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On the maximal lattice of finite cyclic groups

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Abstract

In this paper we introduce the notion of a maximal lattice of groups. We also prove some theorems regarding maximal lattices and based on them develop a method to construct the maximal lattice of any finite cyclic group.

Key words : Cyclic groups of prime order, maximal lattice, L-fuzzy subgroups.

1. Introduction

The initial stages of the development of the subject of *Partially ordered sets* and *lattices* are found in the works of mathematicians like George Boole, Richard Dedekind, Charles Sanders Pierce and Earnest Schröder. Garrett Birkhoff [2] developed this subject into a full-fledged one in 1930's. *Fuzzy set theory* and *fuzzy logic* were developed by Lotfi A. Zadeh in 1965[10]. J.A. Goguen[5] considered a complete and distributive lattice L for the membership set, instead of the closed interval [0,1] used by Zdeh; thus introducing the concept of L-fuzzy set. In 1971 Azriel Rosenfeld[8] fuzzified the theory of groups. Just like fuzzy sets were generalized using lattices as membership sets, fuzzy groups also could be generalized.

Cyclic groups of prime power order are very important in the study of groups because of the fact that they are the building blocks of finite groups. In this paper, we investigate how to assign a *maximal lattice* L to a cyclic group of prime power order using its L-fuzzy subgroups and extend it to any finite cyclic group. We also develop a method to construct maximal lattices and illustrate it with some examples. Terms and notations in Lattice Theory used in this paper are as found in Bernard Kolman[1] and Devey B.A.[3].

2. Basic concepts :

A relation $\leq$ on a set A is called a *partial order* if $\leq$ is reflexive, antisymmetric and transitive. The set A together with the partial order $\leq$ is called a *partially ordered set* (or a *poset*) and is denoted as (A, $\leq$) or simply A. The elements a and b of a poset A are said to be *comparable* if if $a \leq b$ or $b \leq a$. If every pair of elements in a poset A is comparable, we say that A is a *linearly* (*or totally*) *ordered set* and the partial order in this case is called a *linear (total)* order. We also say that such an A is a *chain*. For any set S, its power set *P*(S) together with set inclusin $\subseteq$ is a poset. If $|S|\leq 2$, it is not totally ordered. The set Z^+ of positive integers together with the usual order $\leq$ is a totally ordered set(toset). An element $a \in A$ is called a *maximal element* of A if there is no element c in A such that $a < c$ and an element element $b \in A$ is called a *minimal element* of A if there is no element c in A such that $c < b$. An element $a \in A$ is called a *greatest element* of A if $x \leq a$ for all $x \in A$. An element $a \in A$ is called a *least element* of A if $a \leq x$ for all $x \in A$. The greatest element of a poset, if it exists, is denoted by I and is called the unit element. The least element of a poset, if it exists, is denoted by O and is called the *zero element*.

Consider a poset A and a subset B of A. An element $a \in A$ is called an *upper bound* of B if $b \leq a$ for all $b \in B$. An element $a \in A$ is called a *lower bound* of B if $a \leq b$ for all $b \in B$. An element $a \in A$ is called a *least uper bound* of B or *supremum of B* denoted as lub(B) or sup B or $\vee$ B if a is an upper bound of B and $a \leq a'$, whenever a' is an upper bound of B. An element $a \in A$ is called a *greatest lower bound* of B or *infimum* of B denoted as glb(B) (or inf B or $\wedge$ B), if a is a lower bound of B $a' \leq a$, whenever a' is a lower bound of B. *lattice* is a poset $(L,\leq)$ inwhich every subset {a, b} consisting of two elements has a least upper bound and a greatest lower bound. We denote lub ({*a*,*b*,}) and glb ({*a*,*b*}) by $a \vee b$ and $a \wedge b$ respectively and call them the *join* and *meet* respectively of a and b.

For any non-empty set S,$(P(S),\subseteq)$ is a poset. For any A, B $\in$ P(S), $A\vee B=A\cup B$ and $A\wedge B=A\cap B$ exist and so it is a lattice. Another example of a lattice is $(Z^+,\leq)$ where $\leq$ is defined by $a\leq b$ iff a divides b. Here $a\vee b$=lcm (a,b) and $a\wedge b$=gcd(a,b). Now for any positive integer n, let D_n denote the set of all positive divisors of n. Then D_n together with the relation 'divisibility' is a lattice.

