Ultra Scientist Vol. 25(1)A, 43-46 (2013).

Decomposition of curvature tensor fields in a Tachibana recurrent space of first order

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(Acceptance Date 24th January, 2013)

Abstract

Takano[1] have studied and defined decomposition of curvature tensor in a recurrent space. Sinha and Singh[2] have studied decomposition of recurrent curvature tensor field in a Finsler space. Further, Negi and Rawat[5] studied decomposition of recurrent curvature tensor fields in a Kaehlerian space. Rawat and Silswal[7] studied and defined decomposition of recurrent curvature fields in a Tachibana space.

In the present paper, we have studied the decomposition of curvature tensor field R^h_{ijk} in terms of two vectors and a tensor field. Also several theorems have been established therein.

1. Introduction

An Almost Tachibana space is an Almost Hermite space *i.e.* a 2n- dimensional space with an almost complex structure F_i^h satisfying the conditions

$$F_j^i F_i^h = -A_j^h \tag{1.1}$$

and with a Riemannian metric g_{ji} satisfying

$$F_j^t F_i^s g_{ts} = g_{ji} \tag{1.2}$$

from which

$$F_{ji} = -F_{ij} \tag{1.3}$$

where

$$F_{ji} = F_j^t g_{ti} \tag{1.4}$$

and finally has the property that the skew-symmetric tensor F_{ih} is a killing tensor

$$F_{ih,j} + F_{jh,i} = 0 \tag{1.5}$$

from which

$$F_{i,j}^h + F_{j,i}^h = 0 \tag{1.6}$$

and

$$F_i = -F_{i,j}^j \tag{1.7}$$

where the comma (,) followed by an index denotes the operator of covariant differentiation w.r.t. the metric tensor g_{ij} of the Riemannian space.

If the space satisfies the condition[3]

$$F^h_{i,j} = 0 . \qquad (1.8)$$

then the space is said to be Tachibana space and denoted by T^c_n.

The Riemannian curvature tensor field is defined by

$$R^h_{ijk} = \partial_i \begin{Bmatrix} h \\ jk \end{Bmatrix} - \partial_j \begin{Bmatrix} h \\ ik \end{Bmatrix} + \begin{Bmatrix} h \\ il \end{Bmatrix}\begin{Bmatrix} l \\ jk \end{Bmatrix} - \begin{Bmatrix} h \\ jl \end{Bmatrix}\begin{Bmatrix} l \\ ik \end{Bmatrix},$$

The Ricci tensor and scalar curvature are respectively given by

$$R_{ij} = R^a_{aij} \quad \text{and} \quad R = g^{ij} R_{ij} \qquad (1.9)$$

It is well known that these tensors satisfy the identity

$$R^a_{ijk,a} = R_{jk,i} - R_{ik,j} \qquad (1.10)$$

$$R_{,i} = 2R^a_{i,a} \qquad (1.11)$$

$$F^a_i R_{aj} = -R_{ia} F^a_j \qquad (1.12)$$

and

$$F^a_i R^i_a = R^a_i F^i_a \qquad (1.13)$$

The holomorphically Projective curvature tensor P^h_{ijk} is defined by

$$P^h_{ijk} = R^h_{ijk} + \frac{1}{(n+2)} \left(R_{ik}\, \delta^h_j - R_{jk} \delta^h_i + S_{ik} F^h_j - S_{jk} F^h_i + 2S_{ij} F^h_k \right), \qquad (1.14)$$

where

$$S_{ij} = F^a_i R_{aj} \qquad (1.15)$$

The Bianchi identities are given by

$$R^h_{ijk} + R^h_{jki} + R^h_{kij} = 0 \qquad (1.16)$$

and

$$R^h_{ijk,a} + R^h_{ika,j} + R^h_{iaj,k} = 0 \qquad (1.17)$$

The commutative formulae for the curvature tensor fields are given as follows

$$T^i_{,jk} - T^i_{,kj} = T^a R^i_{ajk} \qquad (1.18)$$

and

$$T^h_{i,ml} - T^h_{i,lm} = T^a_i R^h_{aml} - T^h_a R^a_{iml} \qquad (1.18a)$$

A Tachibana space T^c_n is said to be Tachibana recurrent space of first order if its curvature tensor field satisfies the condition

$$R^h_{ijk,a} = \lambda_a R^h_{ijk} \qquad (1.19)$$

where λ_a is a non-zero vector and is known as recurrence vector field.

