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# Fixed point theorem for expansion mappings of rational type on a implicit relation

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## Abstract

In this note a fixed point theorems on expansion mappings is established in a complete metric space under certain conditions**.** Further a common fixed point theorem for pair of weakly compatible expansion mappings is established. In this theorem the completeness of space is replaced with a set of four alternative conditions for functions satisfying implicit relations. These theorems extended and improve results of S.M. Kang[2], M.A. Khan *et al*.[3], B.E. Rhoades[7] and T.Taniguchi[8].

*Key woards:* Common fixed points, expansion mappings, Complete metric space, weak compatible mappings , implicit relations .

**Mathematics Subject Classification**: 54H25.

## Introduction

Wang *et al*.[9] proved some fixed point theorems on expansion mappings, which correspond some contractive mappings. Further, by using functions, Khan *et al*.[3] generalized the result of [9]. Also Rhoades[7] and Taniguchi[8] generalized the result of [9] for pair of mappings. Kang[2] generalized the result of Khan *et al*.[3], Rhoades[7] and Taniguchi[8] for expansion mappings.

Popa improved results of Jha *et al*.[1,4,5] for Meir and Keeler type mappings by taking weak compatibility property and replacing the completeness of the space with a set of four alternative conditions for functions implicit relations.

The objective of this paper is to prove common fixed point theorem for surjective mappings satisfying some expansion conditions, which extend corresponding result of Kang[2] and Khan *et al*.[3]. In the sequel, we introduce some implicit relations in section 4 which are found to be viable, productive and powerful tool in existence of common fixed point for non-surjective mappings satisfying certain expansion type conditions.

*Preliminaries :*

Throughout this paper, $\Re$ and $\aleph$ denote the set of real numbers and the set of natural numbers, respectively. We use the following definitions in the proof of our main theorems.

*Definition 1.1:* Let $X$ be a topological space and $f : X \to \Re$ a real valued mapping on *X*. Then *f* is called upper semi-continuous on *X* iff $f^{-1}(-\infty, t)$ is open in *X* for every $t \in \Re$. A mapping *f* is called lower semi –continuous if – *f* is upper semi-continuous.

*Definition*[11] *1.2:* The self maps *S* and *T* of a metric space (*X,d*) are said to be weakly compatible if *Sx=Tx* implies *STx=TSx*.

Kang[4] proved the following theorem:

Let $\Re_+$ be the set of all non-negative real numbers and let $\varphi$ denote the family of all real functions $\phi : \Re_+^3 \to \Re_+$ satisfying the following conditions $(C_1)$ and $(C_2)$ according to Khan *et al.*[5]:

$(C_1)$ $\phi$ is lower semi-continuous in each coordinate variable,

$(C_2)$ Let $v, w \in \Re_+$ be such that either $v \geq \phi(v, w, w)$ or $v \geq \phi(w, v, w)$.

Then $v \geq hw$, where $\phi(1, 1, 1) = h > 1$.

*Theorem 1.3:* Let *A* and *B* be surjective mappings from a complete metric space (*X,d*) into itself satisfying

$$d(Ax, By) \geq \phi(d(Ax, x), d(By, y), d(x, y))$$

For all $x, y \in X$ with $x \neq y$, where $\phi \in \varphi$. Then *A* and *B* have a common fixed point in X.

## 2. Main Results

Let $\Re_+$ be the set of all non-negative real numbers and let $\varphi$ denote the family of all real functions $\phi : \Re_+^4 \to \Re_+$ satisfying the following conditions:

$(C_1)$ $\phi$ Is lower semi-continuous in each coordinate variable,

$(C_2)$ $\phi$ Is non-increasing in second and third coordinate variable,

$(C_3)$ Let $v, w \in \Re_+$ be such that $v \geq \phi(w, v, w, v + w)$ or $v \geq \phi(w, w, v, w + v)$. Then $v \geq hw$, where $\phi(1, 1, 1, 1) = h > 1$.

Now we prove the theorem as follows:

*Theorem 2.1:* Let A and B be surjective mappings from a complete metric space (*X,d*) into itself satisfying

$$d(Ax, By) \geq \phi\left(d(x, y), d(Ax, x), d(y, By), \frac{d(Ax, y) + d(By, x)}{1 + d(y, Ax)d(x, By)}\right) \tag{2.1}$$

For all $x, y \in X$ with $x \neq y$, where $\phi \in \varphi$. Then *A* and *B* have a common fixed point in *X*.