A non-empty subset S of a lattice L is called a *sublattice* of L if a, b $\in$ S $\Rightarrow$ $a\wedge b$, $a\vee b\in S$. If $(A,\leq)$ and $(B,\leq)$ are posets, then $(AXB,\leq)$ is a poset, with partial order $\leq$defined by $(a,b) \leq (a',b')$ if $a\leq a'$ in A and $b\leq b'$ in B. The partial order $\leq$defined on the Cartesian product A X B is called the *product partial order.* If $(L_1,\leq)$ and $(L_2,\leq)$ are lattices, then $(L_1 X L_2,\leq)$ is a lattice called *the product lattice of* L_1 and L_2 where the partial order $\leq$ of L is the product partial order. Here $(a_1,b_1)\vee(a_2,b_2)=(a_1\vee a_2, b_1\vee b_2)$ and $(a_1,b_1)\wedge(a_2,b_2)=(a_1\wedge a_2, b_1\vee b_2)$.

2.1. *Definition*[1] *:* A lattice L is called *distributive* if for any elements *a, b* and *c* in L, we have the following distributive properties:

1. $a \wedge(b\vee c)=(a\wedge b) \vee (a\wedge c)$
2. $a \vee(b\wedge c)=(a\vee b) \wedge (a\vee c)$

2.2. *Definition*[1] *:* A lattice L is said to be *complete* if every non-empty subset of it has glb and lub

If L_1 and L_2 are complete lattices, then $L_1 X L_2$ is also a complete lattice with joins and meets being formed co-ordinate wise.

2.3. *Example :* $(P(S),\subseteq)$ is a complete and distributive lattice.

2.4. *Definition*[4] *:* Given a universal set X, a *Fuzz Set* on X (*or a fuzzy subset of* X) is defined as a function $A:X\rightarrow[0,1]$. Its range is denoted as *Im(A)*.

2.5. *Example :* Let X={a,b,c,d,e}. Define $A:\rightarrow[0,1]$ by A(a)=0.1; A(b)=0.2; A(c)=0.3; A(d)=0.4; A(e)=0.5. Then A is fuzzy subset of X and Im(A)={0.1,0.2,0.3,0.4,0.5}.

2.6. *Definition*[5] *:* If L is a lattice and X is a universal set, then an *L-fuzzy set A on X* (or *an L-fuzzy subset A* of X) is a function $A : X\rightarrow L$. We shall write $A\in L^X$ for *A is an L-fuzzy set on X*.

2.7. *Example :* Let X={a,b,c,d} and $L=D_6=\{1,2,3,6\}$ under the relation divisibility. Define $A : X\rightarrow L$ by A(a)=1, A(b)=2, A(c)=3 and A(d)=6. Then A is an L-fuzzy set on X.

3. *Fuzzy Groups :*

3.1. Definition[8] : A fuzzy subset A of a multiplicative group G is said to be a *fuzzy subgroup* of G or a *fuzzy group on G* if for every $x, y\in G$

(i) $A(xy)\geq\min\{A(x),A(y)\}$ and (ii) $A(x^{-1}) =A(x)$.

3.2. *Example :* Let G={1,-1,-i} under multiplication of complex numbers. Define $A : G\rightarrow[0,1]$ by A(1)=1, A(-1)=0.5, A(i)=A (-i)=0.25. Then A is a fuzzy subgroup of G.

3.3. *Definition*[7]. An L-fuzzy subset A of G is called an *L-fuzzy subgroup* of G (or *an L-fuzzy group on* G) if
(i) $A(xy)\geq A(x)\wedge A(y)$, $\forall x,y\in G$, and
(ii) $A(x^{-1}) \geq A(x)$, $\forall x\in G$

It may be recalled that for any positive integer n, $Z_n=\{0,1,2,\ldots,n-1\}$ is a group with respect to addition modulo n.