The following relations follows immediately from (1.19) i.e.

$$R_{ij,a} = \lambda_a R_{ij} \qquad (1.20)$$

Multiplying the above by g^{ij}, we have

$$R_{,a} = \lambda_a R \, . \qquad (1.21)$$

2. Decomposition of curvature tensor field $\boldsymbol{R^h_{ijk}}$:

Consider the decomposition of recurrent curvature tensor field R^h_{ijk} in the following form

$$R^h_{ijk} = P^h X_{,i} Y_{j,k} \qquad (2.1)$$

where two vectors P^h, $X_{,i}$, and a tensor field $Y_{j,k}$ such that

$$P^h \lambda_h = 1. \qquad (2.2)$$

Now, we have the following:

Theorem (2.1) : Under the decomposition (2.1), the Bianchi identities for R^h_{ijk} take the forms

$$X_{,i} Y_{j,k} + X_{,j} Y_{k,i} + X_{,k} Y_{i,j} = 0 \qquad (2.3)$$

and

$$\lambda_a Y_{j,k} + \lambda_j Y_{k,a} + \lambda_k Y_{a,j} = 0 \quad (2.4)$$

Proof : From (1.16) and (2.1), we have

$$P^h\ [X_{,i}Y_{j,k} + X_{,j}Y_{k,i} + X_{,k}Y_{i,j}] = 0 \quad (2.5)$$

Multiplying (2.5) by λ_h and using (2.2), we obtain the required result (2.3).

Again, using (1.17), (1.19) and (2.1), we have

$$X_{,i}P^h[\lambda_a Y_{j,k} + \lambda_j Y_{k,a} + \lambda_k Y_{a,j}] = 0 \quad (2.6)$$

Multiplying (2.6) by λ_h and using (2.2), we have

$$X_{,i}[\lambda_a Y_{j,k} + \lambda_j Y_{k,a} + \lambda_k Y_{a,j}] = 0 \quad (2.7)$$

Since $X_{,i} \neq 0$

$$\therefore\ [\lambda_a Y_{j,k} + \lambda_j Y_{k,a} + \lambda_k Y_{a,j}] = 0.$$

which completes the proof of the theorem.

Theorem (2.2): Under the decomposition (2.1), the tensor field R^h_{ijk}, R_{ij} and $Y_{j,k}$ satisfy the relation

$$\lambda_a R^a_{ijk} = \lambda_i R_{jk} - \lambda_j R_{ik} = X_{,i}Y_{j,k} \quad (2.8)$$

Proof : With the help of (1.10) and (1.19), we have

$$\lambda_a R^a_{ijk} = \lambda_i R_{jk} - \lambda_j R_{ik} \quad (2.9)$$

Multiplying (2.1) by λ_h and using relation (2.2), we have

$$\lambda_h R^h_{ijk} = X_{,i}Y_{j,k} \quad (2.10)$$

In view of equations (2.9) and (2.10), we get the required result (2.8).

Theorem (2.3): Under the decomposition (2.1), the quantities λ_a and P^h behave like the recurrent vectors. The recurrent form of these quantities are given by

$$\lambda_{a,m} = \mu_m \lambda_a \quad (2.11)$$

and

$$P^h_{,m} = -\mu_m P^h \quad (2.12)$$

Proof : Differentiating (2.9) covariantly w.r.t. x^m, and using (2.1) and (2.8), we have

$$\lambda_{a,m}P^a X_{,i}Y_{j,k} = \lambda_{i,m}R_{jk} - \lambda_{j,m}R_{ik} \quad (2.13)$$

Multiplying (2.13) by λ_a , and using (2.1) and (2.9), we have

$$\lambda_{a,m}(\lambda_i R_{jk} - \lambda_j R_{ik}) = \lambda_a(\lambda_{i,m}R_{jk} - \lambda_{j,m}R_{ik}) \quad (2.14)$$

Now, multiplying (2.14) by λ_h , we have

$$\lambda_{a,m}(\lambda_i R_{jk} - \lambda_j R_{ik})\lambda_h = \lambda_a\lambda_h(\lambda_{i,m}R_{jk} - \lambda_{j,m}R_{ik}) \quad (2.15)$$

