r\nu,2\lambda),......\end{matrix}]). \quad (10)$$

Next use $(\alpha)_n = \Gamma(\alpha+n)/\Gamma(\alpha)$ change the order of summation and integration on the left hand side of (10) and replace $(e_j+\delta)$ by e_j and $(f_j+\delta)$ by f_j, to have (7).

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