3.4. *Example.* Let G be the group $Z_6 =\{0,1,2,3,4,5\}$ under addition modulo 6.Take $L=D_6=\{1,2,3,6\}=\{0,a,b,1\}$ where 0 and 1 denote 1 and 6 respectively; and a and b are 2 or 3(interchangeable). Define $A: Z_6 \rightarrow D_6$ by A(0)=1, A(2)=A(4)=a, A(3)=b and A(1)=A (5)=0. Then A is an L-fuzzy group on Z_6.

3.5. *Example.* Let V={e,a,b,c} be the *Klein-4 group*, whose composition table is as follows:

Table (i): Definition of binary operation * on V.

*	e	a	b	c
e	a	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Let $L=D_6= \{0,x,y,1\}$. Define $A:V\rightarrow D_6$ by A(e)=1, A(a)=x,A(b)=y and A(c)=0. Then A is an L-fuzzy group on V.

3.6. *Lemma*[7]. Let G be a group and $A\in L^G$. Then A is an L-fuzzy group on G iff $A_a=\{x/x\in G, A(x)\geq a\}$ is a subgroup of G, $\forall a \in A(G)\cup\{b\in L/b\leq A(e)\}$ ■

3.7. *Notation*[8]. If G is a group and L is a lattice, then (G,L) shall denote the collection of all L-fuzzy groups on G.

3.8. *Example.* Consider the group $<Z_{18},+_{18}>$ and the lattice $D_{12} =\{1,2,3,4,6,12\}$. Define

$A:Z_{18}\to D_{12}$ by $A:0\mapsto 12$, $\{3,15\}\mapsto 2$, $\{6,12\}\mapsto 6$, $9\mapsto 4$, $\{2,4,8,10,14,16\}\mapsto 3$, $\{1,5,7,11,13,17\}\mapsto 1$. Then A is an L-Fuzzy subgroup of Z_{18} as can be verified using the lemma 3.6.

4. L-fuzzy subgroups of finite cyclic groups:

4.1. *Definition*. Let $L=(\{a_1,a_2,\ldots\ldots.a_n\},\leq)$ be a lattice. We say that L is a *finite lattice* containing n points and write $|L|=n$.

4.2. *Example*. $D_6=\{1,2,3,6\}$ is a lattice under divisibility. It is a finite lattice containing four points and so $|D_6|=4$.

4.3. *Definition*. Let G be a group, L be a finite lattice and $A: G\to L$ be an L-fuzzy group. A is said to *saturate* L if $Im(A)=L$. If there is an L-fuzzy group A on G which saturates L, then we say that *G saturates **L***.

4.4. *Example*. Consider $G=\langle Z,+\rangle$ and $L=(\{0,1/3,1/2,1\},\leq)$. Define $A: G\to L$ by $A(0)=1$, $A(x)=1/2$ if $x\in 4Z-\{0\}$, $A(x)=1/3$ if $x\in 2Z-4Z$ and $A(x)=0$ if $x\in Z-2Z$. Then A is an L-fuzzy group on G with $Im(A)=L$. Hence, A as well as G saturates L.

4.5. *Example* .Let $G=\langle Z_4,+_4\rangle$ and $L=(\{0,1\},\leq)$. Define $A: G\to L$ by $A(0)=1$; $A(x)=0$, if $x\neq 0$. Then A is an L-fuzzy group on G which saturates L and so G also saturates L.

4.6. *Example*. Let $G=\langle Z_4,+_4\rangle$ and $L=(\{0,1/2,1\},\leq)$. Define $A: G\to L$ by $A(0)=1$; $A(x)=0$, if $x\neq 0$. Here A is an L-fuzzy group on G with $Im(A)\neq L$. Hence A does not saturate L. But if we define $B: G\to L$ by $B(0)=1$; $B(2)=1/2$ and $B(1)=B(3)=0$, then B is an L-fuzzy group on G which saturates L and hence G also saturates L.

4.7. *Definition*. Let G be a group and L be a finite lattice. A sublattice L_1 of L is said to be a *maximal lattice* saturated by G if there is an $A\in(G,L)$ which saturates L_1 and there is no $B\in(G,L)$ which saturates a sublattice L_2 of L with $|L_2|>|L_1|$.