*Proof:* Let $x_0$ be an arbitrary point in *X*. Since *A* and B are surjective, we choose a point $x_1$ in *X* such that $Ax_1 = x_0$ and for this point $x_1$, there exists a point $x_2$ in *X* such that

$Bx_2 = x_1$. By this way, we can define a sequence $\{x_n\}$ in X such that

$$Ax_{2n+1} = x_{2n} \text{ and } Bx_{2n+2} = x_{2n+1} \qquad (2.2)$$

Suppose that $x_{2n} = x_{2n+1}$ for $n \geq 0$. Then $x_{2n}$ is a fixed point of $A$. If $x_{2n+1} \neq x_{2n+2}$ then from (2.1), we have

$$d(x_{2n}, x_{2n+1}) = d(Ax_{2n+1}, Bx_{2n+2})$$

$$\geq \phi\left( d(x_{2n+1}, x_{2n+2}), d(Ax_{2n+1}, x_{2n+1}), d(x_{2n+2}, Bx_{2n+2}), \frac{d(Ax_{2n+1}, x_{2n+2}) + d(Bx_{2n+2}, x_{2n+1})}{1 + d(x_{2n+2}, Ax_{2n+1})d(x_{2n+1}, Bx_{2n+2})} \right)$$

$$\geq \phi\left( d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+1}), d(x_{2n+2}, x_{2n+1}), \frac{d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})}{1 + d(x_{2n+2}, x_{2n})d(x_{2n+1}, x_{2n+1})} \right)$$

$$\geq \phi(d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+1}), d(x_{2n+2}, x_{2n+1}), d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+1}))$$

This implies, by $(C_3)$

$$d(x_{2n}, x_{2n+1}) \geq hd(x_{2n+1}, x_{2n+2}) \quad .$$

This yields a contradiction, and so $x_{2n+1} = x_{2n+2}$. Thus $x_{2n}$ is a common fixed point of $A$ and B. If $x_{2n+1} = x_{2n+2}$ for some $n \geq 0$, it is similarly verified that $x_{2n+1}$ is a common fixed point of $A$ and $B$. Without loss of generality, we can suppose $x_n \neq x_{n+1}$, for each $n \geq 0$. From (2.1), we have

$$d(x_{2n}, x_{2n+1}) = d(Ax_{2n+1}, Bx_{2n+2})$$

$$\geq \phi\left( d(x_{2n+1}, x_{2n+2}), d(Ax_{2n+1}, x_{2n+1}), d(x_{2n+2}, Bx_{2n+2}), \frac{d(Ax_{2n+1}, x_{2n+2}) + d(Bx_{2n+2}, x_{2n+1})}{1 + d(x_{2n+2}, Ax_{2n+1})d(x_{2n+1}, Bx_{2n+1})} \right)$$

Which implies, from $(C_3)$

$$d(x_{2n+1}, x_{2n+2}) \leq \frac{1}{h} d(x_{2n}, x_{2n+1}).$$

Similarly

$$d(x_{2n+1}, x_{2n+2}) = d(Bx_{2n+2}, Ax_{2n+3})$$

$$\geq \phi\left( d(x_{2n+3}, x_{2n+2}), d(x_{2n+3}, Ax_{2n+3}), d(x_{2n+2}, Bx_{2n+2}), \frac{d(x_{2n+3}, Bx_{2n+2}) + d(x_{2n+2}, Ax_{2n+3})}{1 + d(x_{2n+2}, Ax_{2n+3})d(x_{2n+3}, Bx_{2n+2})} \right)$$

$$\geq \phi(d(x_{2n+3}, x_{2n+2}), d(x_{2n+3}, x_{2n+2}), d(x_{2n+2}, x_{2n+1}), d(x_{2n+3}, x_{2n+2}) + d(x_{2n+2}, x_{2n+1}))$$

Which implies, by $(C_3)$

$$d(x_{2n+1}, x_{2n+2}) \geq hd(x_{2n+2}, x_{2n+3})$$

Or $$d(x_{2n+2}, x_{2n+3}) \leq \frac{1}{h} d(x_{2n+1}, x_{2n+2})$$

Therefore, we obtain

$$d(x_{n+1}, x_{n+2}) \leq \frac{1}{h} d(x_n, x_{n+1})$$

Since $h>1$, by lemma of Jungck[2], $\{x_n\}$ is a Cauchy sequence and hence it converges to some point $z$ in $X$. Consequently, the sub-sequence $\{x_{2n}\}, \{x_{2n+1}\}$ and $\{x_{2n+2}\}$ also converges to $z$.

Since $A$ and $B$ are surjective, there exist two points $v$ and $w$ in $X$ such that $z = Av$ and $z = Bw$. Thus using (2.1), we have

$$d(x_{2n}, z) = d(Ax_{2n+1}, Bw)$$

$$\geq \phi\left( d(x_{2n+1}, w), d(x_{2n+1}, Ax_{2n+1}), d(w, Bw), \frac{d(x_{2n+1}, Bw) + d(w, Ax_{2n+1})}{1 + d(w, Ax_{2n+1}) d(x_{2n+1}, Bw)} \right)$$

$$\geq \phi\left( d(x_{2n+1}, w), d(x_{2n+1}, x_{2n}), d(w, Bw), \frac{d(x_{2n+1}, Bw) + d(w, x_{2n})}{1 + d(w, x_{2n}) d(x_{2n+1}, Bw)} \right)$$

Letting $n \to \infty$ we get

$$0 = d(z, z) \geq \phi\left( d(z, w), d(z, z), d(w, z), \frac{d(z, z) + d(w, z)}{1 + d(w, z) d(z, z)} \right)$$

$$\geq \phi(d(z, w), 0, d(w, z), d(w, z)),$$

Which implies, by $(C_3)$ , $0 \geq hd(z,w)$

So that $z=w$. Similarly, we have $z=v$. Therefore, $A$ and $B$ have a common fixed point in $X$.

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