Since the expression on the right hand side of the above equation is symmetric in *a* and *h*, therefore[4]

$$\lambda_{a,m}\lambda_h = \lambda_{h,m}\lambda_a \quad (2.16)$$

provided $\lambda_i R_{jk} - \lambda_j R_{ik} \neq 0$

The vector field λ_a being a non-zero, we can choose a proportional vector field μ_m such that

$$\lambda_{a,m} = \mu_m \lambda_a \quad (2.17)$$

Further, differentiating (2.2) w.r.t. x^m and using (2.17), we have

$$P^h_{,m} = -\mu_m P^h, \quad (\text{since } \lambda_h \neq 0) \quad (2.18)$$

which proves the theorem.

Theorem (2.4) : Under the decomposition (2.1), the vector $X_{,i}$ and the tensor $Y_{j,k}$ satisfies the relation

$$X_{,i}Y_{j,k}(\lambda_m + \mu_m) = X_{,i}Y_{j,km} + Y_{j,k}X_{i,m} \quad (2.19)$$

Proof: Differentiating (2.1) covariantly w.r.t. x^m and using (1.19), (2.1) and (2.18), we get the required result of the theorem[6].

Theorem (2.5): Under the decomposition (2.1), the curvature tensor and holomorphically projective curvature tensor are equal if

$$Y_{k,m}\{(X_{,i}\delta^h_j - X_{,j}\delta^h_i) + X_l(F^h_j F^l_i - F^h_i F^l_j)\} + 2X_l Y_{j,m}F^h_k F^l_i = 0 \quad (2.20)$$

Proof : Equation (1.14) may be expressed as

$$P_{ijk}^h = R_{ijk}^h + D_{ijk}^h \qquad (2.21)$$

where

$$D_{ijk}^h = \frac{1}{(n+2)}\left(R_{ik}\delta_j^h - R_{jk}\delta_i^h + S_{ik}F_j^h - S_{jk}F_i^h + 2S_{ij}F_k^h\right) \qquad (2.22)$$

Contracting indices h and k in (2.1), we have

$$R_{ij} = P^k X_{,i} Y_{j,k} \qquad (2.23)$$

In view of (2.23), we have

$$S_{ij} = F_i^l P^m X_{,l} Y_{j,m} \qquad (2.24)$$

Making use of relations (2.23) and (2.24) in (2.22), we have

$$D_{ijk}^h = \frac{1}{(n+2)}\left[Y_{k,m}P^m\left\{\left(X_{,i}\delta_j^h - X_{,j}\delta_i^h\right) + X_{,l}\left(F_j^h F_i^l - F_i^h F_j^l\right)\right\} + 2P^m X_{,l} Y_{j,m} F_k^h F_i^l\right] = 0 \qquad (2.25)$$

From (2.21), it is clear that $P_{ijk}^h = R_{ijk}^h$, if $D_{ijk}^h = 0$, which in view of (2.25), becomes

$$Y_{k,m}P^m\left\{\left(X_{,i}\delta_j^h - X_{,j}\delta_i^h\right) + X_{,l}\left(F_j^h F_i^l - F_i^h F_j^l\right)\right\} + 2P^m X_{,l} Y_{j,m} F_k^h F_i^l = 0 \qquad (2.26)$$

Multiplying (2.26) by λ_m and using (2.2), we obtain the required condition (2.20).

Theorem (2.6) : Under the decomposition (2.1), the scalar curvature R, satisfy the relation[8]

$$R_{,k} = g^{ij} X_{,i} Y_{j,k} \qquad (2.27)$$

Proof : Multiplying equation (2.23), we have

$$g^{ij} R_{ij} = g^{ij} P^k X_{,i} Y_{j,k} \qquad (2.28)$$

or $R = P^k g^{ij} X_{,i} Y_{j,k}$ $[\because R = g^{ij} R_{ij}]$ (2.29)

Now, multiplying (2.29) by λ_k, then using (2.2) and (1.21), we have

$$\lambda_{,k} = g^{ij} X_{,i} Y_{j,k}.$$

which completes the proof of the theorem.

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