4.8. *Example*. Consider the sublattices $L_1=\{0,1\}$ and $L_2=\{0,1/2,1\}$ of $L=\{0,1/3,1/2,1\}$ and let $G=\langle Z_4,+_4\rangle$. Define $A: G\to L$ by $A(0)=1$; $A(x)=0$, if $x\neq 0$. Also define $B: G\to L$ by $B(0)=1$; $B(2)=1/2$ and $B(1)=B(3)=0$. Then L_1 is not a maximal lattice of G, because there is L_2 with $|L_2|>|L_1|$ and $B: G\to L$ which saturates L_2. It can be shown that L_2 is a maximal lattice for G.

We may recall that a group G is said to be of prime power order if $|G|=p^n$, for some prime number p and positive integer n.

4.9. *Theorem*[9]. Let G be a cyclic group of prime power order. Then the lattice of all subgroups of G is a chain ■

4.10. *Theorem*. Let G be a cyclic group of prime power order. Then a maximal lattice L_G for G is a chain isomorphic to the chain of all subgroups of G.

Proof: Suppose that the order of G is p^r, where p is a prime and r is a positive integer. Then G has r+1subgroups in all, which form a chain. Let L_1: $G_0=\{e\}\subset G_1\subset G_2\subset \ldots\ldots\ldots\subset G_{r-1}\subset G_r=G$ be this chain of subgroups. Find a subchain $L_2: 0=a_0<a_1<a_2<\ldots\ldots\ldots.<a_{r-1}<a_r=1$ of the chain $L=[0,1]$ and define $A:G\to L$ by

$A(e)=1$
$A(x) = a_{r-1},$ if $x\ G_1-\{e\}$
$= a_{r-2},$ if $x\ G_2- G_1$
………………………..
………………………..
$= a_{1},$ if $x\ G_{r-1}- G_{r-2}$ and
$= 0,$ if $x\ G_{-} - G_{r-1}$.

Then the level set A_{a_i} is a subgroup of G for each a_i , $0 \le i \le r$. Hence, by lemma 3.6, A is an L-fuzzy subgroup of G. It is clear that G saturates L_2. Now we shall show that L_2 is maximal. Suppose that L_2 is not maximal. Then we can find a subchain L_3 of L having more points (between 0 and 1) than L_2 has. Also there is some L-fuzzy subgroup $B:G \rightarrow L$ which saturates L_3. Let b be a point in L_3 other than a_i , $0 \le i \le r$. Then, $a_{j-1} < b < a_j$ for some j. Since L_3 is saturated by B, there exists some subset X $(\neq \emptyset)$ of G such that $B(X)=\{b\}$. Now by lemma 3.6, the level subset B_b is a subgroup of G. Since, $a_{j-1} < b < a_j$, B_b is different from all the subgroups B_{a_i}, $0 \le i \le r$. Thus corresponding to each such b we get one subgroup in addition to the subgroups determined by a_i. This produces more than r+1 subgroups for G, which is impossible since G has precisely r+1 subgroups. This contradiction shows that L_2 is maximal. Thus the maximal lattice $L_G = L_2$.It is easily seen that L_{-2} is isomorphic to L_1 ■

4.11. *Corollary*. A maximal chain for a cyclic group of prime power order is unique up to isomorphism.

Proof: Let G be a group of prime power order and L_1 and L_2 be two maximal chains for G. Then L_1 and L_2 are isomorphic to the chain of subgroups of G and as such they are isomorphic to each other ■

4.12. *Remark*. In view of the above corollary, we can talk of *the maximum chain* for a cyclic group of prime power order.

4.13. *Corollary*. The maximum chain for a group of prime order is $\{0,1\}$.

Proof: Let G be a group of prime order p. Then it is cyclic of order $p^1= p$. Hence by theorem 4.10 it has two points in its chain of subgroups and so is the maximum chain. Hence the result ■

4.14. *Corollary*. For any prime number p, the maximum chain for Z_p is $\{0,1\}$.

4.15. *Corollary*. If p is a prime number and n is any positive integer then the maximum chain for is $0=a_0<a_1 \ldots\ldots <a_n=1$ ■

4.16. *Example*. Z_8 is a prime powered group since $|Z_8|=8=2^3$. Hence, by corollary 4.15.,the maximum chain of Z_8 is $0=a_0<a_1 <a_2<a_3=1$.

It is well-known that every finite cyclic group of order n is isomorphic to Z_n. So, henceforth we represent cyclic groups of order n by Z_n.

4.17. *Proposition*[6]. The group Z_m X Z_n is isomorphic to Z_{mn} if and only if m and n are relatively prime ■

4.18. *Proposition*[6]. The group $\prod_{i=1}^{n} Z_{m_i}$

is cyclic and isomorphic to if and only if the number m_i , for i=1,2…n are pairwise relatively prime ■

4.19. *Proposition*[6]. Suppose $n = p_1^{n_1} p_2^{n_2} \ldots\ldots . p_r^{n_r}$, where p_i's are distinct primes. Then Z_n is isomorphic to $Z_{p_1^{n_1}} X\ Z_{p_2^{n_2}} \ldots\ldots . X\ Z_{p_r^{n_r}}$ ■

4.20. *Theorem*. Suppose $n = p_1^{n_1} p_2^{n_2} \ldots\ldots . p_r^{n_r}$, where p_i's are distinct primes. Then the maximum lattice for $Z_n = Z_{p_1^{n_1}} X\ Z_{p_2^{n_2}} \ldots\ldots . X\ Z_{p_r^{n_r}}$ is the product lattice of the maximum chains for the factors $Z_{p_i^{n_i}}$.

Proof: By theorem 4.10, each $Z_{p_i^{n_i}}$ has a chain of (n_i+1) points as its maximum lattice. Let L be the product of these chains under partial order product. Then L has $\prod_{i=1}^{r}(n_i+1)$ points. Each $Z_{p_i^{n_i}}$ is a cyclic group of prime power order. It has (n_i+1) subgroups. Hence there are $\prod_{i=1}^{r}(n_i+1)$ subgroups of Z_n altogether. Corresponding to each subgroup we can find a point on the lattice L. This is possible because there is a 1-1 correspondence between subgroups of $Z_{p_i^{n_i}}$ and the points of its chain. We can define an L-fuzzy set $A : Z_n \rightarrow L$ in such a manner that the level set A_a for each $a \in L$ is a subgroup of Z_n. Then A is an L-fuzzy subgroup of Z_n. It is obvious that A saturates L. Since each chain is maximum for the corresponding cyclic group, L is maximum for the product group ■

4.21. Example. $Z_6 \cong Z_2$ X Z_3. The maximum chain for both Z_2 and Z_3 is $D_2=\{0,1\}$. Take $Z_6=\{0,1\}$ X $\{0,1,2\}=\{00,01,02,10,11,12\}$. Its composition table is given below in table (ii): Addition on the first digit is modulo 2 and that on second is modulo 3.

Table (ii): Composition table for Z_2XZ_3.

+	00	01	02	10	11	12
00	00	01	02	10	11	12
01	01	02	00	11	12	10
02	02	00	01	12	10	11
10	10	11	12	00	01	02
11	11	12	10	01	02	00
12	12	10	11	02	00	01

Its proper subgroups are $Z_2=\{00,10\}$ and $Z_3=\{00,01,02\}$. Its subgroup lattice is given in figure(i):

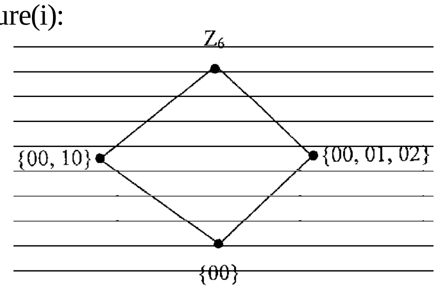

Figure (i): Subgroup lattice of Z_6

Take $L=D_2XD_2$

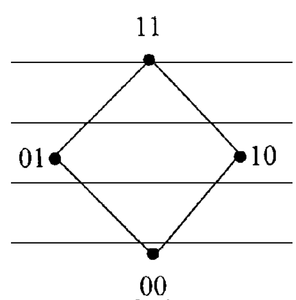

Figure (ii):The product lattice $L=D_2XD_2$

Define A:Z_6 L by A(00) = 11, A(10) = 01, A(01) = A(02) = 10, A(11) = A(12) = 00. This is an L-Fuzzy group and D_2 x D_2 is the maximum lattice for Z_2 x Z_3 =Z_6.

4.22. Example. Z_{30}=Z_2XZ_3XZ_5={0,1} X{0,1,2}X{0,1,2,3,4}={000,001,……..., 123, 124}. It has eight subgroups having 1 element, 2elements, 3elements, 5 elements, 6 elements, 10 elements, 15 elements and 30elements respectively.They are

{*e*}= {000},
B = {000, 100},
C = {000, 010, 020},
D = {000, 001, 002, 003, 004},
E = {000, 010, 020, 100, 110, 120},
F = {000, 001, 002, 003, 004, 100, 101, 102, 103, 104},
G = {000, 010, 020, 001, 011, 021, 002, 012, 022, 003, 013, 023, 004, 014, 024} and
Z_{30}= {000, 001, 002, 003, 004, 010, 011, 012, 013, 014, 020, 021, 022, 023, 024, 100, 101, 102, 103, 104, 110111, 112, 113, 114, 120, 121, 122, 123, 124}

The subgroup lattice can be found to be D_{30}.

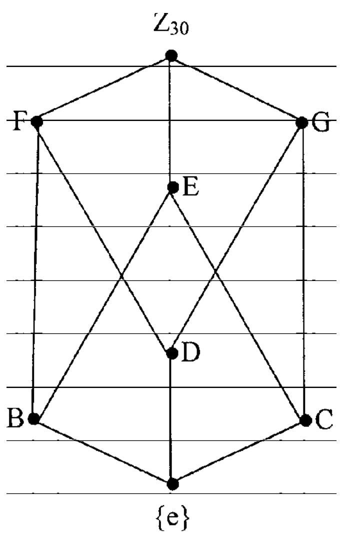

Figure (iii): Subgroup lattice for Z_{30}

Take L= D_{30}=D_2XD_2XD_2={0,1}X{0,1}X{0,1}:

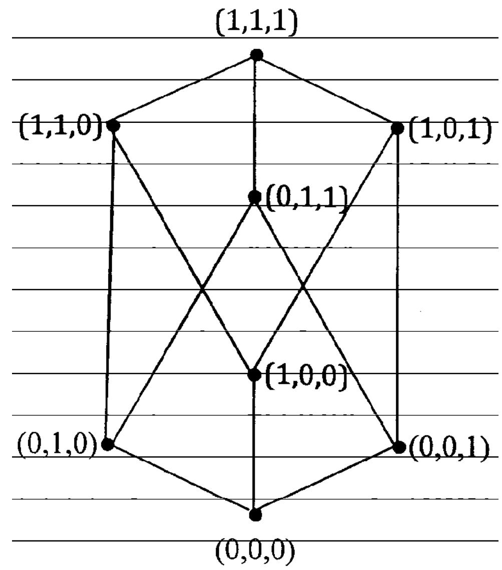

Figure (iv):Product lattice D_2XD_2XD_2

The L-fuzzy subgroup A: Z_6L is defined as follows:
A(x)= (1,1,1) if x ∈{e}={000}
A(x)= (1,1,0) if x∈ B-{e}={100}
A(x)= (1,0,1) if x ∈C-{e}={010,020}
A(x)= (0,1,1) if x∈ D-{e}={001,002,003,004}
A(x)= (1,0,0) if x ∈E-(BC)={110,120}
A(x)= (0,1,0) if x∈ F-(BD)={101,102,103,104}
A(x)= (0,1,0) if x∈ G-(CD)={011,021,012, 022, 013, 023,014,024}
A(x)=(0,0,0) if x∈{ 111,112,113,114,121,122,123,124}
This shows that the maximum lattice for Z_{30}=D_{30}.

5. Concluding remarks

Since any finite cyclic group is isomorphic to Z_n for some n and any n can be factored in the form n=$p_1^{n_1}p_2^{n_2}\dots\dots..p_r^{n_r}$,where $p_1,p_2,\dots..p_r$ are distinct prime numbers, the technique described above can be applied to find the maximum lattice of any finite cyclic group. The method can also be extended to infinite cyclic groups and this is the subject matter of another paper to be communicated soon.

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